Modelling Multivariate Dependence Structures in Insurance and Credit Risk via Copulas

By

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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The fact that I lived thousands of kilometres away from my safety net for almost 10 years implies that you tolerated not having your only daughter by your side for the same amount of time.
This thesis is dedicated to both of you.

List of Publications

- Mohd Ramli S.N., J. Jang. 2014 *Neumann Series on the Recursive Moments of Copula-Dependent Aggregate Discounted Claims.* Risks. 2014; 2(**2**):195-210. {http://www.mdpi.com/2227-9091/2/2/195}

- Mohd-Ramli S.N., J. Jang. *A Multivariate Jump Diffusion Process for Counterparty Risk in CDS Rates*. (Submitted to European Actuarial Journal)

- Mohd-Ramli S.N., J. Jang. *Jump Diffusion Model with Copula Dependence Structure in Defaultable Bond Pricing*. (Submitted to Annals of Actuarial Science)

Abstract

This PhD thesis seeks to offer a new framework that accommodates dependency in pricing an insurance portfolio following the renewal risk model, corporate bonds, as well as credit default swaps (CDS). This will be achieved by combining the approach and methodology of actuarial science with stochastic processes and probability theories, as well as employing a hint of the integral calculus used in the electromagnetic and viscoelasticity fields.

This thesis is a collection of three papers, which are presented in Chapters 2, 3 and 4. While Chapters 3 and 4 can be read in conjunction with each other, Chapter 2 can be read in isolation because it presents a completely different perspective of insurance to the financial perspective taken in the other two articles (Chapter 3 and 4). Nevertheless, the three papers share the same scope, which is the use of copula to capture the dependency between variables. In total, four copulas are explored: the Farlie-Gumbel-Morgenstern (FGM) copula, Gumbel copula, Gaussian copula and Student-t copula. However, only three copulas are compared in each working paper. The first article in Chapter 2 models a continuous time renewal risk process, and uses copulas to capture the dependence structure between the claims inter-arrival time and discounted claims size. The second and third articles work under the framework of a reduced form model and use various copulas to capture the dependence structure between the jump sizes of the intensity processes, each of which is represented by a jump diffusion process.

Taking the insurance perspective, the first article - titled *Neumann Series on the Recursive Moments of Copula-Dependent Aggregate Discounted Claims* - studies the recursive moments of aggregate discounted claims, where the dependence between the interclaim time and the subsequent claim size is considered. Using the general expression for the m^{th} order moment proposed in [1] which takes the form of the Volterra Integral Equation (VIE), we used the method of successive approximation to derive the Neumann series of the recursive moments. We then compute the first two moments of aggregate discounted claims, i.e. its mean and variance, based on the Neumann series expression where the dependence structure is captured by the FGM copula, Gaussian copula and Gumbel copula, with exponential marginal distributions. Insurance premium calculations with their figures are also illustrated.

The second work – titled *A multivariate jump diffusion process for counterparty risk in CDS rates* – considers counterparty risk in CDS rates. To do so, it uses a multivariate jump diffusion process for obligors' default intensity, where jumps (i.e. magnitude of contribution of primary events to default intensities) occur simultaneously and their sizes are dependent. For these simultaneous jumps and their sizes, a homogeneous Poisson process and three copulas, which are Farlie-Gumbel-Morgenstern (FGM), Gaussian and Student-t copulas are used. This project applies copula-dependent default intensities of multivariate Cox process to derive the joint Laplace transform that provides us with joint survival/default probability and other relevant joint probabilities. For that purpose, the piecewise deterministic Markov process (PDMP) theory developed in [2] and the martingale methodology in [3] are used. The survival/default probability is computed using the three copulas and exponential marginal distributions, and the results are applied to calculate CDS rates, assuming deterministic rate of interest and recovery rate. Sensitivity analysis for the CDS rates were also conducted by changing the relevant parameters and providing their figures.

The final article – titled *Jump diffusion model with copula dependence structure in defaultable bond pricing* – studies the pricing of a defaultable bond under various copulas. For that purpose, it used a bivariate jump diffusion process for a bond issuer's default intensity and the short rate of interest. We assume the jumps (i.e. magnitude of contribution of primary events to default intensities) occur simultaneously and their sizes are dependent. For these simultaneous jumps and their sizes, a homogeneous Poisson process and three copulas – FGM copula, Gaussian copula and Student-*t* copula are used, respectively. The joint Laplace transform for the variables' integrated processes is derived to provide the expression for defaultable bond price, using copula-dependent jump sizes. Once again, we apply the piecewise deterministic Markov process (PDMP) theory developed in [2] and the martingale methodology in [3]. Zero coupon defaultable bond prices and their yield are computed using the three copulas and exponential marginal distributions. The model is then used to calibrate zero coupon bonds on one-day basis as well as for an extended period of one year. Calibration results show that the Student-t copula provides the best fit relative to the other two copulas.

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1
Introduction

This thesis examines the dependency of variables using copula with applications in pricing financial products, such as a zero coupon bond and credit default swap (CDS), as well as insurance premium calculation. With copula linking the variables, the contributions of this thesis concentrate on two areas

- multivariate intensity modelling with jump diffusion process and its explicit form of bond price, as well as CDS price
- the explicit form of recursive moments of an aggregate discounted claim, assuming the claim arrival time, following Poisson distribution.

This introduction discusses the motivation and objectives of undertaking the studies in each of the aforementioned research topics and provides an overview of the thesis.

1.1 Overview & Motivation

The increasingly frequent occurrence of catastrophic events implies that the assumption of independence between event occurrence and claim severity is no longer sufficient in insurance risk modeling. This is especially true given its impact on pricing and reserving, capital allocation, solvency, as well as regulatory systems. Examples of this effect include the February 2009, Victorian bushfire in Australia (10,200 insurance claims amounting to approximately AUD 1.2 billion), the February 2011, Christchurch earthquake (USD 13 bn insured economic losses), the 2011 Great Eastern Japanese earthquake (loss amounting to as much as USD 40 billion), as well as the 2012 Hurricane Sandy (an expected loss of USD 25 billion) (see [2, 57]).

At the same time, corporate bonds' default rates have declined since 2009 as the world economy has begun to recover from the global financial crisis in response to government initiatives.

However, the continuing distress in the United States (US) and Eurozone economies may jeopardize the low default rate environment. Hence, it is necessary to develop pricing models for corporate bonds that capture the dependence structure between obligors' default intensity and macroeconomics variables.

With the increasing globalization of business, a shock which initially affects a couple of institutions or a particular region of the economy may spread to the rest of the financial industry and then infect the wider economy. The financial events of late 2008 provide a perfect illustration of this. The mismanagement of subprime mortgages in the US has had far reaching consequences. In the US it has resulted in federal takeover of Fannie Mae and Freddie Mac, the Bank of America takeover of Countrywide Financial Corporation and the bankruptcy of New Century Financial Corporation (see [46], [55], [66] and [68] for instance). The contagion spread with further bankruptcies and default of mortgages, and lenders in US making significant losses. The subprime mortgage meltdown resulted in new ownership of Bears Stern and Merrill Lynch and the bankruptcy of Lehman Brothers. These events have, in turn, caused the worldwide collapse of stock prices and shaken global financial markets further, generating new waves of default and bankruptcy.

1.1.1 Credit Default Swap

Credit Default Swap (CDS) is a bilateral agreement where the protection buyer transfers the credit risk of a reference entity to the protection seller for a specific period, T. The buyer of this protection pays a certain premium, called spread (denoted as $\bar{s}$ in this thesis), to the seller until the maturity of the contract, or until default occurs, whichever is earlier. The spread is paid against the default of the reference entity, reflecting the riskiness of the of the underlying credit. Readers are referred to texts on derivatives e.g. [62] and [13], for a more thorough definition on CDS.

Following the Basel II Accord (2004) (see [5]) that requires banks to set aside a certain amount of capital to cover the risk inherent to their credit portfolios, it is therefore imperative for financial institutions and insurance companies to use a good model in order to forecast company ratings accurately. Market surveys conducted by the International Swaps and Derivatives Association (ISDA) show notional amounts of outstanding interest rate and currency swaps reaching US$866 billion in 1987, US$17.7 trillion in 1995, and US$99.8 trillion in 2002; an astonishing compounded growth rate of 37.2% per year ([16]). The significance of the market for credit instruments was mentioned in [22] pointed out that the nominal, outstanding value of the global over the counter (OTC) derivative at the end of 2008 of US$592,000 billion and the notional amount of outstanding credit derivative swaps (CDS) was US$42,000 billion, compared to the 2008 total world GDP which was about US$61,000 billion.

In the years following the introduction of the Li model (see [49]) that relies on the normal copula and multivariate normal joint distribution that provided a new perspective on credit risk modelling, the credit derivative market grew exponentially to the extent that the market value reached almost tenfold of total world GDP. However, as we have seen in the year 2008, inadequate mathematical modelling caused the American Insurance Group (AIG) not being able to quantify the risks in their CDS portfolio and reduce their CDS exposure much earlier, leading to the collapse of the company before being bailed out by the US government. Hence, with the huge notional amount of derivatives being traded in the world, the need to develop pricing models that capture the dependence structure between obligors as well as incorporating the element of jumps/shocks in the economy becomes inevitable.

1.1.2 Risks: Insurance Risk and Default Risk

The interplay between insurance and finance seen in the structure of financial products such as the CDS implies that the correlation and dependence between the obligors as well as the macroeconomic and market variables are imperative in pricing and reserving.

From the insurance perspective, [23] defines risk as *"a non-negative random variable (RV) which represents the random amount of money paid by an insurance company to indemnify a policyholder, a beneficiary and/or a third party in execution of an insurance contract"*. Despite the premium amount being traditionally governed by the law of large numbers, the need to include dependency in the premium and surplus determination is increasingly important with the complexity of insurance products, and the more frequent occurence of catastrophes. Pricing insurance products using the traditional approach may cause an insurance firm to charge a lower premium amount, and hence not to be fully prepared for potential risks caused by dependency of the variables of interest.

In contrast, default risk describes the potential for a counterparty to fail to meet its obligations, as defined in a financial contract, hence causing losses to the other party. This includes, for example, a bond issuer missing a coupon payment, a debtor failing to repay its loan, or a counterparty of a swap failing to make interest payments. The term default is not confined only to bankruptcy, but also encompasses reduced credit quality. While the former leads to a permanent halt of the entire transaction – that is, the future cash flows will not be paid – the latter leads to an increased probability of the counterparty going bankrupt, and hence is more difficult to assess.

1.2 Literature Review

1.2.1 Copula

Copulas provides the flexibility to choose a variety of marginal distributions for the variables being focused on, as opposed to the typical multivariate distributions that permits marginals of the same type as the joint distribution. This enables examinination of the effect of individual defaults on joint default behaviour using various types of distributions. Analogously, the correlation structure can be varied by choosing different types of copula to quantify the effects of default correlation on a portfolio. Many standard statistical texts offer illustrations of copula scatter plots with various dependence structures, such as in [23], [54] and [58].

Introduced by Abe Sklar in 1959 in [65], the copula is a multivariate distribution function with univariate marginal distribution functions restricted to the n-cube.

Definition 1.2.1. *A copula is a function C of n variables on the unit n-cube $[0,1]^n$ with the following properties:*

- $C(\mathbf{u}) \in [0,1]$

- $C(0,\ldots,u_k,\ldots,u_n) = C(u_1,\ldots,0,\ldots,u_n) = 0$
- $C(1,\ldots,1,u_k,1,\ldots,1) = u_k \ \forall\ k$
- n-increasing

Theorem 1.2.2. ***(Sklar's theorem)*** *Let H denote a n-dimensional distribution function with univariate margins $F_1,\ldots,F_n$. Then there exists a copula C such that for all real $(x_1,\ldots,x_n)$*

$$H(x_1,\ldots,x_n) = C(F_1(x_1),\ldots,F_n(x_n))$$

If $F_1,\cdots,F_n$ are continuous, then C is unique; otherwise, C is uniquely determined on $RanF_1 \times \cdots \times RanF_n$. Conversely, if C is a copula and $F_1,\cdots,F_n$ are distribution functions, then the function H is a joint distribution function with margins $F_1,\cdots,F_n$.

Sklar's theorem implies that if all the margins are **continuous**, the copula is **unique**, and determined uniquely on the ranges of the margins. Also, if $F_1^{-1},\ldots,F_n^{-1}$ denotes the generalised inverses of the marginal distribution functions, then for every $(u_1,\ldots,u_n)$ in the unit n-cube,

$$C(u_1,\ldots,u_n) = F\left(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)\right).$$

The application of copula to represent the dependence structure between variables have also been widely explored in the field of insurance and equity index modelling. In an attempt to represent the dependence structure between the interclaim time and the subsequent claim size, [4] used a Farlie-Gumbel-Mogenstern (FGM) copula to link the exponential interclaim times with generalized Pareto distributions for heavy tailed claim amounts. The FGM copula was used again in CDS pricing to link the default intensity of reference credit, CDS seller and buyer in [51]. Time-varying copulas were used to model international equity market co-movements in [34]. The authors found that the Student-t copula with Student-t marginals is a good candidate for modelling the returns arising in an international equity index portfolio where the extreme losses are known to have a tendency to occur simultaneously. Another study under the reduced form framework, postulated a Gaussian copula on the exponential triggers of the default times in an attempt to model default correlation between the reference credit and the counterparty of a CDS contract [10].

Assuming that there exists dependence structure between the variables being modelled (which are assumed to occur simultaneously), this thesis captures the said structure by using copula. In the first article featured in Chapter 2 of this thesis, the variables are claim severity (or claim size) and inter-claim waiting time. The second and third articles model the default intensity of the obligor's of a CDS contract, as well as the short rate and a bond issuer's default intensity, respectively. The variables in the second and the third articles are assumed to follow the jump diffusion process.

Four copulas were explored which are the FGM, Gumbel, Gaussian and the student-t copula. Their multivariate probability distribution functions and respective scattered plots are given as follows:

- FGM copula

$$c^{FGM}(u_1,\ldots,u_d) = 1 + \sum_{i=1}^{d} \theta_{ij}^{F} \prod_{j=1}^{d} \left(1 - 2u_j\right) \tag{1.1}$$

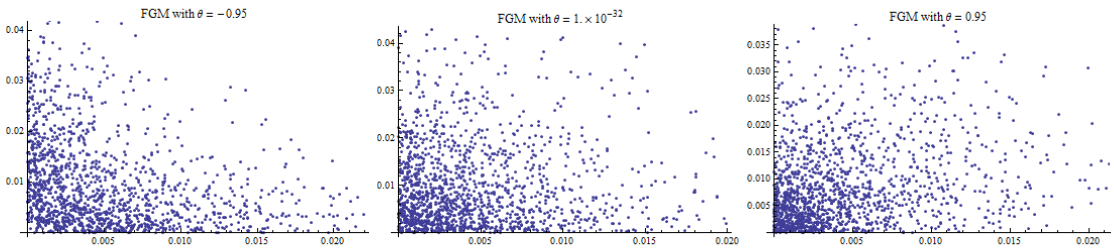

FIGURE 1.1: FGM copula with exponential margins and dependence parameter -0.95, 0, 0.95

- Gumbel copula

$$c_\theta^M(F_X(x), F_W(s)) = \frac{(-\ln u)^\theta}{-u \ln u} \frac{(-\ln v)^\theta}{-v \ln v} \frac{\sqrt[\theta]{(-\ln u)^\theta + (-\ln v)^\theta}}{[(-\ln u)^\theta + (-\ln v)^\theta]^2} \times \frac{\sqrt[\theta]{(-\ln u)^\theta + (-\ln v)^\theta} + \theta - 1}{e^{\sqrt[\theta]{(-\ln u)^\theta + (-\ln v)^\theta}}}. \tag{1.2}$$

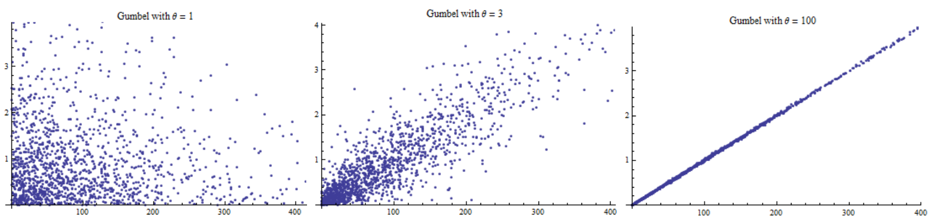

FIGURE 1.2: Gumbel copula with exponential margins and dependence parameters one, three, 100

- Gaussian copula

$$c_\Theta^G(u_1, \ldots, u_d) = |\Theta|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\zeta^{\mathbf{T}}\left(\Theta^{-\mathbf{1}} - \mathfrak{I}^d\right)\zeta\right\} \tag{1.3}$$

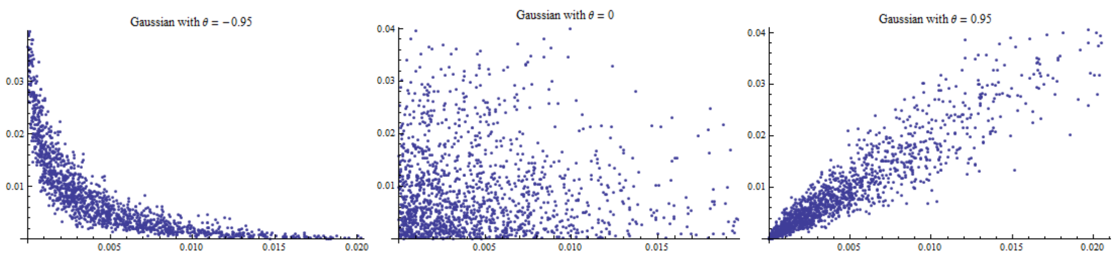

FIGURE 1.3: Gaussian copula with exponential margins and dependence parameter -0.95, 0, 0.95

- Student-t copula

$$c^{t}_{\upsilon,\Theta}(u_1,\dots,u_d) = |\Theta|^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\upsilon+d}{2}\right)\left\{\Gamma\left(\frac{\upsilon}{2}\right)\right\}^{d-1}\left[1+\frac{1}{\upsilon}\zeta^{\mathbf{T}}\,\Theta^{-\mathbf{1}}\zeta\right]^{-\left(\frac{\upsilon+d}{2}\right)}}{\left\{\Gamma\left(\frac{\upsilon+1}{2}\right)\right\}^{d}_{j=1}{}^{d}\left(1+\frac{1}{\upsilon}\zeta_j^2\right)} \tag{1.4}$$

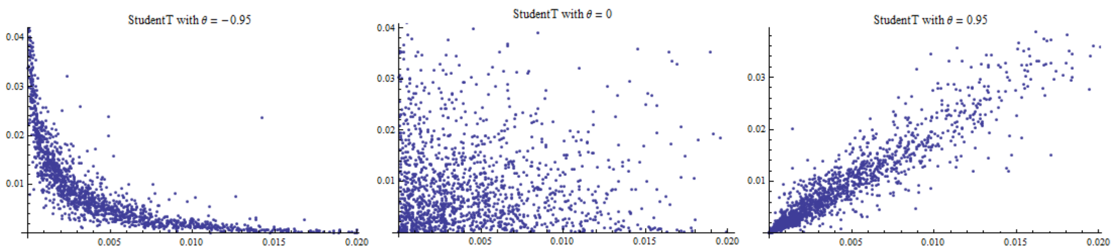

FIGURE 1.4: Student-T copula with exponential margins and dependence parameter -0.95, 0, 0.95

where d is the dimension of the variables, υ is the degrees of freedom and Θ is the covariance matrix containing the dependence measure θ. We also define $\zeta = [\ \zeta_1 \ \cdots \ \zeta_n\ ]^T$ where $\zeta_i = \Phi^{-1}(u_i)$ or $\zeta_i = t_{\upsilon}^{-1}(u_i)$ are the inverse Gaussian or inverse student-t distribution with degrees of freedom υ respectively taken on the variables u_i.

The FGM copula, was used for its simplicity and analytical tractability, for which it was also used in [37] and [51]. Its simplicity allows for the closed form expressions of final results to be easily derived. It was also used to compare the current study's numerical results against their counterparts in [4] and [51]. The Gumbel copula was also chosen, as it could be adopted by an insurance company, that assumes that risks with extreme magnitude have the tendency to occur together (see [21]).

The Gaussian copula, was chosen to examine the effect of elliptical copula on simultaneous jumps in the intensity process of CDS counterparties as well as on the dependence between claim size and inter-claim time, since this has not been explored extensively to the best of the researchers' knowledge. The student-t copula was chosen to incorporate the possibility of having more frequency of higher and/or smaller as well as opposing joint jumps size impact on the obligors' intensity.

Three copulas are compared in each working paper. Chapter 2 calculates the first, second and m^{th} moment of aggregate discounted claims under three copulas: the FGM, Gaussian and Gumbel copula in order to explore the different effect each copula family can cause on the moments. In chapter 3 and 4, another member of the elliptical copula family was employed, which is the student-t copula, instead of the Gumbel copula. This was done in an attempt to investigate how different the bond price and CDS rates are relative to the price and rates under the Gaussian copula, which was widely used prior to the 2008 Global Financial Crisis.

1.2.2 Volterra Integral Equation of the 2nd Kind

The most general form of linear integral equation (IE) is given by:

$$h(T)\Psi(T) = g(T) + \int_a^{b(T)} K(T,s)\Psi(s)ds, \tag{1.5}$$

where $\Psi(T)$ is the solution to the IE that needs to be obtained, $g(T)$ is a given function and $K(T,s)$ is the kernel for the IE. Equation (1.5) can be a homogeneous/non-homogenous, Volterra/Fredholm IE of the 1st/the 2nd kind, for which readers are referred to the conditions given in Section 2.1 of [44]. Linear IE can be solved either numerically using methods, such as the Runge–Kutta and collocation methods (see, e.g., [52] and [14]), or solved explicitly, such as by obtaining its Neumann series via the Picard method of successive approximations or using the Laplace transform method.

If we have $g(T) \neq 0$, $h(T) = 1$, and $b(T) = T$, (1.5) becomes:

$$\Psi(T) = g(T) + \int_a^{T} K(T,s)\Psi(s)ds, \tag{1.6}$$

which is a non-homogeneous Volterra integral equation (VIE) of the second kind. The Volterra IE is widely used in the areas of viscoelasticity and electromagnetic to compute the dynamics of materials that ”contain” memory, other than being useful in renewal theory and demography (see, e.g., [6] and [43], as well as the references therein for a more rigorous treatment on VIEs).

A unique and continuous solution, $\Psi(T)$, is obtainable if there is a combination of a continuous kernel, $K(T,s)$, in the region $a \leq s \leq T \leq b(T)$ with a function, $g(T)$, that is continuous in the region $a \leq T \leq b(T)$, even though it is not a requirement for the kernel function, $K(T,s)$, to be continuous (see page 1 of [26] and page 5 of [64]). For the case of a discontinuous kernel function, we need to check if $K(T,s)$ fulfills the three regularity conditions set on page 3 of [43], and, hence is an L^2-function.

In this thesis, as we assume that the claim size and the inter-claim time are continuous r.v.'s, and by corollary 2.2.6 of [58] on copula continuity, $g(T)$ is therefore a continuous function for $s \in [0,T]$ and $x \in [0,\infty]$, since it is the sum and product of continuous functions. The kernel function is also continuous, as it is an exponential function given by:

$$K(T,s) = e^{-(\beta+m\delta)(T-s)}. \tag{1.7}$$

Additionally, it is a bounded function in the square $\Pi = \{(T,s) : a \leq T \leq b(T), a \leq s \leq T\}$.

Now, the recursive moment equation resulting from the technique used in [48] and [4] takes the form of a VIE of the second kind, which is widely used in the fields of mathematical physics, such as the electromagnetic and viscoelasticity fields, to represent the dynamics of materials that contain memory (refer to [6], [15], [44], [59] and [64]). Chapter 2 uses the same technique and then extends the recursive moments obtained in [4], so that it can be applied to any continuous bivariate distribution to accommodate the dependency between the two variables. To do so, the recursive expression of the moments were solved using the Neumann series obtained via the Picard method of successive approximations, upon which a selection of bivariate distributions could be applied, including bivariate copula.

1.2.3 Reduced Form Model vs. Structural Model

When selecting the modelling approach to be used in chapters 3 and 4, two types of model were considered, which are the structural and the reduced-form model. This study took a similar approach to [37] and [51] by assuming that the default intensities are driven by Cox process to price the default risk embedded in corporate bond, inspired by the similarity between the claims arrival process and default time arrival.

Corporate debt valuation models can be divided into two main approaches which are the structural approach and reduced form approach. The first class of models under the structural approach views the firm's liabilities as contingent claims issued against the firm's assets, with all the payoffs to the firms's liabilities in bankruptcy completely specified (see the seminal work in [53] and [8]). In other words, bankruptcy is viewed as the event when the firm's value hits a pre-specified boundary. The view undertaken in this class of models was then simplified in [50] and [33], in which the cash flows to risky debt in the event of bankcruptcy were exogenously specified as a given fraction of each promised dollar in the event of bankcruptcy. This is to avoid the need to understand the complex priority structure of payoffs to firm's liabilities in the event of bankruptcy. In [60], the bond prices follow a structural default model with jumps computed with Monte Carlo simulation based on Brownian bridge algorithm.

Even though structural models elegantly link an event of default to the value of a firm's financial assets, the hypothesis that both the recovery rate and default probability depend on the value of the firm causes this approach to be relatively inflexible because the necessary company information may not be available to some investors ([17]). Further, modelling the value of the asset as a diffusion process in continuous time gives a typical hump-shaped credit spread curve, with zero intercept, which is an unrealistic suggestion that implies *a default event is predictable by knowing the information available at any time t*. This approach may also neglect other factors that could trigger the default of the firm, which would result in a much smaller credit spread generated than those actually perceived by the market players. Thus, this study choose to work under the reduced form framework which is generally mathematically tractable, thereby making it convenient to calibrate to market data. From the practical perspective, this approach is also preferred by investors who do not have full access to a firm's information since the reduced form approach has flexibility in terms of the default information being embedded in the observed securities price.

In contrast to the structural approach, under which default correlation is modelled via asset correlation, the reduced form approach introduces the correlation aspect through a model in which the default of one obligor triggers the default of another, albeit suffering from a lack of clear economic rationale that could be used to describe the nature of a particular process ([3]). Previous studies of the reduced form approach have taken several directions in the attempt to incorporate default correlation and multiple defaults. For instance, [47] prepared a convenient framework that allowed for dependencies between default intensities and state variables to analyze financial instruments subject to credit risk through counterparty default and to analyze derivatives with credit risk variable as the underlying. Under a generalized K-states Markovian model, Cox process was used to model the (stopping) time when the rating changed until the issuer went default in the last state. One of the earliest papers to use the term reduced form approach, [25] treated default as an unpredictable event governed by external hazard rate process. The researchers showed that a contingent claim that is subject to default risk can be priced just like the default-free claim simply by replacing the short rate r_t with the default-adjusted short rate process $R_t = r_t + h_t l_t$ under an equivalent martingale measure in an arbitrage free framework. In [40], the then existing reduced-form model was extended and the concept of counterparty risk was introduced to capture the economy-wide and inter-firm linkages by including jumps in the default intensities that follows a Cox process.

The direction in which copula has been incorporated in the reduced form approach was initiated by [49] whereby the Gaussian copula is used to derive the joint probabilities of the obligors. Apart from incorporating the copula into the reduced form approach, many authors such as [31, 36, 51, 63], also include jump elements. This approach was also used by [41] to achieve a closed form solution of catastrophe bond price. In [1], the reduced form approach was combined with the Hawkes process to model the asset returns and subsequently derived the closed-form expressions for observable moments of log returns.

Another approach is the hybrid of the structural and reduced form approach, developed in [39] whereby the bankcruptcy process is modelled as a continuous time Markov process with discrete state space representing the firm's credit ratings. This model originates from the Jarrow and Turnbull (1995) model that takes the reduced form approach promoted in [38]. The hybrid approach further simplifies the view taken in the structural models by specifying the credit event exogenously and allowing the bankcruptcy assumptions to be imposed only on observables (i.e. the firm's credit ratings) as opposed to firm's asset values.

The work in chapter 3 extends the martingale approach in [36] to a multivariate dimension to capture the dependence structure between the obligors' default intensity, each taking a form of a model under the reduced form approach, and then uses it to price the CDS rate. Then in chapter 4 the same approach was applied to a bivariate dimension in capturing the dependence structure between the short rate and the counterparties' default intensity, as opposed to the independent structure between the bond issuer and interest rate in [37].

1.2.4 Jump Diffusion Model

This study concentrates on a very specific vector of intensity process: the multivariate jump diffusion process. In this process, the intensities are triggered by primary events that result in simultaneous positive jumps in intensity processes. These include events such as oil and commodity prices, governments fiscal and monetary policies, the release of corporate financial reports, political and social decisions, rumours of mergers and acquisitions among firms, the collapse and bankruptcy of firms, the September 11 World Trade Centre catastrophe, Hurricane Katrina and so forth. As time passes, default intensity processes decrease as all firms in the market do their best to avoid bankruptcy after the arrival of a primary event. This decrease continues until another event occurs that again results in simultaneous positive jumps in intensity processes.

By using the jump diffusion process to represent interest rate, asset returns as well as default intensity (such as the work by [19], [24], [45], [51] and [56]), the effects of shocks on the variable being modelled could be captured. The surprise elements that caused the effects of shocks could come from both the demand and supply sides of the economy as well as catastrophe events. Readers are referred to [67] and [45] for an elaboration on the various motivation of using a jump diffusion process.

Let $N_1(t), t \geq 0$ be a homogenous Poisson process with a unit intensity. We also let $B(t), t \geq 0$ be a process independent of $N_1(t)$ where $B(0) = 0$, $\mathbb{P}(B(t) < \infty) = 1$ for any $t > 0$ with non-decreasing and right continuous path. A Cox process $N(t)$ is defined as the superposition of $N_1(t)$ and $B(t)$, i.e. $N(t) = N_1(B(t))$. Readers are referred to e.g. [7] for a more thorough discussion on Cox process.

Numerous papers have examined modelling for the dependence of default intensities via a Cox process or point process for the purpose of derivative pricing (such as [20], [63], [42], [71], [32] and [61]). The use of jump diffusion model in pricing the CDS instrument, without using copula, was also explored in [12]. The analytical expression for CDS price offered in the literature was obtained using the Jamshidian option decomposition trick as in [35].

In this thesis, we work under the probability space $(\Omega, \mathscr{J}, \mathbb{P})$ consisting of the sample space Ω, the σ-algebra $\mathscr{J}$ and the probability measure $\mathbb{P}$. The variables being modelled in chapter 3 and 4 (i.e. the short rate and default intensity of the financial obligors) is assumed follow the jump diffusion model defined as below:

$$dX_t^{(i)} = c^{(i)}\left(b^{(i)} + a^{(i)}X_t^{(i)}\right)dt + \sigma^{(i)}\sqrt{X_t^{(i)}}dW_t^{(i)} + dC_t^{(i)} \tag{1.8}$$

Under this setting,

- $c^{(i)}b^{(i)}$ represents the long term mean level of the variable being modelled
- $c^{(i)}a^{(i)}$ represents the drift coefficient, which is the speed at which the variable is driven back to its long term mean with $a^{(i)} < 1$.
- $\sigma^{(i)}$ is the diffusion coefficient; and
- $W_t^{(i)}$ is a standard Brownian motion governing variable $X^{(i)}$

This study also defines

$$C_t^{(i)} = \sum_{j=1}^{M_t} Y_j^{(i)}$$

as a pure jump process with M_t being the number of jumps up to time t and $Y_j^{(i)}, j = 1, 2, \cdots, M_t$ being their sizes. It is assumed that Y_j's occur simultaneously and that they are independent and identically distributed (i.i.d) with distribution function $F(y^{(i)})$. In order to ensure positivity, the condition $2c^{(i)}b^{(i)} > \sigma^{(i)}$ has to be fulfilled, just like the seminal Cox Ingersoll Ross (1985) model [18].

Following the definition of the Cox process, we define the default arrival time as

$$\tau^{(i)} = \inf\left\{t : N_t^{(i)} = 1 \,\middle|\, N_0^{(i)} = 0\right\}$$

for $i = 1, \cdots, n, r$. This is equivalent to the first jump time of the Cox process $N_t^{(i)}$ $(i = 1, 2, \cdots, n, r)$ where $i = 1, 2, \cdots, n$ indicates the obligor involved in the financial contract and r indicates the short rate.

Alternatively, the default event can also be seen as the first time t when the integrated hazard rate $\int_0^t X_u^{(i)} du$ breaches a certain threshold level $U^{(i)}$ that remains unknown to the economy prior to default. Stated simply, a high value of $\int_0^t X_u^{(i)} du$ implies that default will happen soon (only a further small value of t is needed to breach the threshold level $U^{(i)}$).

Some literature has taken a different approach by manipulating the components of the jump diffusion model. Taking the jump component as zero, one ends up with the Cox-Ingersoll-Ross model [18]. This model has been used with slight modification in option pricing such as in [9–11]. In other instances, some literature modelled the concerned variables using the shot noise process by letting $\sigma^{(i)} = 0$ (e.g. [20], [27–30]).

In Chapter 3, the jump diffusion model is used to represent the default intensity of CDS counterparties while in Chapter 4 it is used to represent the default intensity of a bond issuer and the short rate. In both chapters, the diffusion term is allowed to be non-zero in an attempt to add the element of firm specific default risk. This section ends with figure 1.5 illustrating the simulated jump diffusion process with dependence structure captured by a student-t copula. Illustrations of the jump diffusion processes with other copula dependence structure are available in Chapters 3 and 4.

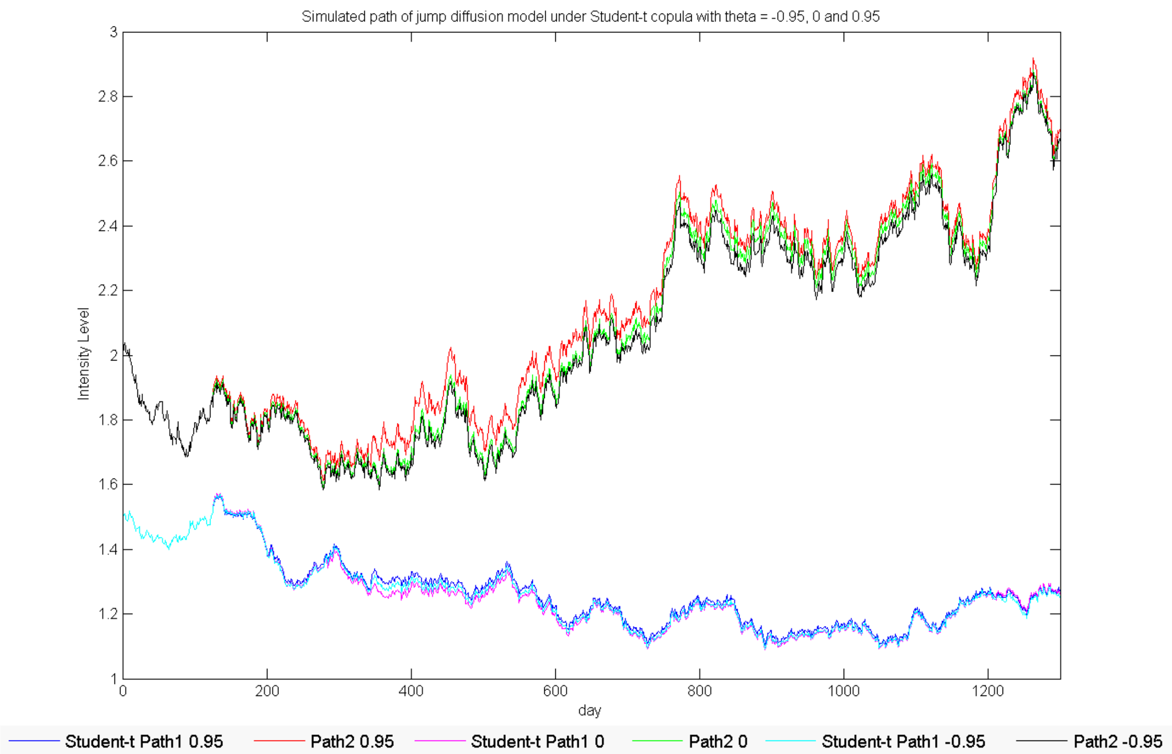

FIGURE 1.5: Simulated paths of jump diffusion process with dependence structure capture by student-t copula

1.2.5 Numerical Computation

This study computed the values of moments, CDS rates, bond prices and yields only up to $\theta = \pm 0.95$ for the case of FGM, Gaussian and student-t copulae. For values of θ nearing the tail side of the elliptical copulae - that is $|\theta| > 0.95$ - the values of moments, CDS rates, and bond prices and yields showed a non-stable behaviour. Hence, due to time constraint, this study was unable to cover the numerical computation side of the study extensively until the point ± 0.999 as was initially intended.

Mathematica in-built default integration strategy was used in the computation of numerical integration, which is the global adaptive strategy, given by the command 'GlobalAdaptive'. With this strategy, the integral subregion with the largest error estimate was divided recursively into two equal parts, and the integral and error estimates were conducted for each half. The global error was expected to decrease monotonically as the number of integration steps increased. The strategy was combined with the additional integration procedure, 'MaxErrorIncreases', which allowed the error estimates to be reduced monotonically by increasing the number of integration steps. Instead of the default value of MaxErrorIncrease of 2,000 steps for multidimensional integrals, the value of 16,000 steps was used in an attempt to balance accuracy with the time available.

For the bond price calibration, the Mathematica in-built function, 'NMinimize' was used, which is useful in determining the numerical value of the global minimum of a non-linear programming problem. This is typically done by allowing both decrease and increase of the objective function. Depending on the nature of the objective function and constraints, the function operates using the linear programming, the Nelder-Mead, the differential evolution or the simulated annealing algorithms. Readers are referred to [69] and [70] for more information. Without specifying any algorithms, this study used 'NMinimize' to minimize the squared difference between the model price and the market price, subject to the model constraints implied in section 2 of Chapter 4 as well as non-negativity of the volatilities of the elliptical copula.

1.3 Structure of the Thesis

This thesis consists of three research papers that showcase the application of copulas in capturing the dependence structure in the fields of insurance and applied finance.

Chapter 2 showcases the working paper titled *"Neumann Series on the Recursive Moments of Copula-Dependent Aggregate Discounted Claims"*, which has been published in the special edition of the journal *Risks: Application of Stochastic Processes in Insurance*. Prior to the publication, this working paper has been presented at the following conferences:

- 2013 PhD AFAS-Econ Workshop, Macquarie University, 24 September 2013
- Higher Degree Research EXPO 2013, Macquarie University, 5 to 7 November 2013
- Quantitative Methods in Finance Conference, 17 to 20 December 2013 (hosted by University of Technology, Sydney)

Chapter 3 presents the working paper titled *A multivariate jump diffusion process for counterparty risk in CDS rates*, which has been submitted to the European Actuarial Journal and is currently under review. The working paper was presented at the following conferences:

- Higher Degree Research EXPO 2012, Macquarie University, 12 to 13 November 2012
- 48^{th} Actuarial Research Conference 2013, Temple University, USA, 31 July to 3 August 2013

Chapter 4 presents the working paper titled *Jump Diffusion Model with Copula Dependence Structure in Corporate Bond Pricing*, which has been submitted to the Annals of Actuarial Science and is currently under review. The working paper has been presented at the following conferences:

- Higher Degree Research EXPO 2011, Macquarie University, 10 to 11 November 2011
- Australasian Actuarial Education and Research Symposium, 2011, ANU, 1 to 2 December 2011
- Bachelier Finance Society 7th World Congress (BFS) 2012, 19 to 22 June 2012
- International Conference on Computing, Mathematics Statistics 2013, Malaysia, 28 to 29 August 2013

Additionally, the working paper was accepted for presentation at the AFIR Colloqium 2012, Mexico City, 1 to 4 October 2012.

Finally, chapter 5 summarizes the thesis with the conclusion from each article as well as the potential direction of future research.

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2

Neumann Series on the Recursive Moments of Copula-Dependent Aggregate Discounted Claims

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This article has been published in the special issue of *Risks: Application of Stochastic Processes in Insurance*. It can be accessed at www.mdpi.com/2227-9091/2/2/195. The article is presented in its entirety here and hence contains repetitions of certain segments of the Introduction presented in Chapter 1.

Abstract We study the recursive moments of aggregate discounted claims, where the dependence between the inter-claim time and the subsequent claim size is considered. Using the general expression for the $m-th$ order moment proposed in [12], which takes the form of the Volterra integral equation (VIE), we used the method of successive approximation to derive the Neumann series of the recursive moments. We then compute the first two moments of aggregate discounted claims, i.e., its mean and variance, based on the Neumann series expression, where the dependence structure is captured by a Farlie–Gumbel–Morgenstern (FGM) copula, a Gaussian copula and a Gumbel copula with exponential marginal distributions. Insurance premium calculations with their figures are also illustrated.

Keywords: aggregate discounted claims; moments; copulas; Volterra integral equation; Neumann series; insurance premium

2.1 Introduction

As the occurrence of catastrophe events becomes more frequent, the assumption of independence between event occurrence and claim severity is no longer sufficient in insurance risk modeling, given its impact on pricing and reserving, capital allocation, solvency, as well as regulatory systems. The February 2009, Victorian bushfire in Australia (10,200 insurance claims amounting to approximately AUD 1.2 billion), the February 2011, Christchurch earthquake (USD 13 billion insured economic losses), the 2011 Great Eastern Japanese earthquake (loss amounting to as much as USD 40 billion), as well as the 2012 Hurricane Sandy (an expected loss of USD 25 billion) are the examples of this effect (see [1, 2]).

In dealing with the dependency between the inter-claim arrivals and claim sizes, various approaches have been proposed in previous studies that can be noticed in [3–11], as well as the references therein. Regardless of the model used, we notice that previous research focused on either examining the expression of the moments of the aggregate discounted claims, $Z(t)$, as can be seen in [6, 11–14], or by finding the related ruin measures and the ruin probability expressions, just like in [3–5, 10, 15].

Assuming the Poisson claim arrival process with claim sizes following mixed exponential distributions, [7] obtained the explicit expressions of the actuarial net premiums and the variances of the discounted aggregate claims from the Laplace transform of the distribution of the shot noise process, which was derived using the martingale approach. The first two moments of the aggregate discounted claims were obtained in [9] assuming the dependency between the claim sizes and the rates of claim occurrence affected by a Markovian environment, called the circumstance process. A delayed renewal process was also explored in [12–15], as well as [11] to accommodate the epochs between claim arrival and the observation of the risk process.

The asymptotical behaviour of a conditional tail probability dependence structure of claim sizes given the inter-claim arrival time was studied in [5, 8]. Assuming that the conditional tail of claim size given the inter-claim time satisfies a certain condition for a bounded inter-claim time and a really huge claim size, [5] obtained the asymptotic tail probabilities of the discounted aggregate claims. Three copulas were indicated as fulfilling this assumption, which are the Farlie–Gumbel–Morgenstern (FGM) and the Frank and Ali–Mikhail–Haq (AMH) copulas, and the Weibull claim size was paired with exponential inter-claim arrival time in their numerical example. On the other hand, [8] explored the analytical properties related to the same dependence structure described by the survival copulas, such as their local and global uniformity.

Conditioning on the first arrival and using a renewal theory argument, [12] derived a useful expression for the $m-th$ recursive moment, whereby the inter-claim arrival time and the claim severity are assumed to be independent. The same conditioning argument was then applied in [6, 10], assuming the FGM copula and then solved using the Laplace transform approach. More recently, [11] also adopted the same technique to derive the recursive moments of a Sparre Andersen risk process assuming a fairly general dependence structure between the inter-claim time and subsequent claim size variables, providing a simplified moments expression for assuming Erlang weights. Four types of copula were showcased in their examples of joint distribution between the said variables, which are the polynomial copula, the Bernstein copula, the generalized FGM copula, the extended FGM copula (references for these copulas are available in Section 3 of [11]).

The recursive moment equation resulting from the technique used in [6, 12] takes the form of a Volterra integral equation of the second kind, which is widely used in the fields of mathematical physics, such as the electromagnetic and viscoelasticity fields, to represent the dynamics of materials that contain memory (refer to [16–20]). We are interested in using the same technique and then extend the recursive moments obtained in [6], so that it can be applied to any continuous bivariate distribution to accommodate the dependency between the two variables. To do so, we solve the recursive expression of the moments using the Neumann series obtained via the Picard method of successive approximations, upon which a selection of bivariate distributions can be applied, including bivariate copula.

This article is structured as follows. Section 2.2 will introduce the general framework of the continuous time renewal risk model together with its recursive moments with exponentially distributed inter-claim time and general claim size distribution. The dependency between the claim size and inter-claim time are then specified using a bivariate copula. For that purpose, we consider three copulas, which are the FGM copula, the Gaussian copula, which is a type of elliptical copula, and the Gumbel copula, an Archimedean type of copula, which is a natural candidate to represent an extreme value copula that caters for the one-sided dependence structure (see [21]).

In Section 2.3, we introduce the Volterra integral equation, which will be solved using the successive approximations method, leading to the Neumann series expression of the recursive moments, which is the main result of this paper. The Neumann series expression of the recursive moments allows the flexibility to capture various dependence structures provided by copula probability density functions (pdf).

Section 2.4 starts with the comparison between the value of moments obtained by our Neumann series expression assuming the FGM copula and the closed form solution by [6]. We then present the numerical analysis, showing the value of moments across the dependence parameter for each copula considered, assuming an exponentially distributed claim size. The illustration and comparison of moments, as well as premium values under the standard deviation principle are also included in this section. Section 2.5 concludes the article.

2.2 Model Setup

We consider a continuous time renewal risk model as in [6], whereby $\underline{Z} = \{Z(t)\}_{t\geq 0}$ with:

$$Z(t) = \begin{cases} \sum_{i=1}^{N(t)} e^{-\delta T_i} X_i & \text{if } N(t) > 0 \\ 0 & \text{if } N(t) = 0 . \end{cases}$$

In this model, $\underline{N} = \{N(t)\}_{t\geq 0}$ is a homogeneous Poisson process and X_i is a non-negative random variable (r.v.) representing the claim amount occurring at time T_i for $i = 1, 2, \ldots, N(t)$. The instantaneous rate of net interest, δ, is assumed to be deterministic.

We also define the inter-claim time variable r.v. W_j as:

$$W_j = \begin{cases} T_j & \text{for } j = 1, \\ T_j - T_{j-1} & \text{for } j = 2, 3, \ldots . \end{cases}$$

The variables, X_j and W_j, are assumed to be continuous. In this study, we relax the independent assumption between the inter-claim time, W_j, and the claim size, X_j, and we let $\{(X_j,W_j)\}_{j\in\mathbb{N}}$ to form a sequence of independent and identically distributed (i.i.d) random vectors, whose components are dependent.

2.2.1 Recursive Moments of Aggregate Discounted Claims

Conditioning on the arrival of the first claim as in [6], [12] and [10], and knowing that $\mathbf{E}(X^m|W = s) = \int_0^\infty x^m f_{X|W=s}(x)dx$ for $m \geq 1$, we have the general form of the $m-th$ moments of aggregate discounted claim as the following:

$$\begin{aligned} \mu_Z^{(m)}(T) = \mathbf{E}[Z^m(T)] &= \int_0^T f_W(s) e^{-m\delta s} \mathbf{E}(X^m|W = s)ds + \int_0^T f_W(s) e^{-m\delta s} \mu_Z^{(m)}(T-s)ds \\ &+ \sum_{j=1}^{m-1} \binom{m}{j} \int_0^T f_W(s) e^{-m\delta s} \mathbf{E}(X^j|W = s) \mu_Z^{(m-j)}(T-s)ds \\ &= \int_0^T \int_0^\infty e^{-m\delta s} x^m f_{X,W}(x,s)\, dxds + \int_0^T e^{-m\delta s} f_W(s) \mu_Z^{(m)}(T-s)ds \\ &+ \sum_{1\leq j<m} \binom{m}{j} \int_0^T \int_0^\infty e^{-m\delta s} x^j f_{X,W}(x,s)\, \mu_Z^{(m-j)}(T-s)dxds, \end{aligned} \tag{2.1}$$

where $f_{X,W}(x,s)$ is the bivariate probability density function (pdf) of the pair, X_j and W_j.

In this study, the joint pdf is described via a copula, $C_\theta(u,v)$, whose pdf is given by $c_\theta(u,v) = \frac{\partial^2}{\partial u \partial v} C_\theta(u,v)$ with dependence parameter θ (see, e.g., [22] and [23] for a general review on copulas). The bivariate pdf of (X,W) at (x,s) can be represented as:

$$f_{X,W}(x,s) = c_\theta(F_X(x), F_W(s)) f_X(x) f_W(s),$$

where $f(\cdot)$ and $F(\cdot)$ are the marginal pdf and cdf for r.v.'s X and W.

Since the jump occurrences are assumed to follow a Poisson distribution, we therefore have an exponentially distributed inter-claim arrival time, i.e., $W \sim Exp(\beta)$. Upon replacing $f_W(s) = \beta e^{-\beta s}$, we obtain:

$$\begin{aligned}
\mu_Z(T) &= \int_0^T \int_0^\infty \beta x e^{-(\beta+\delta)s} f_X(x) c_\theta \left(F_X(x), F_W(s)\right) dxds + \beta \int_0^T e^{-(\beta+\delta)s} \mu_Z(T-s) ds \\
&= C(T) + \beta \int_0^T e^{-(\beta+\delta)s} \mu_Z(T-s) ds \\
&= C(T) + \beta \int_0^T e^{-(\beta+\delta)(T-s)} \mu_Z(s) ds
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
\mu_Z^{(m)}(T) &= \beta [\int_0^T \int_0^\infty x^m e^{-(\beta+m\delta)s} f_X(x) c_\theta \left(F_X(x), F_W(s)\right) dxds \\
&+ \sum_{1 \le j < m} \binom{m}{j} \int_0^T x^{-(m-j)} e^{-[\beta+m\delta]s} c_\theta \left(F_X(x), F_W(s)\right) f_X(x) \mu_Z^{(m-j)}(T-s) dxds] \\
&+ \beta \int_0^T e^{-(\beta+m\delta)s} \mu_Z^{(m)}(T-s) ds \\
&= C^{(m)}(T) + \beta \int_0^T e^{-(\beta+m\delta)s} \mu_Z^{(m)}(T-s) ds \\
&= C^{(m)}(T) + \beta \int_0^T e^{-(\beta+m\delta)(T-s)} \mu_Z^{(m)}(s) ds
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
C^{(m)}(T) &= \beta [\int_0^T \int_0^\infty x^m e^{-(\beta+m\delta)s} f_X(x) c_\theta \left(F_X(x), F_W(s)\right) dxds \\
&+ \sum_{1 \le j < m} \binom{m}{j} \int_0^T x^{-(m-j)} e^{-[\beta+m\delta]s} c_\theta \left(F_X(x), F_W(s)\right) f_X(x) \mu_Z^{(m-j)}(T-s) dxds]
\end{aligned} \tag{2.4}$$

for $m = 2, 3, \cdots$.

2.2.2 Copula Used

We are interested to calculate the first, second and $m-th$ moment of aggregate discounted claims under three copulas: the FGM copula, the Gaussian copula and the Gumbel copula. Their respective pdfs are given by:

$$c_\theta^F \left(F_X(x), F_W(s)\right) = 1 + \theta(1 - 2F_X(x))(1 - 2F_W(s)), \tag{2.5}$$

$$c_\theta^G \left(F_X(x), F_W(s)\right) = \frac{1}{\sqrt{(1-\theta^2)}} e^{-\frac{\theta(2\Phi^{-1}(F_X(x))\Phi^{-1}(F_W(s)) - \theta(\Phi^{-1}(F_X(x))^2 + \Phi^{-1}(F_W(s))^2))}{2(\theta^2-1)}}, \tag{2.6}$$

$$c_\theta^M(F_X(x),F_W(s)) = \frac{(-\ln u)^\theta}{-u\ln u}\frac{(-\ln v)^\theta}{-v\ln v}\frac{\sqrt[\theta]{(-\ln u)^\theta+(-\ln v)^\theta}}{[(-\ln u)^\theta+(-\ln v)^\theta]^2} \\ \times \frac{\sqrt[\theta]{(-\ln u)^\theta+(-\ln v)^\theta}+\theta-1}{e^{\sqrt[\theta]{(-\ln u)^\theta+(-\ln v)^\theta}}}. \tag{2.7}$$

The FGM copula is used in this study due to its simplicity and analytical tractability. It is also used to verify our numerical results in Section 4.1 with [6]. The well-known elliptical family member, the Gaussian copula, is chosen as, to the best of our knowledge, the effect of elliptical copula in terms of the dependence between claim size and inter-claim time have not been explored extensively. The Gumbel copula is also chosen, since it could be adopted by an insurance company, that assumes that risks with extreme magnitude, having the tendency to occur together, as pointed out by De Matteis in [21]. Many standard statistical texts offer illustrations of copula scatter plots with various dependence structure, for which we refer to [22], [24] and [23].

2.3 Linear Integral Equations

The most general form of linear integral equation (IE) is given by:

$$h(T)\Psi(T) = g(T) + \int_a^{b(T)} K(T,s)\Psi(s)ds, \tag{2.8}$$

where $\Psi(T)$ is the solution to the IE that we need to obtain, $g(T)$ and $b(T)$ are given functions and $K(T,s)$ is the kernel for the IE. Equation (2.8) can be a homogeneous/non-homogenous, Volterra/Fredholm IE of the 1st/the 2nd kind, for which readers are referred to the conditions given in Section 2.1 of [18]. Linear IE can be solved either numerically using methods, such as the Runge–Kutta and collocation methods (see, e.g., [26] and [25]), or solved explicitly, such as by obtaining its Neumann series via the Picard method of successive approximations or using the Laplace transform method.

2.3.1 Volterra IE of the 2nd Kind

If we have $g(T) \neq 0$, $h(T) = 1$, and $b(T) = T$, (2.8) becomes:

$$\Psi(T) = g(T) + \int_a^T K(T,s)\Psi(s)ds, \tag{2.9}$$

which is a non-homogeneous Volterra integral equation of the second kind. The Volterra IE is widely used in the areas of viscoelasticity and electromagnetic to compute the dynamics of materials that "contain" memory, other than being useful in renewal theory and demography (see, e.g., [16] and [27], as well as the references therein for a more rigorous treatment on Volterra integral equations).

We easily notice that the moments provided by Equations (2.2) and (2.3) take the form of (2.9) and attempt to derive the explicit solution of the recursive expressions using Neumann series in the next subsection.

A unique and continuous solution, $\Psi(T)$, is obtainable if we have a combination of a continuous kernel, $K(T,s)$, in the region $a \leq s \leq T \leq b(T)$ with a function, $g(T)$, that is continuous in the region $a \leq T \leq b(T)$, even though it is not a requirement for the kernel function, $K(T,s)$, to be continuous (see page 1 of [28] and page 5 of [20]). For the case of a discontinuous kernel function, we need to check if $K(T,s)$ fulfills the three regularity conditions set on page 3 of [27], and, hence is an L^2-function.

In the case of the first and second moments, $\mu_Z(T)$ and $\mu_Z^{(2)}(T)$, the function, $g(T)$, is represented by the following equations, respectively:

$$\int_0^T \int_0^\infty e^{-(\beta+\delta)s} x f_X(x) c_\theta \left(F_X(x), F_W(s)\right) dxds, \tag{2.10}$$

$$\begin{aligned} &\int_0^T \int_0^\infty e^{-(\beta+2\delta)s} x^2 f_X(x) c_\theta \left(F_X(x), F_W(s)\right) dxds \\ +2&\int_0^T \int_0^\infty e^{-(\beta+2\delta)s} x f_X(x) c_\theta \left(F_X(x), F_W(s)\right) \mu_Z(T-s) dxds, \end{aligned} \tag{2.11}$$

where $F_W(s) = 1 - e^{-\beta s}$ is the inter-claim time cdf.
As X and W are continuous r.v.'s, and by corollary 2.2.6 of [23] on copula continuity, $g(T)$ is the continuous function for $s \in [0,T]$ and $x \in [0,\infty]$, since it is the sum and product of continuous functions. The kernel function is also continuous, as it is an exponential function given by:

$$K(T,s) = e^{-(\beta+m\delta)(T-s)}. \tag{2.12}$$

Additionally, it is a bounded function in the square $\Pi = \{(T,s) : a \leq T \leq b(T), a \leq s \leq T\}$.

2.3.2 Neumann Series

In this section, we will find the Neumann series of the Volterra IE assuming the exponentially distributed inter-claim arrival time and a general claim size with continuous pdf. To do so, we start with a proposition from Chapter 3 of [27], which used the Picard method of successive approximation.

Proposition 2.3.1. ***Neumann Series for a Volterra IE of the 2nd Kind***

For the Volterra IE of the 2nd kind, as in (2.9)*, where $g(T)$ and $K(T,s)$ are L^2-functions, its*

Neumann series is given by:

$$\begin{aligned}\Psi(T) &= g(T) + \sum_{n=1}^{\infty} \lambda^n \int_a^T K(T,s)\Psi(s)ds \\ &= g(T) + \lambda \int_a^T \sum_{n=1}^{\infty} \lambda^{n-1} K_n(T,s) g(s)ds \\ &= g(T) + \lambda \int_a^T \Gamma(T,s;\lambda) g(s)ds, \end{aligned} \tag{2.13}$$

where $\Gamma(T,s;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(T,s)$ *is the unique resolvent kernel and* $K_n(T,s)$ *is the n-th-iterated kernel function satisfying the recurrence formula:*

$$K_n(T,s) = \int_s^T K(T,u) K_{n-1}(u,s)du \tag{2.14}$$

with $K_1(T,s) = K(T,s)$.

In order to prove our theorem, it is necessary to find the resolvent kernel, which is obtained in the following lemma.

Lemma 2.3.2. *Consider the kernel function given by* (2.12). *For* $m = 1,2,\ldots$, *its resolvent kernel is therefore given by:*

$$\Gamma(T,s;\lambda) = e^{-m\delta(T-s)}. \tag{2.15}$$

Proof: Using (2.14), we obtain $\mathrm{K}_2(T,s), K_3(T,s), \cdots, K_{n+1}(T,s)$ starting from $K(T,s) = K_1(T,s) = e^{-(\beta+m\delta)(T-s)}$. Letting $(T-s) = -(s-T)$ and since $(s-T)^n = [-(s-T)]^n$ for even n, the resolvent kernel is then obtained by summing up $K_m(T,s)$ as follows:

$$\begin{aligned}\Gamma(T,s;\lambda) &= e^{-(m\delta+\beta)(T-s)} \sum_{n=1}^{\infty} \frac{[-(s-T)\beta]^n}{n!} \\ &= e^{-(m\delta+\beta)(T-s)} e^{-\beta(T-s)} \\ &= e^{-m\delta(T-s)}. \quad \square \end{aligned}$$

Now, we can obtain the expression for the first and second moment, which is the main result of this article.

Theorem 2.3.3. *The explicit solution of the first two moments are given by:*

$$\mu_Z(T) = \int_0^T \int_0^{\infty} e^{-\delta s} x \mathfrak{L}_\theta(x,s) dxds + \beta \int_0^T \int_0^s \int_0^{\infty} e^{-\delta(T-s+u)} x \mathfrak{L}_\theta(x,s) dxduds, \tag{2.16}$$

$$
\begin{aligned}
\mu_Z^{(2)}(T) &= \int_0^T \int_0^\infty e^{-2\delta s} x^2 \mathfrak{L}_\theta(F_X(x), F_W(s))dxds \\
&+ 2\int_0^T \int_0^\infty \int_0^{T-s} \int_0^\infty e^{-2\delta s - \delta\tau} xh\mathfrak{L}_\theta(F_X(x), F_W(s))\mathfrak{L}_\theta(F_X(h), F_W(\tau))dhd\tau dxds \\
&+ 2\beta \int_0^T \int_0^\infty \int_0^{T-s} \int_0^\infty e^{-\delta(T+s-\tau+u)} xh\mathfrak{L}_\theta(F_X(x), F_W(s))\mathfrak{L}_\theta(F_X(h), F_W(u))dhdud\tau dxds \\
&+ \beta \int_0^T \int_0^s \int_0^\infty e^{-2\delta(T-s-\tau)} x^2 \mathfrak{L}_\theta(F_X(x), F_W(s))dxds \\
&+ 2\beta \int_0^T \int_0^{s-\tau} \int_0^\infty \int_0^s \int_0^\infty e^{-2\delta(T-s-\tau)} xh\mathfrak{L}_\theta(F_X(x), F_W(\tau))\mathfrak{L}_\theta(F_X(h), F_W(y))dhdydxd\tau dxds \\
&+ 2\beta^2 \int_0^T \int_0^s \int_0^\infty \int_0^{s-\tau} \int_0^y \int_0^\infty e^{-\delta(2T+u-y-s+\tau)} xh\mathfrak{L}_\theta(F_X(x), F_W(\tau)) \\
&\times \mathfrak{L}_\theta(F_X(h), F_W(u))dhdudydxd\tau dxds, \qquad (2.17)
\end{aligned}
$$

where $\mathfrak{L}_\theta(F_X(x), F_W(u)) = e^{-\beta u} f_X(x) c_\theta(F_X(x), F_W(u))$ *with* $F_W(u) = 1 - e^{-\beta u}$.

Proof: Applying Proposition 2.3.1 and Lemma 2.3.2 to (2.3) with $m = 1, 2$, the results follow. □

Section 2.4.1 will numerically illustrate the computation of the first and second moment under three copulas, assuming that the claim sizes are exponentially distributed, i.e., $X \sim Exp(\alpha)$. We do not proceed to obtain the closed form solution of the Neumann series expression for the higher moments, as it is tedious and time consuming. However, they are obtainable using the results provided in this section.

2.4 Numerical Illustration

We now present numerical illustration of the Neumann series expression for the first two moments. We start our discussion by presenting the scatter plots of each copula in Figures 2.1, 2.2 and 2.3, where the marginals are exponential distribution, which is in line with the assumptions used in the numerical computations of the moments in this section. All computations were done using Mathematica.

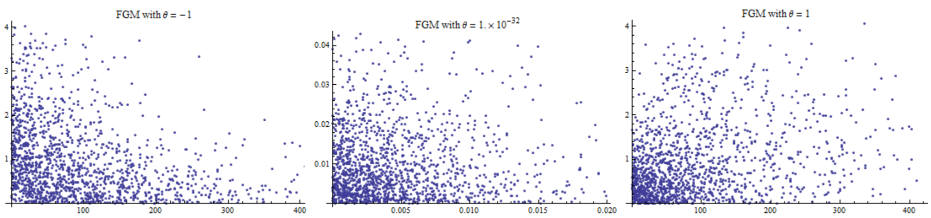

FIGURE 2.1: Farlie–Gumbel–Morgenstern (FGM) copula with exponential margins and dependence parameters -1, zero, one.

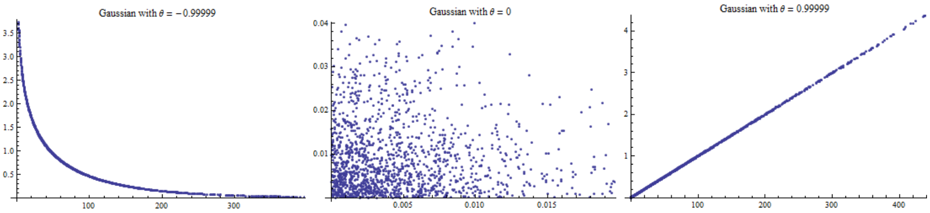

FIGURE 2.2: Gaussian copula with exponential margins and dependence parameters -1, zero, one.

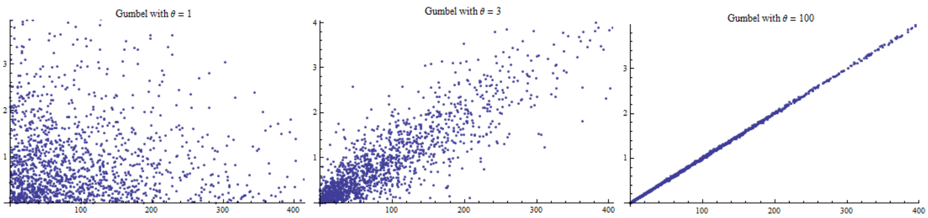

FIGURE 2.3: Gumbel copula with exponential margins and dependence parameters one, three, 100

2.4.1 Numerical accuracy of Neumann series expression for moments

Recall that (2.16) has at most triple integration involved, while (2.17) has up to sextuple numerical integration. This implies that the computation of (2.17) is expected to be close to the solution by [6], due to numerical approximation error, and the values would vary according to selected software packages. To evaluate the performance of the main results, we compare the numerical values returned by our Neumann series (under the column Neumann of Table 2.1), using the FGM copula, with the numerical values given by the closed form solution in [6] (under the column BCLM of Table 2.1).

The values in Table 2.1 were computed using an example of $\delta = 0.04$, $\alpha = 10$, $\beta = 1$, $T = 5$ and $\theta = -0.9, 0, 0.9$. The absolute deviation (Abs. Dev.) figures are obtained by taking the difference between the two columns, BCLM and Neumann (i.e., the absolute value of the solution presented in [6] minus the Neumann series expression), whereas the relative deviation (Rel. Dev.) figures are calculated as $\frac{Abs.Dev.}{BCLM}$.

Our calculations showed that the Neumann series expression for the first moment gives the same value as the closed form solution presented in [6]. On the other hand, the second moment gives a slightly different value at $\theta = -0.9$ and 0.9, when the r.v.'s, X and W, are highly dependent. After a close scrutiny of the programming messages, we noticed that this is caused by numerical approximation errors of the quadruple, quintuple and sextuple integrations that are not present in the calculation of the first moment. To improve the accuracy of the Neumann series expression for higher order moments, the reader can use other software packages or use Monte Carlo simulation.

TABLE 2.1: Moment verification: the case of the FGM copula. Abs. Dev., absolute deviation; Rel. Dev., relative deviation.

Moment	θ	BCLM	Neumann	Abs. Dev.	Rel. Dev.
$\mu_Z(5)$	-0.9	0.475231	0.475231	0	0
	0	0.453173	0.453173	0	0
	0.9	0.431115	0.431115	0	0
$\mu_Z^{(2)}(5)$	-0.9	0.332023	0.329774	0.002249	0.006774
	0	0.287786	0.287786	0	0
	0.9	0.245457	0.247706	0.002249	0.009163

2.4.2 Moments of the Aggregate Discounted Claims

Setting $\delta = 0.04$, $\alpha = 0.01$ and $\beta = 1$ for the case of exponential claim inter-arrival time and exponential claim sizes, respectively, we show the values of moments of the aggregate discounted claims for each copula used in this study. We present the values of the first and second moments of the compound distribution, i.e. $\mu_Z(5)$ and $\mu_Z^{(2)}(5)$, as well as the variance under each copula in Table 2.2- 2.4. The term 'spread', which is defined as the difference between the values returned by θ at both ends, i.e., $\theta_{-0.95} - \theta_{0.95}$ for FGM and Gaussian, and $\theta_1 - \theta_{100}$ for Gumbel, are also shown in Tables 2.2–2.3.

TABLE 2.2: Values of $\mu_Z(5)$ for various copula.

θ	FGM	θ	Gaussian	θ	Gumbel
-0.95	455.543	-0.95	513.470	1	453.173
-0.9	455.419	-0.9	511.887	5	360.864
-0.5	454.421	-0.5	488.903	15	267.995
0	453.173	0	453.173	30	148.317
0.5	451.926	0.5	409.481	50	104.457
0.9	450.928	0.9	368.612	75	21.486
0.95	450.803	0.95	363.124	100	10.712
Spread	4.74		150.346		431.687

Our calculations showed that all copula exhibit decreasing values as θ increases, in line with [11]. Intuitively, a negative dependence structure represented by the pair of short inter-claim waiting time (or frequent claim occurrence within a given time period) with huge claim size will only prompt the insurer to charge a higher premium as opposed to the positive dependence structure.

As we have expected, the values of moments do not vary much across θ when calculated under the FGM copula, as opposed to the Gaussian and Gumbel copulas. Being an extreme copula, values of the first moment calculated under the Gumbel copula also showed the

TABLE 2.3: Values of $\mu_Z^{(2)}(5)$ for various copula.

θ	FGM	θ	Gaussian	θ	Gumbel
-0.95	336,551.170	-0.95	409,852.140	1	287,784.972
-0.9	332,022.549	-0.9	405,315.216	3	148,590.220
-0.5	312,126.218	-0.5	357,029.617	5	139,437.15
0	287,785.862	0	287,785.862	40	21,804.385
0.5	264,034.461	0.5	212,119.492	75	1,088.013
0.9	245,457.386	0.9	149,988.657	80	229.608
0.95	241,329.490	0.95	136,249.086	100	178.443
Spread	95,221.68		273,603.054		287,606.529

TABLE 2.4: Values of $\mathbf{Var}(5)$ for various copula.

θ	FGM	θ	Gaussian	θ	Gumbel
-0.95	128,940.627	-0.95	146,200.971	1	82,420.094
-0.9	124,616.083	-0.9	143,286.915	3	13,837.353
-0.5	105,627.773	0.5	118,003.474	5	9,214.320
0	82,420.094	0.	82,420.094	40	5,597.566
0.5	59,797.351	0.5	44,444.803	75	626.365
0.9	42,121.325	0.9	14,113.850	80	112.770
0.95	38,106.145	0.95	4,390.216	100	61.436

widest spread of the first moment.

Table 2.5 shows the values of the first moment as a function of α and β, respectively, for which we use the Gaussian copula at $\theta = -0.9$. It shows that increasing the inter-claim waiting time parameter, β, results in increasing the mean value of the aggregate discounted claims, and *vice versa* in the case of the claim size parameter. Given an average value of inter-claim arrival time, β, the mean of aggregate discounted claims gets lower as we have a lower average claim size, given by $\frac{1}{\alpha}$. On the other hand, given an average value of the claim size, the mean of the aggregate discounted claims gets bigger as the inter-claim arrival time gets shorter, which implies more frequent claim occurrences. This scenario is illustrated in Figure 2.4 for $\theta = -0.9$ (left hand side of the diagram), as well as $\theta = 0$ (right hand side of the diagram).

2.4.3 Premium Calculation under FGM, Gaussian and Gumbel copulas

We now compute the loaded premium related to the risk of an insurance portfolio represented by $Z(T)$, where the dependence structure is captured by a copula. For that purpose, the first two moments will be useful in the premium calculation based on the expected value

TABLE 2.5: Values of $\mu_Z(5)$ under the Gaussian copula at $\theta = -0.9$.

β =1	$\mu_Z(5)$	$\alpha = 1$	$\mu_Z(5)$
$\alpha = 0.01$	511.887741	$\beta = 0.01$	0.165673
$\alpha = 0.1$	51.188786	$\beta = 0.1$	0.884887
$\alpha = 1$	5.118887	$\beta = 1$	5.118887
$\alpha = 10$	0.511838	$\beta = 10$	45.911776
$\alpha = 15$	0.341257	$\beta = 15$	67.571651

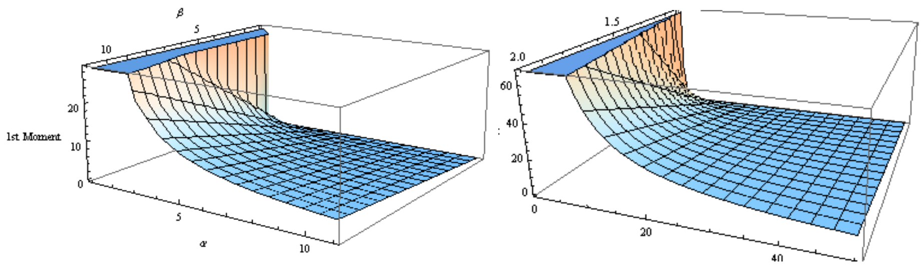

FIGURE 2.4: Sensitivity of the first moment under the Gaussian copula at $\theta = 0$ and $\theta = -0.9$ with respect to claim size and inter-claim time averages.

principle, the variance principle, as well as the standard deviation (SD) premium principle, as the following:

$$\Pi(T) = \mathbf{E}[Z(T)] + \kappa \mathbf{E}[Z(T)],$$

$$\Pi(T) = \mathbf{E}[Z(T)] + \kappa \mathbf{V}ar[Z(T)],$$

$$\Pi(T) = \mathbf{E}[Z(T)] + \kappa \sqrt{\mathbf{V}ar[Z(T)]}.$$

Table 2.6 exhibits the loaded premium according to the SD principle under the three copulas considered, with $\kappa = 0.1$, while Figure 2.5 and Figure 2.6 illustrate the range of premiums under the copulas studied according to the SD premium principle.

TABLE 2.6: Loaded premium according to the SD principle under various copulas.

θ	FGM	θ	Gaussian	θ	Gumbel
-0.95	491.55	-0.95	551.71	1	481.88
-0.9	490.72	-0.9	549.74	3	378.85
-0.5	486.92	-0.5	523.25	5	370.46
0	481.88	0	481.88	40	134.79
0.5	476.38	0.5	430.56	75	23.99
0.9	471.45	0.9	380.49	80	11.87
0.95	470.32	0.95	369.75	100	11.60
Spread	21.23		181.96		470.28

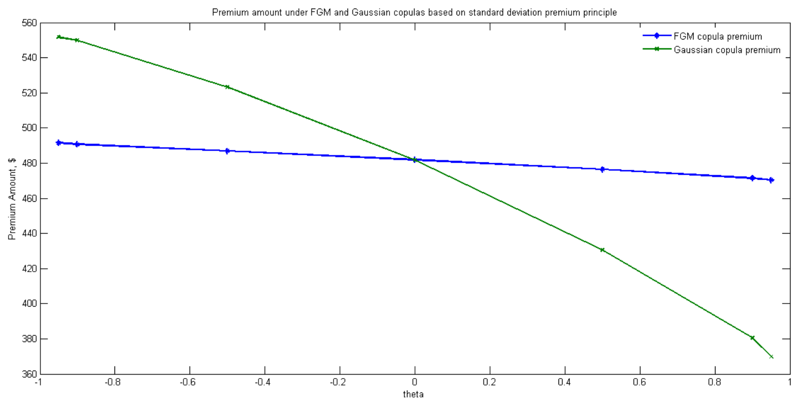

FIGURE 2.5: The loaded premium under FGM and Gaussian copulas based on the SD premium principle.

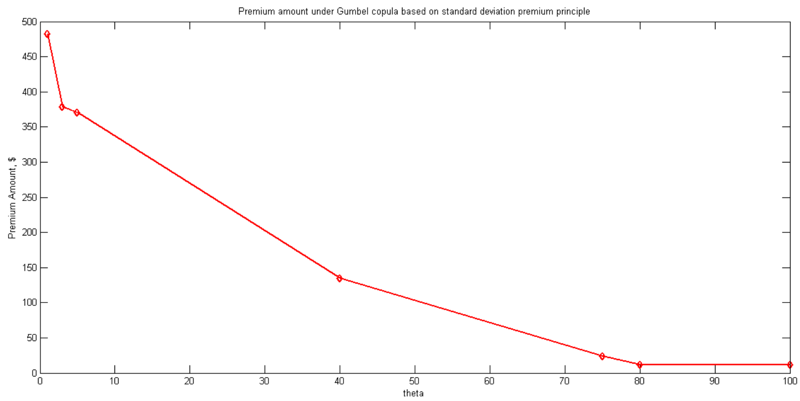

FIGURE 2.6: The loaded premium under the Gumbel copula based on the SD premium principle.

2.5 Conclusion

In this paper, we utilized copulas to capture the dependence structure between the inter-claim arrival time and claim sizes in classical actuarial risk theory. To do so, we represented the expression for the $m-th$ order moment proposed in [12] and [6] in the form of the Volterra integral equation (VIE) of the second kind, which is widely used in renewal theory, demographics, electromagnetism and viscoelasticity.

We derived the Neumann series expression for this recursive equation using the Picard method of successive approximations, based on which we computed the first two moments of the aggregate discounted claims. For the dependence structure between the inter-claim arrival time and claim sizes, we used a Farlie–Gumbel–Morgenstern copula, a Gaussian copula and a Gumbel copula with exponential marginal distributions. We showed the values of moments of the aggregate discounted claims, as well as the loaded premium for each copula used in this study.

It would be of interest to derive the expression for (2.2) and (2.3) using other joint pdfs between X and W. Other copulas with different claim size distributions for X may be considered in the proposed approach, which we leave for further research. We can also consider the Monte Carlo simulation, as well as other numerical methods to solve the VIE (such as Runge–Kutta and the collocation methods), as the next objective of further research to deal with the computation of higher moments.

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3

A Multivariate Jump Diffusion Process for Counterparty Risk in CDS rates

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This article has been submitted for publication in the *European Actuarial Journal*. The article is presented in its entirety here and hence contains repetitions of certain segments of the Introduction presented in Chapter 1.

Abstract We consider counterparty risk in CDS rates. To do so, we use a multivariate jump diffusion process for obligors' default intensity, where jumps (i.e. magnitude of contribution of primary events to default intensities) occur simultaneously and their sizes are dependent. For these simultaneous jumps and their sizes, a homogeneous Poisson process. We apply copula-dependent default intensities of multivariate Cox process to derive the joint Laplace transform that provides us with joint survival/default probability and other relevant joint probabilities. For that purpose, the piecewise deterministic Markov process (PDMP) theory developed in [7] and the martingale methodology in [6] are used. We compute survival/default probability using three copulas, which are Farlie-Gumbel-Morgenstern (FGM), Gaussian and Student-t copulas, with exponential marginal distributions. We then apply the results to calculate CDS rates assuming deterministic rate of interest and recovery rate. We also conduct sensitivity analysis for the CDS rates by changing the relevant parameters and provide their figures.
Keywords multivariate jump diffusion process; multivariate Cox process; joint survival/default probability; copulas; counterparty risk; CDS rate

3.1 Introduction

In practice, the insolvency of one firm can cause an increase in other firms' default intensities due to business links or ties between firms. The mismanagement of subprime mortgages in the US in the year 2007 which had far reaching consequences provide a perfect illustration in this effect, and thereby emphasizing the importance for incorporating shocks and dependence structure in financial modeling.

The jump diffusion process that has been used to represent variables such as the default intensity, asset returns as well as interest rate (such as the work by [10], [26], [28], [30] and [5]) allows us to capture the effects of shocks. Shock elements can arrive due to primary events such as oil and commodity prices, governments fiscal and monetary policies, the release of corporate financial reports, political and social decisions, rumours of mergers and acquisitions among firms, the collapse and bankruptcy of firms, the September 11 World Trade Centre catastrophe and Hurricane Katrina. Each of these events cause jumps in the variable being modelled. Readers are referred to [36] and [26] for a further discussion of the various motivations for using a jump diffusion process.

This paper is based on the jump diffusion approach for the case when the firms in the complementary or substitute industry/sector are affected by a common external event. Numerous papers have examined the modelling for the dependence of default intensities via a point process for the purpose of pricing derivative instruments (such as [35], [24], [6], [37], [17] and [32]). The use of univariate jump diffusion model to represent the reference credit intensity in pricing the CDS instrument was also explored in [2]. The analytical expression for CDS and CDS swaptions prices offered in the literature was obtained using the Jamshidian option decomposition trick as in [20].

Besides the construction of a point process, considerable attention was given to the default dependence between the obligors. The work by [11] considered joint jumps in the default intensity for this effect. [25] and [23] developed it further considering the possibility of default-event triggers that cause joint default. Another approach to incorporate default dependence between obligors is through the use of copulas ([27]; [35]; [24], [16] and [28]). The use of FGM copula with multivariate shot noise process has been explored in [22] which was then extended in [28] by adding diffusion term to the intensity processes. Both papers adopted martingale methodology and PDMP technique to derive the survival probability. Using the same methodology and technique, we examine a multivariate default intensity process where the jump occur simultaneously.

We structure the article in the following order: In section 3.2.2 we define the multivariate jump diffusion process for obligors' default intensity and derive the relevant joint Laplace transform using the PDMP theory and the martingale methodology. These joint Laplace transforms then lead us to the joint survival/default probability and other relevant joint probabilities. This is followed by a numerical example showing how the joint probabilities can be generated capturing the dependence structure between the vector of event jumps, using three copulas as examples which are the Farlie-Gumbel-Morgenstern (FGM) copula, Gaussian copula and Student-t copula. In section 3, we then illustrate how this jump diffusion process can be applied to calculate CDS rates considering counterparty risk. For that purpose, we assume that the jumps of default intensities of the CDS seller and reference credit (RC) occur simultaneously and that the dependence structure between their jump sizes are captured by the three copulas. We also assume deterministic short rate of interest and a deterministic recovery rate for simplicity. This is then followed by a sensitivity analysis of the CDS rates with respect to relevant parameters such as the diffusion rate, the constant reversion level, the decay rate at which the default intensity would retract back to the constant reversion level as well as the jump size of both obligors. Section 4 contains some concluding remarks.

3.2 Model Setup and Theoretical Results

For $i = 1,2,\cdots,n$ denoting obligor i involved in the financial transaction, the multivariate default intensity model we consider has the following structure:

$$d\lambda_t^{(i)} = c^{(i)}\left(b^{(i)} + a^{(i)}\lambda_t^{(i)}\right)dt + \sigma^{(i)}\sqrt{\lambda_t^{(i)}}dW_t^{(i)} + dL_t^{(i)}, \quad L_t^{(i)} = \sum_{j=1}^{M_t} X_j^{(i)} \tag{3.1}$$

where

- $\left\{X_j^{(1)},\, X_j^{(2)}, \cdots,\, X_j^{(n)}\right\}_{j=1,2,\cdots}$ is a vector sequence of dependent but not identically distributed random variables with distribution function $F^{(i)}(x)$ $(x > 0)$,
- M_t is the total number of events up to time t,
- $W_t^{(i)}$ is a standard Brownian motion governing obligor i,
- $a < 0$, $b \geq 0$ and $c > 0$ with $c^{(i)}a^{(i)}$ being the rate of exponential decay for obligor $i = 1,2,\cdots,n$ and $c^{(i)}b^{(i)}$ being the constant reversion level for default intensity of obligor i; and
- $\sigma^{(i)} > 0$ is the diffusion coefficient for obligor i.

We also make the additional assumption that the point process M_t is independent of the vector sequence of jump sizes and that the vector sequence $\left\{X_k^{(1)}, X_k^{(2)}, \cdots, X_k^{(n)}\right\}_{k=1,2,\cdots}$ is independent of another vector sequence for $k \neq j$. $L_t^{(i)}$ is a compound process for the default intensity of obligor i.

In this model, the dependence between the intensities $\lambda_t^{(i)}$ comes from the common event arrival process M_t, together with the dependence between the vector of jumps $(X_j^{(1)}, X_j^{(2)}, \cdots, X_j^{(n)})$. We assume that event arrival process M_t, (i.e. the simultaneous jump process) follows a homogeneous Poisson process with frequency ρ and the vector of jumps is modelled using copulas ([31] and [29]) - that is, the joint distribution of the vector $(X_j^{(1)}, X_j^{(2)}, \cdots, X_j^{(n)})$ is assumed to be of the form $C(F^{(1)}, F^{(2)}, \cdots, F^{(n)})$ with C being a given copula.

As specific examples for C in this paper, we use the FGM, the Gaussian and the Student-t copulas which are given in consecutive manner by

$$C^{FGM}(u_1,\ldots,u_n) = \prod_{i=1}^{n}\left(1+\sum_{1\le i<j}^{n}\theta_{ij}(1-u_i)\right) \tag{3.2}$$

$$C^{G}(u_1,\ldots,u_n) = \int_{-\infty}^{\Phi^{-1}(u_1)} \overset{n}{\cdots} \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{1}{2\pi\sqrt{|\Theta|}}\exp\left(-\frac{1}{2}\omega^{\mathbf{T}}\Theta^{-1}\omega\right)dudv \tag{3.3}$$

$$C_{\upsilon}^{t}(u_1,\ldots,u_n) = \int_{-\infty}^{t_{\upsilon}^{-1}(u_1)} \overset{n}{\cdots} \int_{-\infty}^{t_{\upsilon}^{-1}(u_n)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\upsilon)^2|\Theta|}}\left(1+\frac{\eta^{\mathbf{T}}\Theta^{-1}\eta}{\nu}\right)dudv \tag{3.4}$$

where $u_i \in [0,1]$ for $i = 1,\cdots,n$. For the elliptical copulas, the correlation paramater $\theta \in [-1,1]$ is contained in the correlation matrix $\Theta = \begin{bmatrix} 1\cdots & \theta_{1j} & \cdots\theta_{1n} \\ \vdots & \ddots & \vdots \\ \theta_{n1}\cdots & \theta_{nj} & \cdots 1 \end{bmatrix}$. We also define $\omega = [\ \omega_1 \ \cdots \ \omega_n\ ]^T$ and $\eta = [\ \eta_1 \ \cdots \ \eta_n\ ]^T$ where $\omega_i = \Phi^{-1}(u_i)$ and $\eta_i = t_{\upsilon}^{-1}(u_i)$ are the inverse Gaussian and inverse Student-t distribution with degrees of freedom υ respectively taken on the variables u_i. For the marginal distributions of $X_j^{(i)}$ in the vector of jumps $(X_j^{(1)}, X_j^{(2)}, \cdots, X_j^{(n)})$, any continuous distribution can be considered.

With $F^{(i)}(x_j) = 1 - e^{-\mu^{(i)}x_j}\left(\mu^{(i)} > 0,\ x_j > 0\right)$, for $i = 1,2,\cdots,n$ to represent the marginal distribution, the FGM copula, which is illustrated in Figure 3.2, is used in this study for its simplicity and analytical tractability, where it is also used in [22] and [28]. Its simplicity allows for the closed-form expressions of final results to be easily derived. It is also used to compare our numerical results against their counterparts in [28]. The Gaussian copula, shown in Figure 3.2, is chosen so as to examine the effect of elliptical copula on simultaneous jumps in the intensity process as it has not been explored previously in the context of CDS pricing with counterparty risk. We also choose the Student-t copula to incorporate the possibility of having more frequency of higher and/or smaller as well as opposing joint jumps size impact in the obligors' intensity, as shown in Figure 3.2.

The simulated paths of the jump diffusion process under each copula considered in this study with exponential jump size distributions is also shown in Figures 3.2, 3.4 and 3.6, where $\theta = -0.95, 0$ and 0.95.

3.2.1 Survival and Default Probabilities

Now, let us derive the joint survival probability and relevant joint probabilities. To do so, we use a multivariate Cox process $(N_t^{(1)}, \cdots, N_t^{(n)})$ with the integrated default intensities

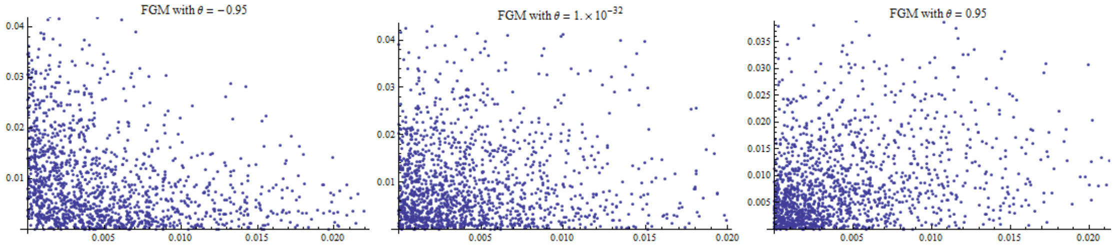

FIGURE 3.1: FGM copula with exponential margins and dependence parameter -0.95, 0, 0.95

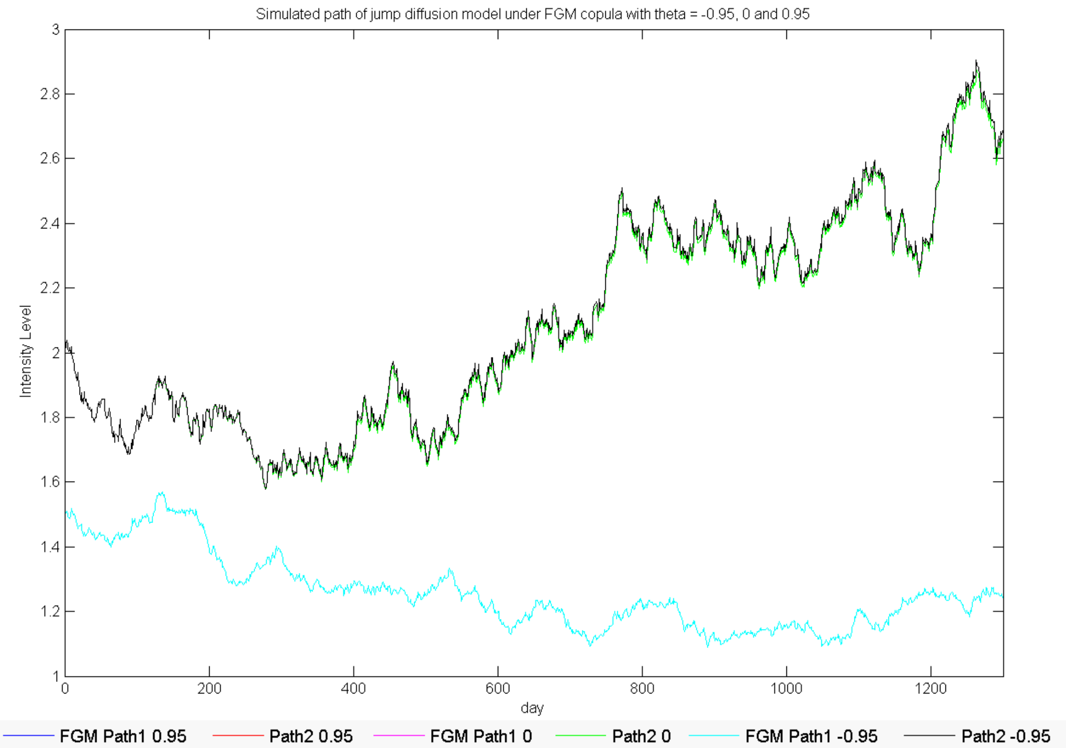

FIGURE 3.2: Simulated paths of jump diffusion process with dependence structure capture by FGM copula

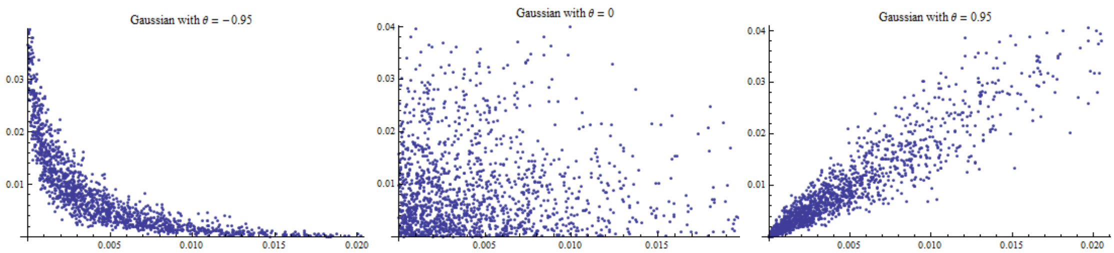

FIGURE 3.3: Gaussian copula with exponential margins and dependence parameter -0.95, 0, 0.95

$\Lambda_t^{(i)} = \int_0^t \lambda_s^{(i)} ds$ $(i = 1, 2, \cdots, n)$ to model the joint default time. We define

$$\tau^{(i)} = \inf\{t : N_t^{(i)} = 1 \,\big|\, N_0^{(i)} = 0\}$$

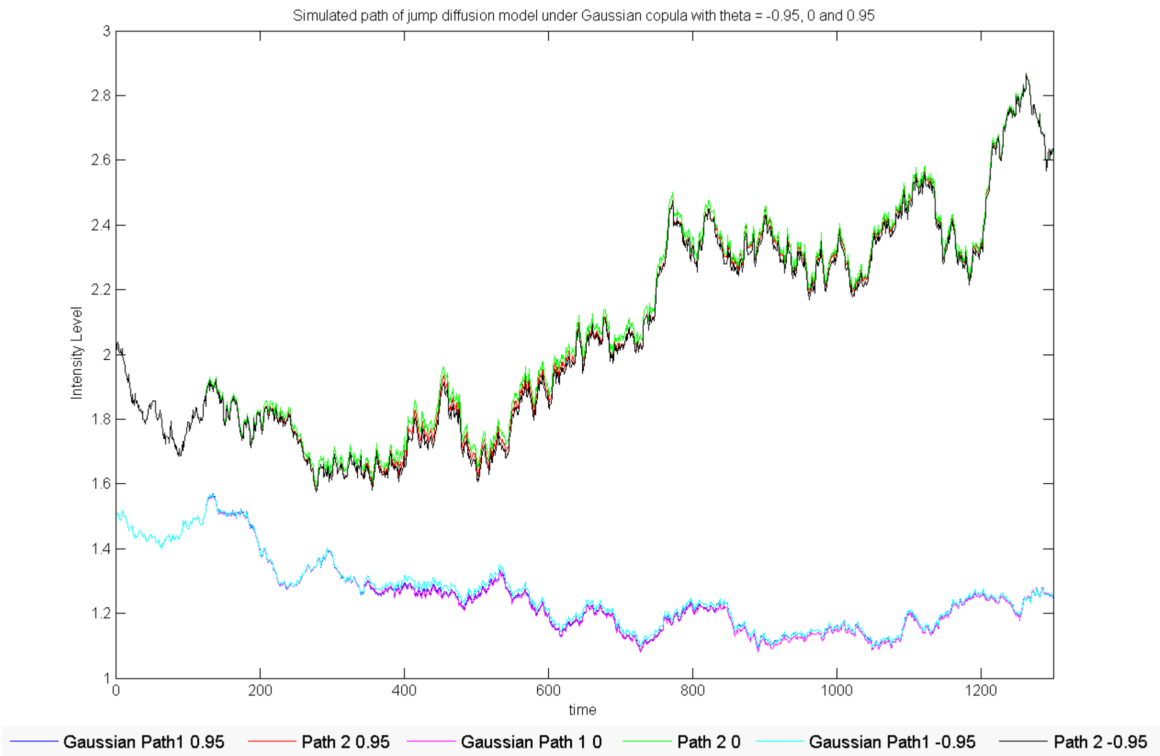

FIGURE 3.4: Simulated paths of jump diffusion process with dependence structure capture by Gaussian copula

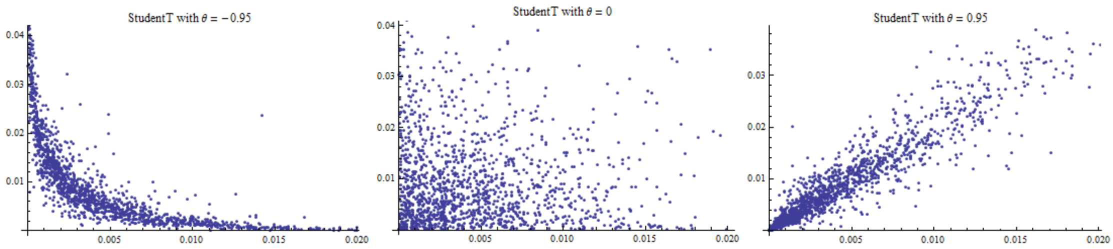

FIGURE 3.5: Student-T copula with exponential margins and dependence parameter -0.95, 0, 0.95

as the default arrival time for the firm $i = 1, \cdots, n$, that is equivalent to the first jump time of the Cox process $N_t^{(i)}$ $(i = 1, 2, \cdots, n)$ respectively.

We derive the joint Laplace transform of the vector $(\Lambda_t^{(1)}, \cdots, \Lambda_t^{(n)})$, i.e.

$$\mathbb{E}\left(e^{-\sum_{i=1}^{n}\gamma^{(i)}\Lambda_t^{(i)}}\middle|\lambda_0^{(1)}, \cdots, \lambda_0^{(n)}\right) \tag{3.5}$$

where $\gamma^{(i)} \geq 0$, as it provides the joint survival/default probabilities by setting $\gamma^{(i)} = 1$ in the equation (3.5) i.e.

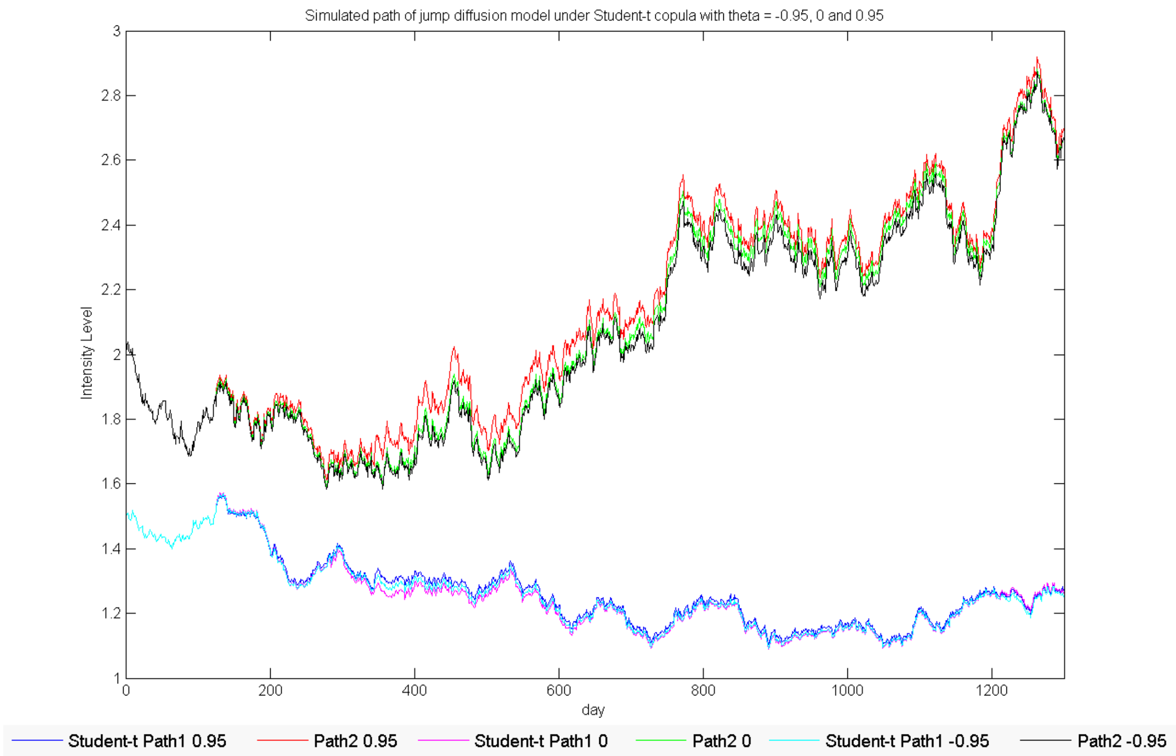

FIGURE 3.6: Simulated paths of jump diffusion process with dependence structure capture by student-T copula

$$
\begin{aligned}
& \Pr\left(\tau^{(1)} > t, \cdots, \tau^{(n)} > t \,\middle|\, \lambda_0^{(1)}, \cdots, \lambda_0^{(n)}\right) \\
= \;& \mathbb{E}\left[e^{-\Sigma_{i=1}^{n} \Lambda_t^{(i)}} \,\middle|\, \lambda_0^{(1)}, \cdots, \lambda_0^{(n)}\right]. \qquad (3.6)
\end{aligned}
$$

Similarly, the expression for joint default probability represented by the following:

$$
\begin{aligned}
& \Pr\left(\tau^{(1)} \leq t, \cdots, \tau^{(n)} \leq t \,\middle|\, \lambda_0^{(1)}, \cdots, \lambda_0^{(n)}\right) \\
= \;& \mathbb{E}\left[(1 - e^{-\Lambda_t^{(1)}}) \cdots (1 - e^{-\Lambda_t^{(n)}}) \,\middle|\, \lambda_0^{(1)}, \cdots, \lambda_0^{(n)}\right]. \qquad (3.7)
\end{aligned}
$$

can be obtained using equation (3.5). For that purpose, the PDMP theory developed by [7] and the martingale methodology by [6] are used.

Analogous to the univariate case in [21], the generator $\mathscr{A}$ of the process

$(\Lambda_t^{(1)},\cdots,\Lambda_t^{(n)},\lambda_t^{(1)},\cdots,\lambda_t^{(n)},t)$ acting on a function $f(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)},\cdots,\lambda^{(n)},t)$ belonging to its domain is given by

$$\begin{aligned}
&\mathscr{A} f(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)},\cdots,\lambda^{(n)},t)\\
= \; &\frac{\partial f}{\partial t}+\sum_{i=1}^{n}\lambda^{(i)}\frac{\partial f}{\partial \Lambda^{(i)}}+\sum_{i=1}^{n}c^{(i)}(b^{(i)}+a^{(i)}\lambda^{(i)})\frac{\partial f}{\partial \lambda^{(i)})}+\frac{1}{2}\sum_{i=1}^{n}\left(\sigma^{(i)}\sqrt{\lambda^{(i)}}\right)^2\frac{\partial^2 f}{\partial \lambda^{(i)2}}\\
&+\rho\left[\begin{array}{c}\int_0^\infty \overset{n}{\cdots}\int_0^\infty f(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)}+x_1,\cdots,\lambda^{(n)}+x_n,t)\frac{\partial^n C(F_{X^{(1)}}(x_1),\cdots,F_{X^{(n)}}(x_n))}{\partial x_1\cdots\partial x_n}\\ \times dx_1\cdots dx_n - f(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)},\cdots,\lambda^{(n)},t)\end{array}\right]
\end{aligned}$$

where $\frac{\partial^n C(F_{X^{(1)}}(x_1),\cdots,F_{X^{(n)}}(x_n))}{\partial x_1\cdots\partial x_n}$ is the joint density of event jump sizes.

For $f(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)},\cdots,\lambda^{(n)},t)$ to belong to the domain of the generator $\mathscr{A}$, it is sufficient that the function $(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)},\cdots,\lambda^{(n)},t)$ is differentiable w.r.t. $\Lambda^{(i)},\lambda^{(i)},t$ for $i=1,\cdots,n$ and that

$$\left\|\begin{array}{c}\int_0^\infty \overset{n}{\cdots}\int_0^\infty f(\cdot,\lambda^{(1)}+x_1,\cdots,\lambda^{(n)}+x_n,\cdot)\frac{\partial^n C(F_{X^{(1)}}(x_1),\cdots,F_{X^{(n)}}(x_n))}{\partial x_1\cdots\partial x_n}dx_1\cdots dx_n\\ -f(\cdot,\lambda^{(1)},\cdots,\lambda^{(n)},\cdot)\end{array}\right\|<\infty.$$

Now we find a suitable martingale to derive the joint Laplace transform of the vector $\left(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)},\cdots,\lambda^{(n)},t\right)$ at time t.

Theorem 3.2.1. *Considering constant $\gamma^{(i)}\geq 0$ and $k^{(i)}\geq 0$,*

$$\begin{aligned}
&\exp\left[-\sum_{i=1}^{n}\left(\gamma^{(i)}\Lambda_t^{(i)}+A^{(i)}(t)\lambda_t^{(i)}+c^{(i)}b^{(i)}\int_0^t A^{(i)}(s)ds\right)\right]\\
&\times\exp\left[\rho\int_0^t[1-\hat{c}\left(A^{(1)}(s),\cdots,A^{(n)}(s)\right)ds\right]
\end{aligned}$$

is a martingale where

$$A^{(i)}(t)=\frac{\left[D^{(i)}+c^{(i)}a^{(i)}\right]+\left[D^{(i)}-c^{(i)}a^{(i)}\right]\exp\left\{D^{(i)}t-k^{(i)}\right\}}{\left(\sigma^{(i)}\right)^2\left(1-\exp\left\{D^{(i)}t-k^{(i)}\right\}\right)} \tag{3.8}$$

with

$$\begin{aligned}
&\hat{c}(\zeta^{(1)},\cdots,\zeta^{(n)})\\
=\;&\int_0^\infty \overset{n}{\cdots}\int_0^\infty e^{-\sum_{i=1}^n\zeta^{(i)}x_i}\frac{\partial^2 C(F_{X^{(1)}}(x_1),\cdots,F_{X^{(n)}}(x_n))}{\partial x_1\cdots\partial x_n}dx_1\cdots dx_n,
\end{aligned}$$

and $D^{(i)}=\sqrt{(c^{(i)}a^{(i)})^2+2(\sigma^{(i)})^2\gamma^{(i)}}$.

Proof. The generator of the process has to satisfy $\mathscr{A}f = 0$ for it to be a martingale. Setting $f = e^{B(t)-\sum_{i=1}^{n}\left[\gamma^{(i)}\Lambda^{(i)}+A^{(i)}(t)\lambda^{(i)}\right]}$ obtains the equation

$$-\sum_{i=1}^{n}\left[\lambda^{(i)}A'^{(i)}(t) - c^{(i)}A^{(i)}(t)\left(b^{(i)}+a^{(i)}\lambda^{(i)}\right) - \lambda^{(i)}\gamma^{(i)}\right]$$
$$-\frac{1}{2}\sum_{i=1}^{n}\left(\sigma^{(i)}\sqrt{\lambda^{(i)}}\right)^2\frac{\partial^2 f}{\partial\lambda^{(i)2}} + B'(t) + \rho[\hat{c}\left(A^{(1)}(t),\cdots,A^{(n)}(t)\right) - 1] = 0$$

and solving it results in

$$\begin{aligned}
A^{(i)}(t) &= \frac{(D^{(i)}+c^{(i)}a^{(i)}) + (D^{(i)}-c^{(i)}a^{(i)})\exp\left(D^{(i)}t-k^{(i)}\right)}{(\sigma^{(i)})^2\left[1-\exp\left(D^{(i)}t-k^{(i)}\right)\right]} \\
\text{and } B(t) &= \sum_{i=1}^{n}c^{(i)}b^{(i)}\int_0^t A^{(i)}(s)ds + \rho\int_0^t[1-\hat{c}(A^{(1)}(s),\cdots,A^{(n)}(s))]ds \\
\text{with } D^{(i)} &= \sqrt{(c^{(i)}a^{(i)})^2 + 2(\sigma^{(i)})^2\gamma^{(i)}} \text{ for } i = 1,\cdots,n.
\end{aligned}$$

Hence the result follows. ■

Using the martingale in Theorem 3.2.1, we can easily obtain the joint Laplace transform of the vector $(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)},\cdots,\lambda^{(n)},t)$ at time t.

Corollary 3.2.2. *Considering constants $\alpha^{(i)} \geq 0$, and $\gamma^{(i)} \geq 0\ \forall i = 1,\cdots,n$ the joint Laplace transform of the vector $(\Lambda^{(1)},\cdots,\Lambda^{(n)},\lambda^{(1)},\cdots,\lambda^{(n)},t)$ is given by*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\sum_{i=1}^{n}\gamma^{(i)}\Lambda_t^{(i)}+\alpha^{(i)}\lambda_t^{(i)}}\middle|\lambda_0^{(1)},\cdots,\lambda_0^{(n)}\right] \\
&= \prod_{i=1}^{n}\left[H^{(i)}(t)^{\frac{2c^{(i)}b^{(i)}}{\sigma^{(i)2}}}\right]e^{-\left(\sum_{i=1}^{n}G^{(i)}(t)\lambda_0^{(i)}+\rho\int_0^t[1-\hat{c}\{G^{(1)}(s),\cdots,G^{(n)}(s)\}]ds\right)} \qquad (3.9)
\end{aligned}$$

where $t > 0$, with

$$G^{(i)}(t) = \frac{\alpha^{(i)}[(D^{(i)}+c^{(i)}a^{(i)}) + (D^{(i)}-c^{(i)}a^{(i)})\exp(-D^{(i)}t)] + 2\gamma^{(i)}(1-\exp\{-D^{(i)}t\})}{\sigma^{(i)2}\alpha^{(i)}[1-\exp(-D^{(i)}t)] + (D^{(i)}-c^{(i)}a^{(i)}) + [D^{(i)}+c^{(i)}a^{(i)}]\exp(-D^{(i)}t)}$$

and

$$H^{(i)}(t) = \frac{2D^{(i)}\exp[-\frac{D^{(i)}+c^{(i)}a^{(i)}}{2}t]}{\sigma^{(i)2}\alpha^{(i)}[1-\exp(-D^{(i)}t)] + (D^{(i)}-c^{(i)}a^{(i)}) + [D^{(i)}+c^{(i)}a^{(i)}]\exp(-D^{(i)}t)}$$

Proof. Set $A^{(i)}(T) = \alpha^{(i)}$ for $i = 1,2,\cdots,n$ using (3.8) where $t < T$, then we have

$$k^{(i)} = D^{(i)}T - \ln\left[\frac{c^{(i)}a^{(i)} + D^{(i)} - \alpha^{(i)}\sigma^{(i)^2}}{c^{(i)}a^{(i)} - D^{(i)} - \alpha^{(i)}\sigma^{(i)^2}}\right]. \tag{3.10}$$

Substitute (3.10) into (3.8) and the martingale in Theorem 3.2.1, the result follows immediately. ■

Corollary 3.2.3. *The joint Laplace transform of the vector* $(\Lambda^{(1)},\cdots,\Lambda^{(n)},t)$ *is given by*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\Sigma_{i=1}^{n}\gamma^{(i)}\Lambda_t^{(i)}}\Big|\lambda_0^{(1)},\cdots,\lambda_0^{(n)}\right] \\
= \;& \exp\left[-\sum_{i=1}^{n} G^{(i)}(t)\lambda_0^{(i)}\right] \times \prod_{i=1}^{n}\left[H^{(i)}(t)\right]^{\frac{2c^{(i)}b^{(i)}}{\sigma^{(i)2}}} \\
& \times \exp\left[-\rho\int_0^t\left[1 - \hat{c}\left\{G^{(1)}(s),\cdots,G^{(n)}(s)\right\}\right]ds\right]
\end{aligned} \tag{3.11}$$

Proof. Equation (3.11) follows immediately if we set $\alpha^{(i)} = 0\ \forall i = 1,\cdots n$ in equation (3.9). ■

Using Corollary 3.2.3, we can easily derive the joint survival/default probability and other relevant joint probabilities. While FGM copula admits a simple analytical expression, the same can not be said for Gaussian and Student-t copulas. Hence, we evaluate the probabilities numerically by replacing the suitable copula formulae in the third component of (3.11). Due to the dependence of simultaneous event jumps of $X^{(i)}$'s with sharing event jump frequency rate ρ, we have that

$$\mathbb{E}\left[e^{-\Sigma_{i=1}^{n}\Lambda_t^{(i)}}\right] \neq \mathbb{E}\left[e^{-\Lambda_t^{(1)}}\right]\mathbb{E}\left[e^{-\Lambda_t^{(2)}}\right]\cdots\mathbb{E}\left[e^{-\Lambda_t^{(n)}}\right].$$

If the event jump $X^{(i)}$ for $i = 1,2,\cdots,n$ occurs by a Poisson process $M_t^{(i)}$ with its frequency rate $\rho^{(i)}$ respectively and everything else is independent of each other, we have the joint survival probability of firm $i = 1,2,\cdots,n$ at time t, which is the product of each marginal survival probability.

3.2.2 Numerical Examples

In this section, we use the results obtained in the previous section to calculate survival/default probabilities and relevant joint probabilities. We assume bivariate dependence structure and a 1-year period ($t_1 = 0, t_2 = 1$) for the simplicity of computation. We also assume constant risk free rate, 0.023 and average annual event occurrence $\rho = 4$ per year. The degrees of freedom used for calculation of CDS rates under the student-t copula is $\nu = 3$. The following table summarizes the parameter values chosen for each obligor:

TABLE 3.1: Parameter values for the intensity process in the hypothetical example

Firms	$c^{(i)}$	$a^{(i)}$	$b^{(i)}$	$\sigma^{(i)}$	$\mu^{(i)}$	$\rho^{(i)}$	$\lambda_0^{(i)}$
Firm 1	0.5	-1	0	0.025	20	4	0.04
Firm 2	0.05	-1	0	0.25	2	4	0.4

In this example, Firm 1 is relatively more robust in terms of shock absorption than its counterpart, Firm 2. The strength of Firm 1 is characterized by a higher decay rate, a lower diffusion parameter, lower initial default intensity as well as higher jump size parameter (hence lower average jump size) as opposed to Firm 2.

From the equations (3.6), (3.7) and relevant probabilities that accounts for the survival of each Firm 1 and Firm 2, given by

$$\begin{aligned}&\Pr\left(\tau^{(1)} > t, \tau^{(2)} < t \,\middle|\, \lambda_0^{(1)}, \lambda_0^{(2)}\right)\\ =\ &\mathbb{E}\left[(1-e^{-\Lambda_t^{(2)}})e^{-\Lambda_t^{(1)}} \,\middle|\, \lambda_0^{(1)}, \cdots, \lambda_0^{(n)}\right],\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&\Pr\left(\tau^{(1)} \leq t, \tau^{(2)} \geq t \,\middle|\, \lambda_0^{(1)}, \lambda_0^{(2)}\right)\\ =\ &\mathbb{E}\left[(1-e^{-\Lambda_t^{(1)}})e^{-\Lambda_t^{(2)}} \,\middle|\, \lambda_0^{(1)}, \cdots, \lambda_0^{(n)}\right],\end{aligned} \tag{3.13}$$

the calculations of the joint survival/default probabilities and relevant joint probabilities are shown in Table 3.3 and 3.4. The individual survival and default probabilities calculated for Firm 1 and Firm 2 are shown in Table 3.2.

TABLE 3.2: Individual survival and default probabilities.

	FGM	Gaussian	Student-t
$\Pr(\tau^{(1)} > 1)$	0.891870	0.891870	0.849264
$\Pr(\tau^{(1)} \leq 1)$	0.108130	0.108130	0.150736
$\Pr(\tau^{(2)} > 1)$	0.322700	0.322700	0.294917
$\Pr(\tau^{(2)} \leq 1)$	0.677300	0.677300	0.705083

While the individual survival and default probability under the Gaussian and FGM copulas are equal, those probabilities in Table 3.2 under the Student-t copula are different as dependent parameter value $\theta = 0$ does not imply the case of independence, in line with [33]. We also found that the Student-t copula returns lower survival probability values and higher default probability values as opposed to its FGM and Gaussian counterparts by 5%. In comparison with the other 2 copulas, the default probability for Firm 2 (the weaker firm) is also greater under Student-t copula, suggesting that dependence structure under a Student-t copula could be a good candidate to depict a riskier environment.

TABLE 3.3: Joint survival and default probabilities.

$\Pr(\tau^{(1)} > 1, \tau^{(2)} > 1)$				$\Pr(\tau^{(1)} \leq 1, \tau^{(2)} \leq 1)$			
θ	FGM	Gaussian	Student-t	θ	FGM	Gaussian	Student-t
−0.95	0.292334	0.290216	0.260738	−0.95	0.077763	0.075646	0.116557
−0.9	0.292393	0.290359	0.260791	−0.9	0.077823	0.075788	0.116611
−0.5	0.292872	0.291711	0.261970	−0.5	0.078032	0.077140	0.117790
0	0.293472	0.293472	0.264586	0	0.078901	0.078901	0.120405
0.5	0.294072	0.295459	0.268433	0.5	0.079502	0.080889	0.124252
0.9	0.294554	0.297207	0.272726	0.9	0.079983	0.082636	0.128546
0.95	0.294614	0.297311	0.273423	0.95	0.080044	0.082740	0.129243

TABLE 3.4: Other relevant joint probabilities.

$\Pr(\tau^{(1)} > 1, \tau^{(2)} < 1)$				$\Pr(\tau^{(1)} < 1, \tau^{(2)} > 1)$			
θ	FGM	Gaussian	Student-t	θ	FGM	Gaussian	Student-t
−0.95	0.599536	0.601654	0.588526	−0.95	0.030367	0.032484	0.034179
−0.9	0.599477	0.601511	0.588472	−0.9	0.030307	0.032342	0.034125
−0.5	0.598998	0.600159	0.587293	−0.5	0.029828	0.03099	0.032946
0	0.598398	0.598398	0.584678	0	0.029229	0.029229	0.030331
0.5	0.597798	0.596411	0.580831	0.5	0.028628	0.027241	0.026484
0.9	0.597316	0.594663	0.576537	0.9	0.028147	0.025494	0.022190
0.95	0.597256	0.594560	0.575841	0.95	0.028086	0.025390	0.021494

Since Firm 1 is relatively stronger than Firm 2, the individual survival probability of Firm 1 is higher than its counterpart under all copula considered (see Table 3.2) with Student-t copula giving the lowest value, (approximately 0.85) whereas the FGM and Gaussian copula return almost 0.90 probability of Firm 1 surviving after 1 year. Hence the joint probabilities given in the FGM and Gaussian columns of Table 3.3 and 3.4 where the survivorship of Firm 2 is concerned, approach the individual survival / default probabilities of Firm 2, which are approximately 0.3 and 0.7 as given in Table 3.2, respectively.

With a low individual default probability within 1 year of Firm 1 under each copula, the joint defaultability of both firms also approaches Firm 1's individual default probability. Combined with the low individual survival probability of Firm 2 within 1 year, the probability that Firm 2 would survive after 1 year with Firm 1 defaulting within the same period, is very low (between 0.02 and 0.03) under each copula.

The results in Table 3.3 and 3.4 also demonstrate that the FGM, Gaussian and Student-t copulas show the same pattern, i.e. either increasing or decreasing as the dependence structure represented by parameter θ progress from negative to positive. We also note that the spread (i.e. the difference between probabilities corresponding to -0.95 and 0.95) is the widest under the Student-t copula (126.8511 bps), followed by Gaussian copula (70.9428 bps) and FGM copula (22.8044 bps).

Table 3.3 shows that joint survival and default probability decrease as the value of copula parameter θ moves from -0.95 to 0.95 as time to default for each firm moves in the same direction. Thus, when $\theta = -0.95$, we can consider applying the results to calculate joint survival and default probability for the firms in the substitute industry/sector. For example when $\theta = -0.95$, consider that Firm 1 produces cars run by petrol and Firm 2 produces cars run by battery. If the oil price surges due to an external event affecting the car manufacturing industry, consumers are likely to begin changing their petrol-run cars to battery-run cars.

In contrast, Table 3.4 show that joint probabilities increase as the value of copula parameter θ becomes -0.95 (or nearly -1) as time to default for each firm moves in the opposite direction. Hence when $\theta = 0.95$ (or nearly 1) we can consider applying the results to calculate joint survival and default probability for the firms in the complementary industry/sector - for instance, Firm 1 being an air-liner and Firm 2 being a chain hotel. An occurrence of a catastrophic event such as the September 11 World Trade Centre attacks or the disappearance of Malaysia Airlines flight MH370 may cause consumers to travel less via air and subsequently causing hotel booking rates to fall.

When comparing joint default probability between complementary industries and substitute industries, it was found that the joint default probability of firms in complementary industries was higher than its counterpart in substitute industries, which is economically intuitive (see $\Pr(\tau^{(1)} \leq 1, \tau^{(2)} \leq 1)$ in Table 3.3). When comparing the joint survival probability between complementary industries and substitute industries, we also found that the joint survival probability of firms in complementary industries was higher than its counterpart in substitute industries, which is also economically intuitive (see $\Pr(\tau^{(1)} > 1, \tau^{(2)} > 1)$ in Table 3.3). The relevant joint probabilities of the firms in substitute industries are higher than their counterparts in complementary industries because it is more likely that one firm will fail (or survive) when the other firm survives (or fails) if they are in substitute industries (see Table 3.4).

3.3 Applications

3.3.1 CDS Pricing Under Counterparty Risk

This section applies the results in Section 3.2 to the pricing of a financial product. For this purpose, the instrument credit default swaps (CDS) is chosen as there are three parties involved in this contract - a **r**eference **c**redit, a CDS **b**uyer and a CDS **s**eller. Note that the dependence is assumed only between the seller and the reference credit and that the buyer is assumed to be default free.

In calculating the CDS rate, we assume that the deterministic instantaneous rate of interest $r = 0.0023$ for a zero-coupon default-free bond. Then its price at time 0, paying 1 at time t is given by $B(0,t) = e^{-rt}$. Just like the previous section, the degrees of freedom used for calculation of CDS rates under the student-t copula is also $\nu = 3$.

We denote the default intensity process of the CDS buyer, seller and reference credit by $\lambda_t^{(b)}$, $\lambda_t^{(s)}$ and $\lambda_t^{(RC)}$ respectively. The CDS rate formula, denoted by $\bar{s}$, as adopted from [34] is given by

$$\bar{s} = (1-\pi)\frac{\sum_{k=1}^{k_N} e^{rc,s}(0,t_{k-1},t_k)}{\sum_{n=1}^{N}(t_{k_n} - t_{k_{n-1}})\overline{B}^b(0,t_{k_n})} \tag{3.14}$$

where

$$\begin{aligned} & e^{rc,s}(0,t_{k-1},t_k) \\ = \; & \mathbb{E}\left[\begin{array}{c} \exp\left(-\int_0^{t_k} r_s ds\right)\left[\exp\left(-\int_0^{t_{k-1}}\lambda_s^{(RC)}ds\right) - \exp\left(-\int_0^{t_k}\lambda_s^{(RC)}ds\right)\right] \\ \times\left[\exp\left(-\int_0^{t_k}\lambda_s^{(s)}ds\right)\right]\Big| r_0,\lambda_0^{(RC)},\lambda_0^{(s)}\Big] \end{array}\right] \end{aligned}$$

$$\overline{B}^b(0,t_{k_n}) = \mathbb{E}\left[\exp\left\{-\int_0^{t_{k_n}}\left(r_s + \lambda_s^{(b)}\right)ds\right\}\Big| r_0,\lambda_0^{(b)}\right] \tag{3.15}$$

and $t_{k_1} < t_{k_2} < \cdots < t_{k_n}$.

We assume that r_t and $\lambda_t^{(i)}$ are independent of each other and the recovery rate is deterministic. To keep the calculation simple, we use the case of 1-year CDS contract with premium paid by the buyer every 6 months, i.e. $N = 2$, $t_0 = 0$, $t_{k_1} = 0.5$, and $t_{k_2} = 1$, as well as recovery rate π. We may also use equation (3.15) to price defaultable bonds as well as credit spread between default-free bond and defaultable bond.

Assuming recovery rate $\pi = 0.5$ with the parameter values used in section 3.2.2, the parameter values for the intensity process of the CDS counterparties are shown in Table 3.5 and the CDS rate values are shown in Table 3.6.

TABLE 3.5: Parameter values for the intensity process of the CDS counterparties

Counterparty	$c^{(i)}$	$a^{(i)}$	$b^{(i)}$	$\sigma^{(i)}$	$\mu^{(i)}$	Jump frequency
CDS Seller	0.5	-1	0	0.025	10	4
Reference Credit	0.05	-1	0	0.25	2	4
CDS Buyer	0.2	-1	0	0.1	7	3

TABLE 3.6: CDS rates computed under various copulas dependence structure.

θ	FGM	Gaussian	Student-t
-0.95	0.347543	0.348826	0.354295
-0.9	0.347509	0.348744	0.354262
-0.5	0.347231	0.347905	0.353553
0	0.346884	0.346884	0.351978
0.5	0.346535	0.345732	0.349662
0.9	0.346256	0.344721	0.347077
0.95	0.346221	0.344718	0.346658
Spread (bps)	13.2196	41.0831	76.3648

As opposed to the elliptical copulas, the CDS rates under FGM copula do not show much difference as the dependence parameter varies from negative to positive dependence, parallel with the finding in [28]. This is shown by the value of spread of only 13.2196 bps (given by $0.347543 - 0.346221$) as compared to the Gaussian copula (41.0831 bps) and Student-t copula (76.3648 bps). We also note that the CDS rates show a decreasing pattern under all copulas considered as θ varies from negative correlation to positive correlation, which is a similar pattern to that seen in the survival probabilities (see Figure 3.7).

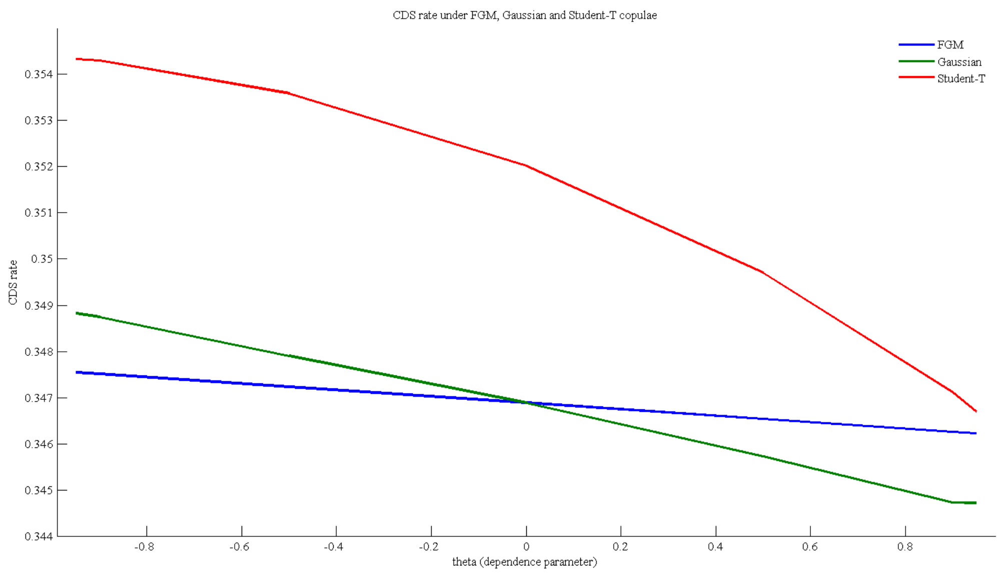

FIGURE 3.7: CDS rates under FGM, Gaussian and Student-t copulas.

3.3.2 CDS rates calculation: Sensitivity analysis

In this section, we conduct sensitivity analysis of CDS rates with respect to the seller's and reference credit's jump size rate, frequency rate, diffusion rate, decay rate and the reversion level. Since the patterns of CDS rates sensitivity analysis are the same under all copulas, only the findings under Student-t copula are presented here and we refer the readers to Appendix B for the rest of findings under the Gaussian and FGM copulas.

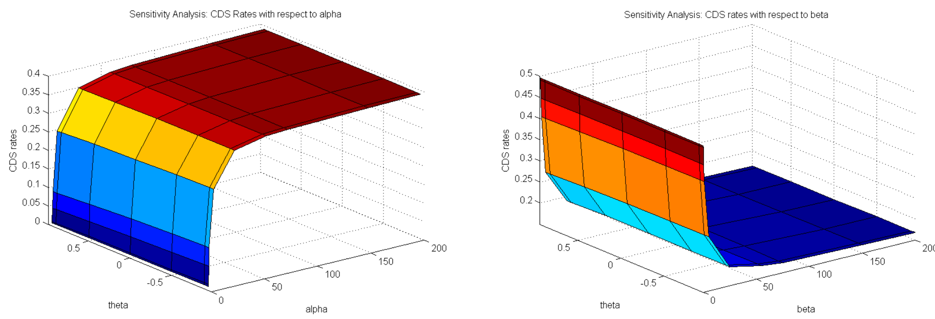

FIGURE 3.8: Sensitivity of CDS rates under Student-t copula with respect to seller's (left) and RC's (right) jump size jump size, $\mu^{(s)}$ and $\mu^{(RC)}$ respectively.

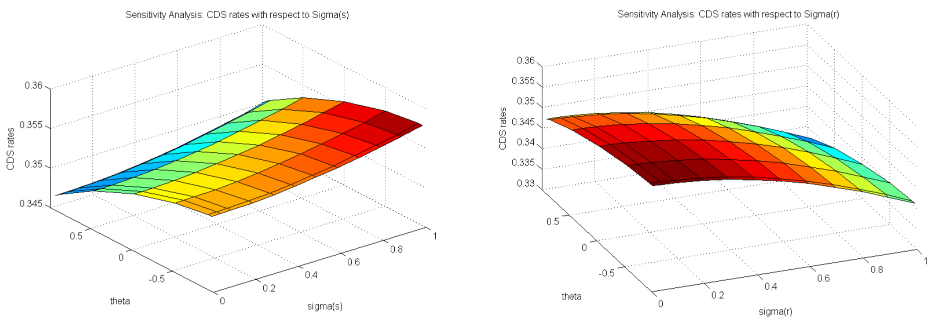

FIGURE 3.9: Sensitivity of CDS rates under Student-t copula with respect to seller's and RC's diffusion rates, i.e. $\sigma^{(s)}$ and $\sigma^{(r)}$ respectively.

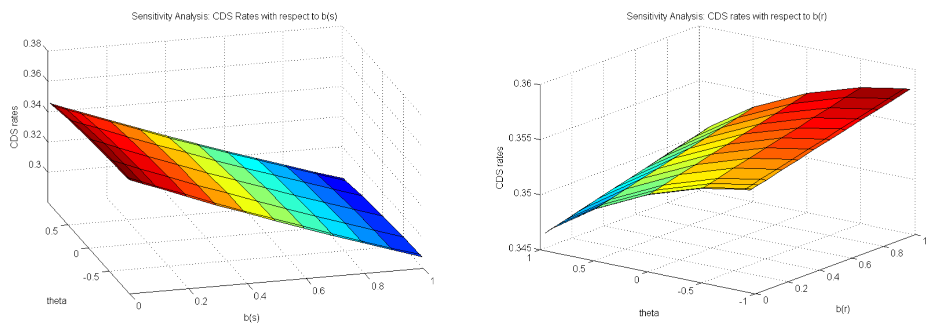

FIGURE 3.10: Sensitivity of CDS rates under Student-t copula with respect to the constant reversion level of seller, $b^{(s)}$, and RC $b^{(RC)}$, with $c^{(s)} = c^{(RC)} = 1$ and $a^{(s)} = a^{(RC)} = -1$.

As shown in Figure 3.8 and Figure 3.11, the CDS rate is converging to 0 as the values of $\mu^{(RC)}$ and $c^{(RC)}$ are increased. In contrast, CDS rate also converge to 0 as the value of $\mu^{(s)}$

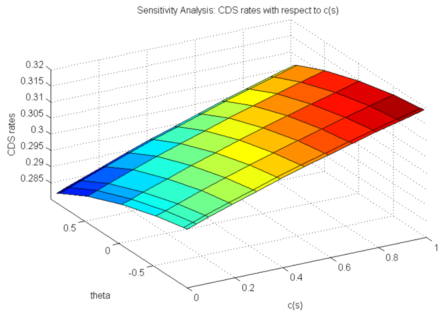

FIGURE 3.11: Sensitivity of CDS rates under Student-t copula with respect to seller's and RC's decay rate, $c^{(s)}$ and $c^{(r)}$ respectively, where $b^{(s)} = b^{(RC)} = 1$ and $a^{(s)} = a^{(RC)} = -1$.

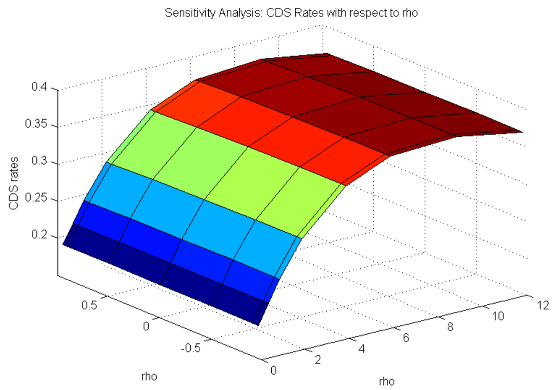

FIGURE 3.12: Sensitivity of CDS rates under Student-t copula with respect to frequency of yearly jump events, ρ.

and $c^{(s)}$ are decreased for the CDS seller. These findings are similar to that of [28] in which increasing the value of the jump size and decay rate parameter, $c^{(i)}$ for $i = s, rc$ will result in a monotonically increasing value of CDS rates (for changes in $\mu^{(s)}$ and $c^{(s)}$) and decreasing (for changes in $\mu^{(RC)}$ and $c^{(RC)}$). Intuitively, from the CDS buyers' point of view, a CDS contract is more attractive when the CDS seller is less likely to default. As long as the CDS seller's credit is strong enough, they can hedge against the default risk of the reference credit using a CDS contract. Hence the lower the CDS rate, the more likely the CDS seller defaults. The worst case scenario for the CDS buyer is when both the reference credit and the CDS seller default.

Figure 3.9 shows a decreasing CDS rates as we increase the value of $\sigma^{(RC)}$, as well as an increasing CDS rates as we increase the value of $\sigma^{(s)}$. Intuitively, an increasing values of reference credit's diffusion rate $\sigma^{(RC)}$ will reduce the CDS rates because the CDS contract is deemed as less safe since the defaultability of the reference credit becomes more certain, thereby reducing the survival probability of the reference credit, as can be seen in equation (3.15). In contrast, while it is slightly difficult to see the intuition behind the increasing CDS rates as we increase the seller's diffusion rate $\sigma^{(s)}$, closely examining the numerator of CDS rate (equation (3.15)) easily verifies that the changes in numerator moves in upward direction as we increase seller's diffusion rate $\sigma^{(s)}$, bearing in mind that

$$\mathbb{E}\left[\exp\left(-\int_0^{t_k}\lambda_s^{(s)}ds\right)\Big|r_0,\lambda_0^{(RC)},\lambda_0^{(s)}\right]$$

have the same form as the default free bond price, as presented in Table 4.3 of [21], which increased as σ increased.

We also found that the CDS rates show a monotonically increasing and decreasing behaviours with respect to changes in $\sigma^{(s)}$ and $\sigma^{(RC)}$ respectively. These are different from the findings shown in Section 4 of [28] which presented a graph showing instability in the values of the CDS rates resulting from the changes in the two parameters.

Comparing to other parameters of each counterparty, the constant reversion level parameters $b^{(s)}$ and $b^{(RC)}$ give an opposite direction of changes in the CDS rates, as in Figure 3.10. Even though the default threshold level will be discernible only after default occurs, higher $b^{(s)}$ and $b^{(RC)}$ implies that the default is more likely to happen. Therefore, assuming that the seller has strong credibility, higher $b^{(RC)}$ allows the seller to demand the buyer to pay higher premium for the CDS contract as the default event is likely to happen. This is parallel to the justification of insurers demanding higher premium from smokers for a life insurance contract as opposed to a non-smoker. On the other hand, higher $b^{(s)}$ implies that the seller is likely to default. Hence, the CDS rates decrease since reduced credibility of the seller will make the CDS contract less attractive and induce the CDS buyer to obtain the protection from another seller.

By changing the values of the event jump frequency, ρ, we notice that the value of the CDS rates will increase up to a certain threshold level under all copula, and decrease thereafter. For the case of student-t copula, this can be seen in Figure 3.12 (refer to Table 3.7), whereas Table 3.8 and 3.9 show the CDS rates under the other two copulas. This implies that while initially the seller was able to withstand the default risk of the reference credit, its ability to absorb that risk declines as the event jumps occur more frequently. This is not examined extensively in the section 4 of [28] where they presented a table showing an increasing values of the CDS rates under the FGM copula, only up to $\rho = 3.9$ (refer to Table 2 of [28]) for $\theta = 1$. When the jump occurrence is too frequent to the extent that it affects both the CDS seller and reference credit, there is an increasing chance of both counterparties going bust. As aforementioned, this would be the worst scenario for the CDS buyers and would subsequently make the CDS rates less valuable from the buyers' perspective.

3.4 Conclusion

For default intensities modeling, we used the multivariate jump diffusion process in which jumps (i.e. the magnitude of contribution of primary events to default intensities) occur simultaneously and their sizes are dependent. We then used a homogeneous Poisson

TABLE 3.7: CDS rates under student-t copula with respect to various ρ. Note: * Diff = $\bar{s}_{\theta_{-0.95}} - \bar{s}_{\theta_{0.95}}$. Difference unit in bps.

θ / ρ	0.05	0.5	1	2	4	6	9	12
-0.95	0.1941	0.2207	0.2499	0.2967	0.3543	0.37986	0.3842	0.3681
-0.9	0.1941	0.2207	0.2499	0.2966	0.3543	0.37983	0.3842	0.3681
-0.5	0.194	0.2205	0.2496	0.2961	0.3536	0.379159	0.3837	0.3677
0	0.1939	0.2201	0.2488	0.2948	0.352	0.377666	0.3826	0.367
0.5	0.1938	0.2194	0.2477	0.293	0.3497	0.375456	0.3809	0.3659
0.9	0.1936	0.2187	0.2464	0.291	0.3471	0.372971	0.379	0.3645
0.95	0.1936	0.2186	0.2462	0.2907	0.3467	0.372566	0.3787	0.3643
Diff	4.6032	21.029	37.57	59.961	76.365	72.94624	55.611	37.689

TABLE 3.8: CDS rates under Gaussian copula with respect to various ρ.

θ / ρ	0.05	0.5	1	2	4	6	9	12
-0.95	0.1866	0.2114	0.2392	0.2855	0.3488	0.3844	0.4057	0.4053
-0.9	0.1866	0.2113	0.2391	0.2855	0.3487	0.3843	0.4056	0.4053
-0.5	0.1865	0.2111	0.2388	0.2848	0.3479	0.3835	0.405	0.4048
0	0.1864	0.2109	0.2383	0.2841	0.3469	0.3824	0.4041	0.4042
0.5	0.1864	0.2106	0.2378	0.2832	0.3457	0.3813	0.4032	0.4035
0.9	0.1863	0.2103	0.2373	0.2824	0.3447	0.3803	0.4023	0.4029
0.95	0.1863	0.2103	0.2373	0.2824	0.3447	0.3803	0.4024	0.4029
Diff	2.2554	10.402	18.806	30.736	41.049	41.117	33.656	24.489

TABLE 3.9: CDS rates under FGM copula with respect to various ρ.

θ / ρ	0.05	0.5	1	2	4	6	9	12
-0.95	0.186485	0.211	0.2386	0.2846	0.3475	0.3831	0.4047	0.4046
-0.9	0.186483	0.211	0.2386	0.2845	0.3475	0.3831	0.4046	0.4045
-0.5	0.186468	0.211	0.2384	0.2843	0.3472	0.3828	0.4044	0.4044
0	0.186448	0.2109	0.2383	0.2841	0.3469	0.3824	0.4041	0.4042
0.5	0.186429	0.2108	0.2381	0.2838	0.3465	0.3821	0.4039	0.404
0.9	0.186414	0.2107	0.238	0.2836	0.3463	0.3818	0.4036	0.4038
0.95	0.186412	0.2107	0.238	0.2836	0.3462	0.3818	0.4036	0.4038
Diff	0.72677	3.3517	6.0593	9.9014	13.22	13.237	10.83	7.8761

process to count simultaneous event jumps in default intensities, and applied the FGM copula, Gaussian copula and Student-t copula were used, assuming exponential marginal distributions to model the dependence structure between event jump sizes. We also presented the simulated paths of the jump diffusion intensity processes under the three copulas with various dependence parameter values, θ.

By applying copula-dependent default intensity to the multivariate Cox process, we derived the joint survival/default probability and other relevant joint probabilities via the joint Laplace transforms for which the PDMP theory and standard martingale methodology were used. We then showed an example to calculate joint survival/default probability, with an application to CDS rate considering counterparty risk. We also conduct sensitivity analyses with respect to the parameter values involved.

In this study, the multivariate jump diffusion process examined was used to model counterparty risk in CDS rates. This process also has the potential to be applicable to a variety of problems where multiple transition rates are involved in the realms of economics, finance and insurance, which could be the object of further research.

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4

Jump Diffusion Model with Copula Dependence Structure in Defaultable Bond Pricing

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This article has been submitted for publication in the *Annals of Actuarial Science*. The submitted article is presented in its entirety here and hence contains repetitions of certain segments of the Introduction presented in Chapter 1.

Abstract We study the pricing of a defaultable bond under various copulas. For that purpose, we use a bivariate jump diffusion process for a bond issuer's default intensity and the short rate of interest. We assume two jumps in this process occur simultaneously and their sizes are dependent. For these simultaneous jumps and their sizes, a homogeneous Poisson process and three copulas, which are a Farlie-Gumbel-Mogenstern (FGM) copula, a Gaussian copula and a t-copula are used, respectively. We derive the joint Laplace transform for their integrated processes that provides us with the expression for defaultable bond price, using copula-dependent jump sizes. To do so, the piecewise deterministic Markov process (PDMP) theory and the martingale methodology in are used. We compute zero coupon defaultable bond prices and their yields using the three copulas and exponential marginal distributions. We then use the model to calibrate zero coupon bonds traded in different markets. We notice that Student-t copula provides the best fit relative to the other two copulas.
Keywords Bivariate jump diffusion model, Default intensity, Short rate of interest, Copulas, Corporate bond pricing

4.1 Introduction

Corporate bonds' default rates have declined since 2009 when the world economy began to recover from the global financial crisis in response to governments' initiatives. However, continuing distress in the US and Eurozone economies may jeopardize the low default rate environment. Hence, it is necessary to develop pricing models for corporate bonds that capture the dependence structure between obligors' default intensity and macroeconomics variables.
Corporate debt valuation models can be divided into two main approaches: the structural approach and the reduced form approach. The first class of models under the structural approach views a firm's liabilities as contingent claims issued against the firm's assets, with all the payoffs to the firms's liabilities in bankruptcy completely specified (see seminal work in [32] and [2]). That is, bankruptcy is viewed as the event when the firm's value hits a prespecified boundary. The view undertaken in this class of models was then simplified in [29] and [14], whereby the cash flows to risky debt in the event of bankruptcy were exogenously specified as a given fraction of each promised dollar in the event of bankcruptcy. This was to avoid the need to understand the complex priority structure of payoffs to a firm's liabilities in the event of bankruptcy. In [36], the bond prices following a structural default model with jumps were computed with Monte Carlo simulation based on Brownian bridge algorithm.

In contrast, we are working under the reduced form approach by introducing the correlation aspect through a model in which the default of one obligor triggers the default of another. Previous studies of the reduced form approach have taken several directions in researchers' attempt to incorporate default correlation and multiple defaults (see e.g. [3], [30],[13] and [18]). A convenient framework that allows for dependencies between default intensities and state variables was prepared in [26], whereby the Cox processes were used to model the (stopping) time when the rating changed until the issuer went default in the last state of a generalized K-states Markovian model. [10], one of the earliest papers to promote the term 'reduced-form' approach, treated default as an unpredictable event governed by external hazard rate process. The article showed that a contingent claim that is subject to default risk can be priced just like the default-free claim simply by replacing the short rate with the default-adjusted short rate process under an equivalent martingale measure in an arbitrage free framework. This model was extended in [22] and the author introduced the concept of counterparty risk to capture the economy-wide and inter-firm linkages by including jumps in the default intensities that follows a Cox process.

Another approach is the hybrid of the structural and the reduced form approach, developed in [21] whereby the bankruptcy process was modelled as a continuous time Markov process with discrete state space representing the firm's credit ratings. This model originated from the Jarrow and Turnbull (1995) model that took the reduced form approach promoted in [20]. The hybrid approach further simplifies the view taken in the structural models by specifying the credit event exogenously and allow the bankruptcy assumptions to be imposed only on observables (i.e. firm's credit ratings) as opposed to firm's asset values. Another hybrid example can also be found in [15] where they provided an explicit formula for defaultable bond and credit default swap using partial differential equation method assuming expected and unexpected default in the case of stochastic default intensity.

Besides the construction of a point process, considerable attention is given to the default dependence. The work by [10] considered joint jumps in the default intensity for this effect, while [24] and [22] developed it further considering the possibility of default-event triggers that cause joint default. Another approach to incorporate default dependence between related parties is through the use of copulas ([28]; [38]; [23], [12] and [30]). The use of FGM copula with multivariate shot noise process was explored in [19], and extended in [30] by adding diffusion term to the intensity processes. Both papers adopted martingale methodology and PDMP technique to derive the survival probability.

The remaining of the article is organized as follows: Section 4.2 defines the bivariate jump diffusion process for short rate and firm's default intensity, whereby it is assumed that the jumps of default intensity and short rate occur simultaneously, and that the dependence structure between their jump sizes was captured by the three copulas. The relevant joint Laplace transforms are derived using the PDMP theory and martingale methodology. These joint Laplace transforms then lead to the expression of the bond price. This is followed by a numerical example in Section 4.3 showing the computation of bond prices and their yields, while capturing the dependence structure between the vector of jumps, using three copulas as examples – the FGM, Gaussian and Students t-copula. Section 4.4, conducts one-day calibrations of zero coupon bonds data dated 30 October 2012, issued by three corporations, i.e. Microsoft Inc., NAB and Eskom Holdings, under each copula considered. This is followed by a one-year calibration of the zero coupon bond issued by Microsoft Inc. under the student-t copula, from 22 June 2010 - 30 June 2011. Section 4.5 presents some concluding remarks.

4.2 Model Setup

For $i = 1$ (bond issuer) and 2 (short rate), the bivariate jump diffusion model considered has the following structure:

$$dX_t^{(i)} = c^{(i)}\left(b^{(i)} + a^{(i)}X_t^{(i)}\right)dt + \sigma^{(i)}\sqrt{X_t^{(i)}}dW_t^{(i)} + dL_t^{(i)}, \quad L_t^{(i)} = \sum_{j=1}^{M_t} Y_j^{(i)} \tag{4.1}$$

where

- $\left\{Y_j^{(1)}, Y_j^{(2)}\right\}_{j=1,2}$ is a vector sequence of dependent but not identically distributed random variables with distribution function $F^{(i)}(y)$ $(y > 0)$,
- M_t is the total number of events up to time t,
- $W_t^{(i)}$ is a standard Brownian motion governing process $X_t^{(i)}$,
- $a < 0$, $b \geq 0$ and $c > 0$ with $c^{(i)}a^{(i)}$ being the rate of exponential decay of $X_t^{(i)}$ and $c^{(i)}b^{(i)}$ being the constant reversion level of process $X_t^{(i)}$,

- $\sigma^{(i)} > 0$ is the diffusion coefficient for $X_t^{(i)}$.

We also make the additional assumption that the point process M_t is independent of the vector sequence of jump sizes and that the vector sequence $\left\{Y_j^{(1)}, Y_j^{(2)}\right\}_{j=1,2}$ are independent of another vector sequence for $k \neq j$. $L_t^{(i)}$ is a compound process for $Y_t^{(i)}$.

In this model, the source of dependence between variables $X_t^{(1)}$ and $X_t^{(2)}$ is from the common event arrival process M_t, together with the dependence between the vector of jumps $\left(Y_j^{(1)}, Y_j^{(2)}\right)$. We assume that the event arrival process M_t, i.e. simultaneous jump process follows a homogeneous Poisson process with frequency ρ and the vector of jumps is modelled using copulas (see e.g. [31] and [35]) – that is, the joint distribution of the vector $\left(Y_j^{(1)}, Y_j^{(2)}\right)$ is assumed to be of the form $C(F^{(1)}, F^{(2)})$ with C being a given copula. Other than bond pricing, copulas have also been applied widely in capturing the dependence structure embedded in insurance portfolio as well as other financial products such as the CDS and indices (see [19], [30], [1], [16] and [33, 34]).

As specific examples for C in this paper, we use the Farlie-Gumbel-Morgenstern, the Gaussian and the Student-t copulas which are given in consecutive manner by:

$$C^{FGM}(u_1, u_2) = \left[1 + \theta\left(1 - u_1\right)\left(1 - u_2\right)\right] u_1 u_2 \tag{4.2}$$

$$C^{G}(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{|\Theta|}} \exp\left(-\frac{1}{2}\omega^{\mathbf{T}}\Theta^{-1}\omega\right) dudv \tag{4.3}$$

$$C_{\upsilon}^{t}(u_1, u_2) = \int_{-\infty}^{t_{\upsilon}^{-1}(u_1)} \int_{-\infty}^{t_{\upsilon}^{-1}(u_2)} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\upsilon)^2|\Theta|}} \left(1 + \frac{\eta^{\mathbf{T}}\Theta^{-1}\eta}{\nu}\right) dudv \tag{4.4}$$

where $u_i \in [0,1]$ for $i = 1,2$, and the correlation parameter $\theta \in [-1,1]$. For the elliptical copulas, the correlation parameter is contained in the correlation matrix $\Theta = \begin{bmatrix} 1 & \theta \\ \theta & 1 \end{bmatrix}$.
We also define ω=$[\ \omega_1 \quad \omega_2\ ]^T$and η=$[\ \eta_1 \quad \eta_2\ ]^T$ where $\omega_i = \Phi^{-1}(u_i)$ and η_i= $t_{\upsilon}^{-1}(u_i)$ are the inverse Gaussian and inverse Student-t distribution with degrees of freedom υ respectively, taken on the variables u_i. For the marginal distributions of $Y_j^{(i)}$ in the vector of jumps $(Y_j^{(1)}, Y_j^{(2)})$, any continuous distribution can be considered.

Using $F^{(i)}\left(y_j\right) = 1 - e^{-\mu^{(i)} y_j}\left(\mu^{(i)} > 0,\ y_j > 0\right)$, for $i = 1,2$, the FGM copula, which is illustrated in Figure 4.1, is used in this study for its simplicity and analytical tractability, where it is also used in [19] and [30] in the context of CDS pricing with counterparty risk. Its simplicity allows for the closed-form expressions to be easily derived. The Gaussian copula, shown in Figure 4.3, is chosen so as to examine the effect of elliptical copula on simultaneous jumps between the default intensity and short rate of interest in the context of defaultable bond pricing. We also choose the Student-t copula to incorporate the possibility of having more frequency of

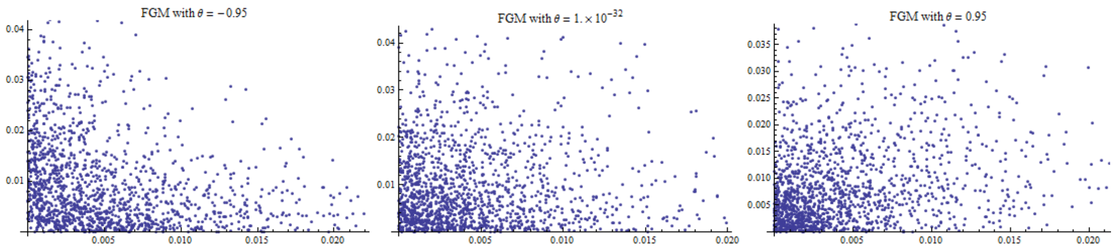

FIGURE 4.1: FGM copula with exponential margins and dependence parameter -0.95, 0, 0.95

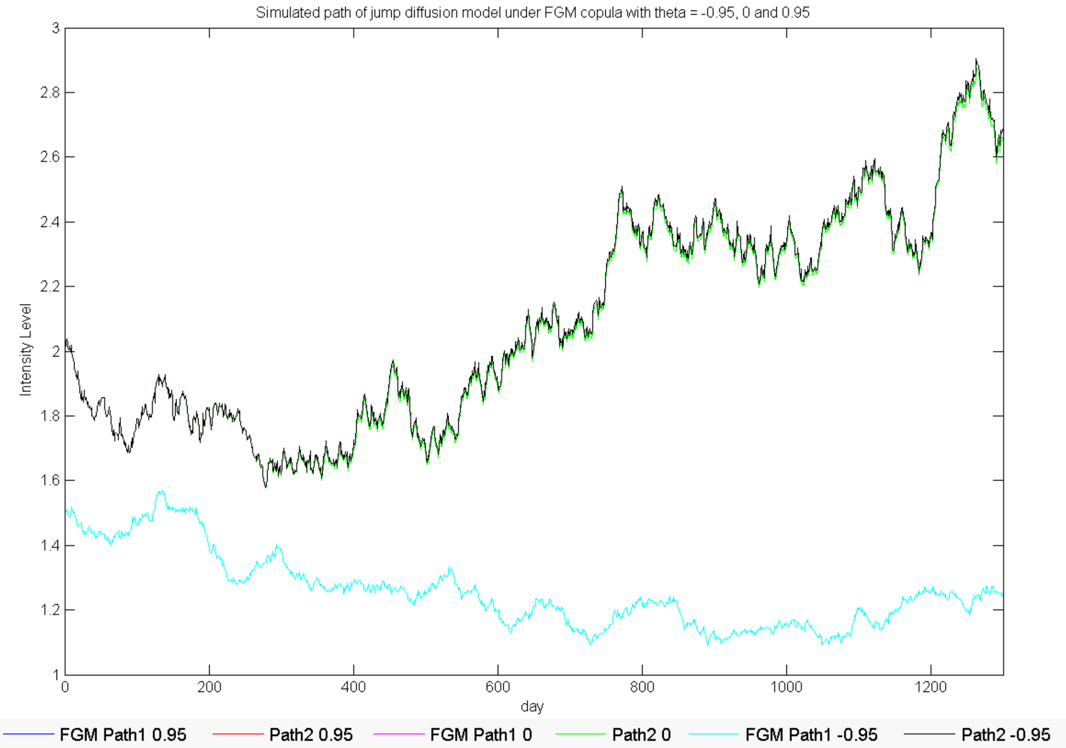

FIGURE 4.2: Simulated paths of jump diffusion process with dependence structure capture by FGM copula

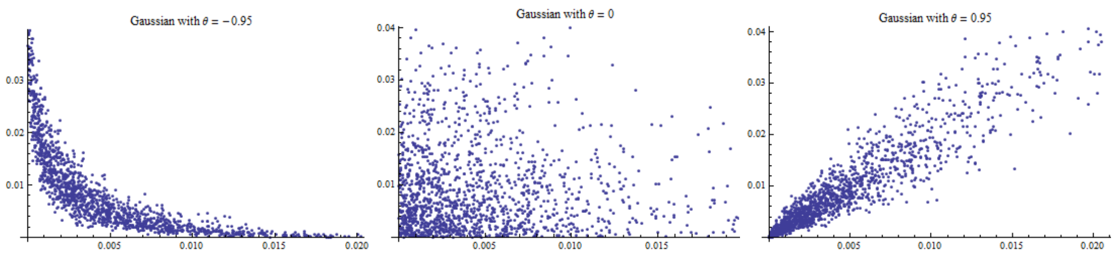

FIGURE 4.3: Gaussian copula with exponential margins and dependence parameter -0.95, 0, 0.95

higher and/or smaller as well as opposing joint jumps size impact between the default intensity

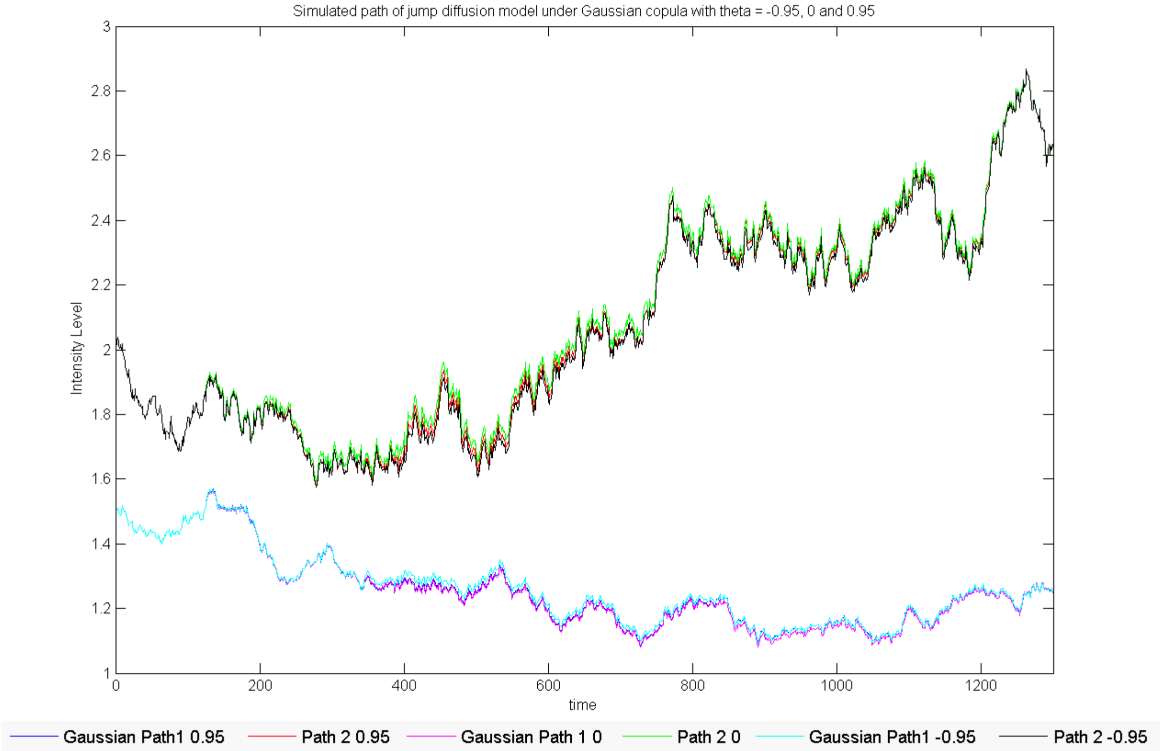

FIGURE 4.4: Simulated paths of jump diffusion process with dependence structure capture by Gaussian copula

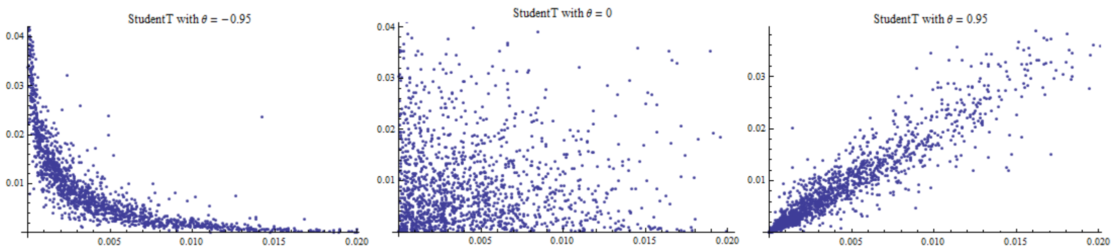

FIGURE 4.5: Student-T copula with exponential margins and dependence parameter -0.95, 0, 0.95

and short rate of interest, as shown in Figure 4.5.

The simulated paths of the jump diffusion process under each copula considered in this study with exponential jump size distributions is also shown in Figures 4.2, 4.4 and 4.6.

4.2.1 The Joint Laplace Transform of the Distribution of the Integrated Process

We start with defining $\Psi_t^{(i)} = \int_0^t X_s^{(i)} ds$, for $i = 1, 2$, to represent the integrated process up to time t.

Let us now derive the joint Laplace transform of the vector $(\Psi^{(1)}, \Psi^{(2)}, X^{(1)}, X^{(2)}, t)$. To do

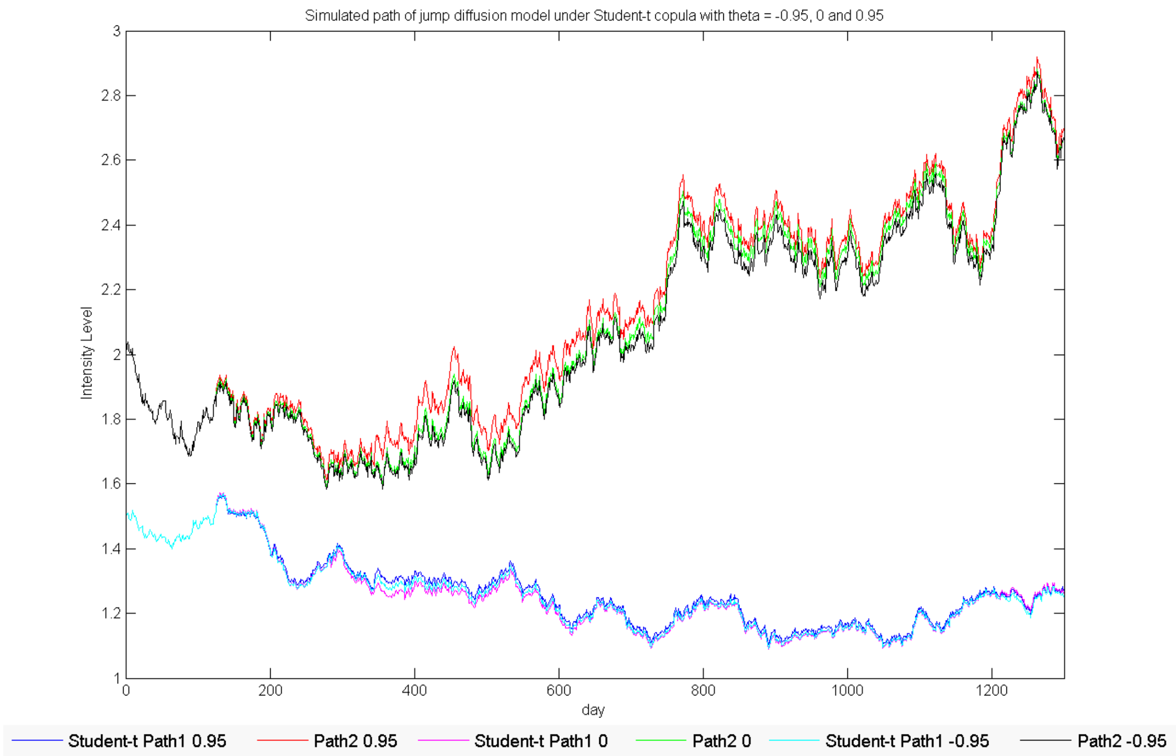

FIGURE 4.6: Simulated paths of jump diffusion process with dependence structure capture by student-T copula

so, we use the PDMP theory developed in [9] and the martingale methodology developed in [8]. Analogous to the univariate case in [18], the generator $\mathscr{A}$ of the process $(\Psi_t^{(1)}, \Psi_t^{(2)}, X_t^{(1)}, X_t^{(2)}, t)$ acting on a function $f(\Psi^{(1)}, \Psi^{(2)}, X^{(1)}, X^{(2)}, t)$ belonging to its domain is given by

$$\begin{aligned}
&\mathscr{A} f(\Psi^{(1)}, \Psi^{(2)}, X^{(1)}, X^{(2)}, t) \\
= \ &\frac{\partial f}{\partial t} + \sum_{i=1}^{2} X^{(i)} \frac{\partial f}{\partial \Psi^{(i)}} + \sum_{i=1}^{2} c^{(i)}(b^{(i)} + a^{(i)} X^{(i)}) \frac{\partial f}{\partial X^{(i)}} + \frac{1}{2} \sum_{i=1}^{2} \left(\sigma^{(i)} \sqrt{X^{(i)}} \right)^2 \frac{\partial^2 f}{\partial X^{(i)2}} \\
&+ \rho \left[\begin{array}{c} \int_0^\infty \int_0^\infty f(\Psi^{(1)}, \Psi^{(2)}, X^{(1)} + y_1, X^{(2)} + y_2, t) \frac{\partial^2 C(F_{Y^{(1)}}(y_1), F_{Y^{(2)}}(y_2))}{\partial y_1 \partial y_2} dy_1 dy_2 \\ - f(\Psi^{(1)}, \Psi^{(2)}, X^{(1)}, X^{(2)}, t) \end{array} \right]
\end{aligned}$$

where $\frac{\partial^2 C(F_{Y^{(1)}}(y_1), F_{Y^{(2)}}(y_2))}{\partial y_1 \partial y_2}$ is the joint density of event jump sizes.

For $f(\Psi^{(1)}, \Psi^{(2)}, X^{(1)}, X^{(2)}, t)$ to belong to the domain of the generator $\mathscr{A}$, it is sufficient that the function $(\Psi^{(1)}, \Psi^{(2)}, X^{(1)}, X^{(2)}, t)$ is differentiable w.r.t. $\Psi^{(i)}$, $X^{(i)}$, and t, for $i = 1, 2$, and that

$$\left\| \begin{array}{c} \int_0^\infty \int_0^\infty f(\cdot, X^{(1)} + y_1, X^{(2)} + y_2, \cdot) \frac{\partial^2 C(F_{Y^{(1)}}(y_1),, F_{Y^{(2)}}(y_2))}{\partial y_1 \partial y_2} dy_1 dy_2 \\ - f(\cdot, X^{(1)}, X^{(2)}, \cdot) \end{array} \right\| < \infty.$$

To derive the joint Laplace transform of the vector $\left(\Psi^{(1)},\Psi^{(2)},X^{(1)},X^{(2)},t\right)$, we begin with finding a suitable martingale.

Theorem 4.2.1. *Considering constant $\gamma^{(i)} \geq 0$ and $k^{(i)} \geq 0$,*

$$\exp\left(-\sum_{i=1}^{2}\gamma^{(i)}\Psi_t^{(i)} + A^{(i)}(t)X_t^{(i)} + c^{(i)}b^{(i)}\int_0^t A^{(i)}(s)ds\right)$$
$$\times \exp\left[\rho\int_0^t [1-\hat{c}\left(A^{(1)}(s),A^{(2)}(s)\right)ds\right]$$

is a martingale where

$$A^{(i)}(t) = \frac{\left[D^{(i)}+c^{(i)}a^{(i)}\right]+\left[D^{(i)}-c^{(i)}a^{(i)}\right]\exp\left\{D^{(i)}t-k^{(i)}\right\}}{\left(\sigma^{(i)}\right)^2\left(1-\exp\left\{D^{(i)}t-k^{(i)}\right\}\right)} \tag{4.5}$$

with

$$\hat{c}(\zeta^{(1)},\zeta^{(2)}) = \int_0^\infty\int_0^\infty e^{-\sum_{i=1}^2\zeta^{(i)}y_i}\frac{\partial^2 C(F_{Y^{(1)}}(y_1),F_{Y^{(2)}}(y_2))}{\partial y_1\partial y_2}dy_1dy_2, \tag{4.6}$$

and $D^{(i)} = \sqrt{(c^{(i)}a^{(i)})^2+2(\sigma^{(i)})^2\gamma^{(i)}}$.

Proof. The generator of the process has to satisfy $\mathscr{A}f = 0$ for it to be a martingale. Setting $f = e^{B(t)-\sum_{i=1}^2\left[\gamma^{(i)}\Psi^{(i)}+A^{(i)}(t)X^{(i)}\right]}$ we obtain the equation

$$-\sum_{i=1}^{2}\left[X^{(i)}A'^{(i)}(t)-c^{(i)}A^{(i)}(t)\left(b^{(i)}+a^{(i)}X^{(i)}\right)-X^{(i)}\gamma^{(i)}\right]$$
$$-\frac{1}{2}\sum_{i=1}^{2}\left(\sigma^{(i)}\sqrt{X^{(i)}}\right)^2\frac{\partial^2 f}{\partial X^{(i)2}}+B'(t)+\rho[\hat{c}(A_1(t),A_2(t))-1]=0$$

and solving it we get

$$\begin{aligned}
A^{(i)}(t) &= \frac{(D^{(i)}+c^{(i)}a^{(i)})+(D^{(i)}-c^{(i)}a^{(i)})\exp\left(D^{(i)}t-k^{(i)}\right)}{(\sigma^{(i)})^2\left[1-\exp\left(D^{(i)}t-k^{(i)}\right)\right]}\\
\text{and } B(t) &= \sum_{i=1}^{2}c^{(i)}b^{(i)}\int_0^t A^{(i)}(s)ds+\rho\int_0^t[1-\hat{c}(A^{(1)}(s),A^{(2)}(s)]ds\\
\text{with } D^{(i)} &= \sqrt{(c^{(i)}a^{(i)})^2+2(\sigma^{(i)})^2\gamma^{(i)}} \text{ for } i=1,2.
\end{aligned}$$

Hence the result follows. ■

Using the martingale in Theorem 4.2.1, we can easily obtain the joint Laplace transform of the vector $(\Psi^{(1)},\Psi^{(2)},X^{(1)},X^{(2)},t)$ at time t.

Corollary 4.2.2. *Considering constants $\alpha^{(i)} \geq 0$, and $\gamma^{(i)} \geq 0$ for i=1,2 the joint Laplace transform of the vector $(\Psi^{(1)}, \Psi^{(2)}, X^{(1)}, X^{(2)}, t)$ is given by*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\sum_{i=1}^{2}(\gamma^{(i)}\Psi_t^{(i)}+\alpha^{(i)}X_t^{(i)})}\Big|X_0^{(1)}, X_0^{(2)}\right] \\
&= \prod_{i=1}^{2}\left[H^{(i)}(t)^{\frac{2c^{(i)}b^{(i)}}{\sigma^{(i)2}}}\right] e^{-\left(\sum_{i=1}^{2}G^{(i)}(t)X_0^{(i)}+\rho\int_0^t[1-\hat{c}\{G^{(1)}(s),G^{(2)}(s)]ds\right)}
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
&G^{(i)}(t) \\
&= \frac{\alpha^{(i)}[(D^{(i)}+c^{(i)}a^{(i)})+(D^{(i)}-c^{(i)}a^{(i)})\exp(-D^{(i)}t)]+2\gamma^{(i)}(1-\exp\{-D^{(i)}t\})}{\sigma^{(i)2}\alpha^{(i)}[1-\exp(-D^{(i)}t)]+(D^{(i)}-c^{(i)}a^{(i)})+[D^{(i)}+c^{(i)}a^{(i)}]\exp(-D^{(i)}t)}
\end{aligned}$$

and

$$\begin{aligned}
&H^{(i)}(t) \\
&= \frac{2D^{(i)}\exp[-\frac{D^{(i)}+c^{(i)}a^{(i)}}{2}t]}{\sigma^{(i)2}\alpha^{(i)}[1-\exp(-D^{(i)}t)]+(D^{(i)}-c^{(i)}a^{(i)})+[D^{(i)}+c^{(i)}a^{(i)}]\exp(-D^{(i)}t)}
\end{aligned}$$

Proof. Set $A^{(i)}(T) = \alpha^{(i)}$ for $i = 1,2$ using (4.5) where $t < T$, then we have

$$k^{(i)} = D^{(i)}T - \ln\left[\frac{c^{(i)}a^{(i)}+D^{(i)}-\alpha^{(i)}\sigma^{(i)^2}}{c^{(i)}a^{(i)}-D^{(i)}-\alpha^{(i)}\sigma^{(i)^2}}\right]. \tag{4.8}$$

Substitute (4.8) into (4.5) and the martingale in Theorem 4.2.1, the result follows immediately. ∎

Corollary 4.2.3. *The joint Laplace transform of the vector $(\Psi^{(1)}, \Psi^{(2)}, t)$ is given by*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\sum_{i=1}^{2}\gamma^{(i)}\Psi_t^{(i)}}\Big|X_0^{(1)}, X_0^{(2)}\right] \\
&= \exp\left[-\sum_{i=1}^{2}J^{(i)}(t)X_0^{(i)}\right]\times\prod_{i=1}^{2}\left[\Xi^{(i)}(t)\right]^{\frac{2c^{(i)}b^{(i)}}{\sigma^{(i)2}}} \\
&\quad\times\exp\left[-\rho\int_0^t\left[1-\hat{c}\left\{J^{(1)}(s),J^{(2)}(s)\right\}\right]ds\right]
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
&J^{(i)}(t) \\
&= \frac{2\gamma^{(i)}(1-\exp\{-D^{(i)}t\})}{(D^{(i)}-c^{(i)}a^{(i)})+[D^{(i)}+c^{(i)}a^{(i)}]\exp(-D^{(i)}t)}
\end{aligned}$$

and

$$\begin{aligned}&\Xi^{(i)}(t)\\ = \quad &\frac{2D^{(i)}\exp[-\frac{D^{(i)}+c^{(i)}a^{(i)}}{2}t]}{(D^{(i)}-c^{(i)}a^{(i)})+[D^{(i)}+c^{(i)}a^{(i)}]\exp(-D^{(i)}t)}\end{aligned}$$

Proof. Equation (4.9) follows immediately if we set $\alpha^{(i)}=0$ for $i=1,2$ in equation (4.7). ■

The above joint Laplace transform expression will be used for bond price computation in the next section. While FGM copula admits a simple analytical expression, the same can not be said for Gaussian and Student-t copulas. Hence, for these elliptical copulas, we evaluate the bond price numerically.

4.2.2 The Expression for Defaultable Bond Price

The expression for defaultable bond price can be derived easily using Corollary 4.2.3.

Corollary 4.2.4. *The defaultable bond price is given by*

$$\begin{aligned}&\mathbb{E}\left[e^{-\Sigma_{i=1}^{2}\Psi_t^{(i)}}\middle|X_0^{(1)},X_0^{(2)}\right]\\ = \quad &\exp\left[-\sum_{i=1}^{2}\Theta^{(i)}(t)X_0^{(i)}\right]\times\prod_{i=1}^{2}\left[\Upsilon^{(i)}(t)\right]^{\frac{2c^{(i)}b^{(i)}}{\sigma^{(i)2}}}\\ &\times\exp\left[-\rho\int_0^t\left[1-\hat{c}\left\{\Theta^{(1)}(s),\Theta^{(2)}(s)\right\}\right]ds\right] \qquad (4.10)\end{aligned}$$

where

$$\begin{aligned}&\Theta^{(i)}(t)\\ = \quad &\frac{2(1-\exp\{-\Delta^{(i)}t\})}{\Delta^{(i)}-c^{(i)}a^{(i)})+[\Delta^{(i)}+c^{(i)}a^{(i)}]\exp(-\Delta^{(i)}t)}\end{aligned}$$

and

$$\begin{aligned}&\Upsilon^{(i)}(t)\\ = \quad &\frac{2\Delta^{(i)}\exp[-\frac{\Delta^{(i)}+c^{(i)}a^{(i)}}{2}t]}{\Delta^{(i)}-c^{(i)}a^{(i)})+[\Delta^{(i)}+c^{(i)}a^{(i)}]\exp(-\Delta^{(i)}t)}\end{aligned}$$

with $\Delta^{(i)}=\sqrt{(c^{(i)}a^{(i)})^2+2(\sigma^{(i)})^2}$.

Proof. Equation (4.10) follows immediately if we set $\gamma^{(i)}=1$ for i=1, 2 in equation (4.9). ■

In an analogous manner, it is also possible to find the expression for default free bond price and the expression for bond price under the celebrated Cox-Ingersoll-Ross model.

Corollary 4.2.5. *The expression for the default -free bond price is given by*

$$
\begin{aligned}
& \mathbb{E}\left[e^{-\Psi_t^{(2)}} \middle| X_0^{(2)}\right] \\
= \ & \exp\left[-\Theta^{(2)}(t)X_0^{(2)}\right] \times \left[\Upsilon^{(2)}(t)\right]^{\frac{2c^{(2)}b^{(2)}}{\sigma^{(2)2}}} \\
& \times \exp\left[-\rho\int_0^t \left[1-\hat{h}\left\{\Theta^{(2)}(s)\right\}\right]ds\right]
\end{aligned}
\tag{4.11}
$$

where

$$
\hat{h}(\Theta^{(2)}) = \int_0^\infty e^{-\Theta^{(2)}y_2} dF_{Y^{(2)}},
$$

which can be easily obtained from Corollary 2.2 in [18].

Proof. Equation (4.11) follows immediately if we set $\gamma^{(1)} = 0$ and $\gamma^{(2)} = 1$ in equation (4.9). ■

The corresponding expression of (4.11) under the Student-t copula can also be obtained, by setting $\theta = 0$ in (4.6). Note that $\theta = 0$ does not imply the case of independence for Student-t copula, in line with [37].

If we set $\rho = 0$ in (4.11), we have the bond price expression under the celebrated Cox-Ingersoll-Ross (1985) model in [6]. Due to the dependence of simultaneous event jumps of $Y^{(i)}$'s with sharing event jump frequency rate ρ, we have that

$$
\mathbb{E}\left[e^{-\Sigma_{i=1}^2 \Psi_t^{(i)}}\right] \neq \mathbb{E}\left[e^{-\Psi_t^{(1)}}\right]\mathbb{E}\left[e^{-\Psi_t^{(2)}}\right].
$$

If the event jump $Y^{(i)}$ for $i = 1,2$ occurs by a Poisson process $M_t^{(i)}$ with its frequency rate $\rho^{(i)}$ respectively and everything else is independent of each other, the expression of the defaultable bond price, that is simply the product of the bond issuer's survival probability and the discount factor.

4.3 Bond Price and Term Structure Analyses

Now we examine the behaviour of the defaultable zero coupon bond prices under three different copulas mentioned in section 4.2. For simplicity, we assume that the jump sizes of both the bond issuer's default intensity ($i = 1$) and the market short rate ($i = 2$) are represented by exponential distributions. The hypothetical defaultable bond pays redemption value \$100 at maturity. Computation was done with Mathematica.

The defaultable bond price values are computed using (4.10), and the simple bond yield d_t is obtained using the following formula:

$$
d_t = \left(\frac{Future\ Value}{P_t}\right)^{\frac{1}{T-t}} - 1.
$$

We examine two scenarios whereby the exponential jump size parameters, $\mu^{(1)}$ and $\mu^{(2)}$ are assigned the values ($\mu_t^{(1)} = 100, \mu_t^{(2)} = 200$) and ($\mu_t^{(1)} = 5, \mu_t^{(2)} = 10$). The first set of parameters represents a safer environment due to low average jump sizes (i.e. $\frac{1}{100}$ and $\frac{1}{200}$), while the second set denotes a relatively riskier environment with high average jump sizes (i.e. $\frac{1}{5}$ and $\frac{1}{10}$). Assuming an average jump occurrences of 4 times per year (i.e. $\rho = 4$), the value of other parameters are summarized in Table 4.1.

TABLE 4.1: Parameter values of bond issuer's default intensity and short rate

	$c^{(i)}$	$a^{(i)}$	$b^{(i)}$	$\sigma^{(i)}$	$\rho^{(i)}$	$\lambda_0^{(i)}$
Issuer (1)	0.15	-1	0	0.12	4	0.05
Short rate (2)	0.2	-1	0	0.1	4	0.0023

With the chosen parameters, we now examine the behaviour of bond prices with one-year maturity across θ. Table 4.2 exhibits the bond price for each scenario under the three copulas considered:

TABLE 4.2: Zero coupon bond price under various copulas for $t = 1$

	$\mu_t^{(1)} = 100$ and $\mu_t^{(2)} = 200$			$\mu_t^{(1)} = 5$ and $\mu_t^{(2)} = 10$		
θ	**FGM**	**Gaussian**	**Student-t**	**FGM**	**Gaussian**	**Student-t**
-0.95	92.6268	92.5294	89.4241	57.7970	57.3593	51.0189
-0.9	92.6269	92.5295	89.5639	57.8096	57.3873	51.0301
-0.5	92.6274	92.6261	89.5779	57.9103	57.6685	51.2747
0	92.6281	92.6281	90.0017	58.0364	58.0364	51.8668
0.5	92.6288	92.6305	90.0297	58.1628	58.4698	52.7912
0.9	92.6293	92.6328	90.1210	58.2642	58.8652	53.8830
0.95	92.6293	92.6338	90.1511	58.2768	58.8701	54.0655
Range	0.2523	10.4383	72.7039	47.9825	151.0781	304.6528

We also compute the yield for bonds priced under all three copulas. Table 4.3 shows the yield for bonds maturing in one year under the copulas considered for both sets of jump sizes ($\mu_t^{(1)} = 100, \mu_t^{(2)} = 200$) and ($\mu_t^{(1)} = 5, \mu_t^{(2)} = 10$).

The term "Range" in Table 4.2 is defined as the difference between the bond prices given by $\theta^{0.95}$ and $\theta^{-0.95}$ in basis point (bps). We see that as the dependence structure θ progressed from negative to positive, the bond price figures in Table 4.2 demonstrate an increasing pattern while the bond yield figure in Table 4.3 show a decreasing pattern under all copulas considered.

In comparison with the other two copulas, the bond price values are the lowest under the Student-t copula, suggesting that a dependence structure under the Student-t copula could be a good candidate to depict a riskier environment. Analogously, the bond yield is highest under the Student-t copula and lowest under the FGM copula.

It is also worth noting that the computations under the Student-t copula does not give the same values of bond prices and yields as the Gaussian and FGM copula when $\theta = 0$. In contrast to the general theorem of copula, the Student-t copula does not give an independence case when the dependence parameter $\theta = 0$, and hence would not result in product copula, as noted in [37].

TABLE 4.3: Zero coupon bond yield under various copulas for $t = 1$

	$\mu_t^{(1)} = 100$ and $\mu_t^{(2)} = 200$			$\mu_t^{(1)} = 5$ and $\mu_t^{(2)} = 10$		
θ	**FGM**	**Gaussian**	**Student-t**	**FGM**	**Gaussian**	**Student-t**
-0.95	7.9601%	8.0738%	11.8267%	73.0193%	74.3395%	96.0056%
-0.9	7.9600%	8.0736%	11.6522%	72.9817%	74.2546%	95.9626%
-0.5	7.9594%	7.9609%	11.6346%	72.6808%	73.4049%	95.0280%
0	7.9586%	7.9586%	11.1090%	72.3056%	72.3056%	92.8017%
0.5	7.9578%	7.9558%	11.0744%	71.9311%	71.0286%	89.4254%
0.9	7.9572%	7.9531%	10.9619%	71.6321%	69.8796%	85.5872%
0.95	7.9571%	7.9520%	10.9248%	71.5947%	69.8655%	84.9609%

When comparing the bond yield across θ for both scenarios, it is noticed that the yield for the case of $\mu_t^{(1)} = 5$ and $\mu_t^{(2)} = 10$ are much higher than the yields given by the case of $\mu_t^{(1)} = 100$ and $\mu_t^{(2)} = 200$ for all copula. This is not surprising as lower exponentially distributed jump size parameters indicate a higher average jump size, thereby indicating a relatively unsafe market environment.

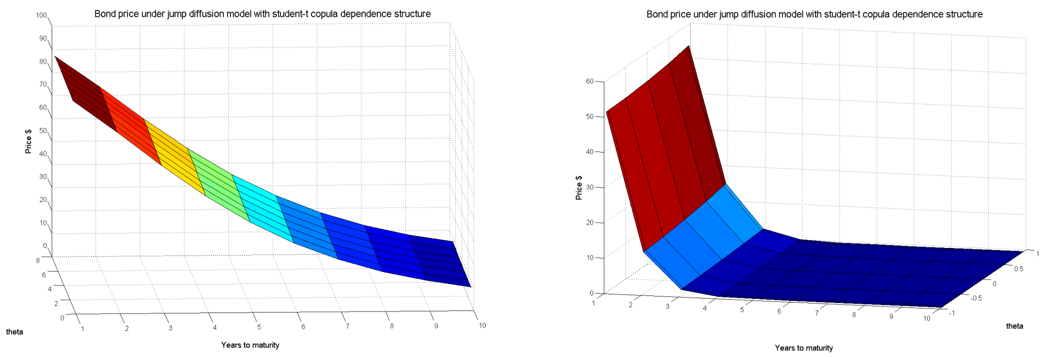

FIGURE 4.7: Bond price as a function of θ and maturity under the jump diffusion model with Student-t copula dependence structure and jump sizes $(\mu_t^{(1)} = 100, \mu_t^{(2)} = 200)$ (left) and $(\mu_t^{(1)} = 5, \mu_t^{(2)} = 10)$ (right)

Figure 4.7 and 4.8 show the bond price and bond yield under the jump diffusion process with dependence structure captured by the Student-t copula as a function of maturity $(T - t)$ (on the x-axis) and θ (on the y-axis). Under both scenarios of $(\mu_t^{(1)} = 100,\ \mu_t^{(2)} = 200)$ and $(\mu_t^{(1)} = 5,\ \mu_t^{(2)} = 10)$, the bond price decreased and yield increased as maturity increased.

Since the bond price and bond yield under the Gaussian and the FGM copula have a similar pattern, we show the diagrams in Appendix C.

4.4 Data & Model Calibration

In this section, we use the model in equation (4.1) with the three copulas considered in Section 4.2 (i.e. equations (4.2), (4.3) and (4.4)) and we calibrate the model to the market price of

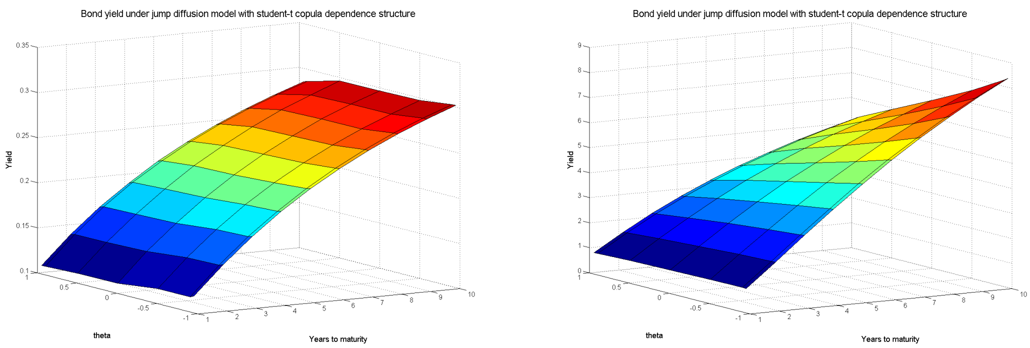

FIGURE 4.8: Bond yield as a function of θ and maturity under the jump diffusion model with Student-t copula dependence structure and jump sizes ($\mu_t^{(1)} = 100, \mu_t^{(2)} = 200$) (left) and ($\mu_t^{(1)} = 5, \mu_t^{(2)} = 10$) (right)

zero coupon bonds traded in various markets. These zero coupon bonds were issued by three corporations with various Moody's ratings and were obtained from the Bloomberg terminal on 30 October 2012. The information of each bond is given in Table 4.4. Following the one-day calibration, we move on to calibrate the model in equation (4.1) with the Student-t copula in equation (4.4) to the daily market price of the zero coupon bond issued by Microsoft Inc. for an extended time period of one year.

TABLE 4.4: Three zero coupon bonds issued by Microsoft Inc, National Australia Bank (NAB) and Eskom Ltd.

Issuer	Microsoft Inc.	NAB	Eskom Holdings
Country	USA	Australia	South Africa
Sector/Industry	Technology	Financial Services	Energy
Maturity (years)	3	5	20
Price	\$102.463	\$96.406	\$60.853
Yield (%)	-3.942	1.9057	8.1377
Rating (Moody's)	Aaa	Aa2	Baa3
$T-t$ (years)	0.6210	1.94935	6.165

In total, 15 parameters need to be calibrated under the FGM and Gaussian copulas, and an extra parameter under the Student-t copula which is the degree of freedom (DoF). Parallel with the assumption made in [25], it was also assumed that each jump size should not exceed 100%, which is a reasonable assumption for the market short rate and daily default intensity. Microsoft Inc. was chosen in this study to represent issuers with strong credibility and National Australia Bank represents issuers from the Australian financial industry. The selection of Eskom Holdings, a major electricity supplier in South Africa, aims to represent issuers from emerging markets.

We refer to [7] for issues related to calibration as well as numerical implementation of the calibration method in the framework of jump diffusion model. Using the in-built function NMinimize in Mathematica (refer to Numerical Nonlinear Global Optimization at https://reference.wolfram.com/language/ref/NMinimize.html and http://reference.wolfram.com/language/tutorial/

ConstrainedOptimizationGlobalNumerical.html for more information), we find, at each sample point, the set of calibrated parameters τ that give the global minimum point of the following objective function

$$\tau = \underset{\tau}{\text{argmin}} \frac{\left(P\left(T;\tau\right) - P_M\left(T\right)\right)^2}{P_M\left(T\right)^2}, \tag{4.12}$$

where $P_M(T)$ denotes the market price and $P(T;\widehat{\tau})$ is given by equation (4.10), subject to the constraints implied by the model as defined in Section 4.2 as well as the volatility parameters of an elliptical copula, ϕ_i being non-negative for $i = 1,2$. While calibrations are normally done with financial instruments having various maturities such as in [13] and [7], we only calibrate the objective function to one zero coupon bond at a time since only one zero coupon instrument is issued by Microsoft Inc. and NAB. Hence the summation sign is not required in our objective function.

4.4.1 One-day Calibration

We present the values of calibrated parameters for each zero coupon bond considered in Tables 4.5, 4.6 and 4.7. As in Section 4.3, the bond issuer's default intensity is denoted by 1 and market short rate by 2. We denote the decay rate by parameter $c^{(i)}a^{(i)}$ and the constant reversion level by $c^{(i)}b^{(i)}$.

The results from the one-day calibrations suggest that calibrating the jump diffusion model under the Student-t copula dependence structure gives a better fit for all three zero coupon bonds chosen (Microsoft Inc, NAB and Eskom Holdings). Calibrating under the Student-t copula consistently shows the least error relative to the other two copulas for all the bonds considered.

As this study examine the use of jump diffusion model with copula dependence structure in defaultable bond pricing, we will emphasize the analyses of our results from the perspective of dependence measure. We note that the zero coupon bond issued by Microsoft Inc. showed the value of θ of nearly 0. It is not surprising to find that the defaultability of a strong firm such as Microsoft Inc. to be less dependent on the market short rate. It is also interesting to note the positive and negative θ values between the market short rate and NAB's default intensity. Being a financial institution, it is possible for a bank's default rate to have positive and negative relationship with the driver of its source of income, i.e. the market short rate. The relationship between interest rate and defaultability of a bank is rather inexplicit, as banks could adopt different strategy to survive given an interest rate environment. Finally, the utility company Eskom Holdings, shows a positive dependency between the market short rate and its default intensity ranging from approximately 0.2 to 0.6. This is expected because an increase in interest rate would adversely affect consumers' spending ability, and hence their ability to pay utility bills. Nevertheless, a better perspective on the dependency between a firms default rate and the market short rate would be obtainable if all the bonds issued by the firm itself were calibrated, for an extended period of time. Hence, in the next section, we calibrate the Microsoft Inc. zero coupon bond for an extended period of time.

TABLE 4.5: Calibrated parameters for zero coupon bond issued by Microsoft Inc.

Issuer	Microsoft Inc.($102.463, -3.9420%, 0.621 years, Aaa)		
Parameters	Gaussian	FGM	t-copula
$c^{(1)}a^{(1)}$	-2.0431	-2.066994	-2.30065
$c^{(1)}b^{(1)}$	11.38226	2.315928	4.359838
$X^{(1)}$	0.000494	$3.7*10^{-05}$	0.176910
$c^{(2)}a^{(2)}$	-0.859803	-0.007487	-0.442561
$c^{(2)}b^{(2)}$	2.049926	0.008974	0.760246
$X^{(2)}$	0.043001	0.025072	0.047887
$\mu^{(1)}$	5.776155	2.109325	2.661011
$\mu^{(2)}$	1.371715	1.460639	2.609246
θ	-0.194220	0.046990	0.069294
ρ	2.002222	2.316447	2.045547
$\sigma^{(1)}$	0.499999	0.286300	0.478281
$\sigma^{(2)}$	0.355271	0.273952	0.360174
ϕ_1	1.453423	NA	0.149088
ϕ_2	1.872341	NA	0.889886
DOF	NA	NA	3.944993
Error	1.941907	24.262991	0.352635
Implied Price	$104.405	$73.484	$102.816
Implied Yield	-6.7062%	64.2368%	-4.3734%

4.4.2 One-year Calibration: Microsoft Inc. Zero Coupon Bond

In the previous one-day calibrations, Student-t copula consistently returned the least error for all three bonds. Therefore, we now perform daily calibration on the zero coupon bond issued by Microsoft Inc. for an extended period of one year, assuming that the dependence structure is captured by the Student-t copula. The jump diffusion model is calibrated to 268 data points of Microsoft Inc. bond price dated from 22 June 2010 to 30 June 2011.

The calibrated price is illustrated in Figure 4.9, while Table 4.8 exhibits summary statistics of the average, standard deviation, minimum and maximum value of each calibrated parameter. We also compute the relative error of the calibrated data points, given by $\frac{(P(T;\hat{\tau})-P_M(T))}{P_M(T)}$, as displayed in Figure 4.10.

The one-year calibration of the Microsoft Inc. bond shows that, the average one-year absolute error is higher than the one-day calibration counterpart, i.e. 1.2659 as opposed to 0.352635. This is due to some calibrations showing very high error, where the model price was much higher than the market price, as presented by a few high spikes in Figure 4.9. However, a closer look at the relative error of 268 data points show that calibration of 264 data points return a relative error of less than 5%, 261 calibrations have a relative error of less than 4%, 255 calibrations have a relative error of less than 3%, 244 calibrations have a relative error of less than 2% and 213 calibrations have a relative error of less than 1%. We can therefore speak of good fits of the model.

We also note that, the average θ value is almost 0, indicating very minimal dependence

TABLE 4.6: Calibrated parameters for a zero coupon bond issued by NAB

Issuer	NAB ($96.406, 1.9057%, 1.94935 years, Aa2)		
Parameters	Gaussian	FGM	t-copula
$c^{(1)}a^{(1)}$	-5.520332	-6.351866	-0.622036
$c^{(1)}b^{(1)}$	6.503219	46.01755	14.58675
$X^{(1)}$	0.336710	0.282795	0.994426
$c^{(2)}a^{(2)}$	-1.769685	-13.34258	-0.372379
$c^{(2)}b^{(2)}$	4.566944	3.987893	0.584545
$X^{(2)}$	0.034749	0.025226	$1.55 * 10^{-06}$
$\mu^{(1)}$	1.000001	2.934222	2.743003
$\mu^{(2)}$	2.369157	2.497194	2.203662
θ	0.432409	-0.999999	-0.860249
ρ	2.000008	2.000000	2.243709
$\sigma^{(1)}$	0.337009	0.464926	0.499912
$\sigma^{(2)}$	0.108473	0.134244	0.364607
ϕ_1	$2.95 * 10^{-05}$	NA	4.470327
ϕ_2	0.706481	NA	1.996665
DOF	NA	NA	2.000024
Error	0.384317	0.313358	0.262975
Implied Price	$96.0217	$96.0926	$96.669
Implied Yield	2.1044%	2.0657%	1.7531%

between the jump sizes of Microsoft Inc.'s default intensity, $X^{(1)}$, and the market short rate, $X^{(2)}$. This is possibly due to the fact that being a strong firm, Microsoft Inc.'s defaultability is less dependent on the market short rate. We show the daily changes of each parameter in the Appendix C.

4.5 Conclusion

This paper examined a bivariate jump diffusion model whose jump sizes are dependent. The variables were the default intensity of a bond issuer $X_t^{(1)}$ and the short rate of interest $X_t^{(2)}$, whose jump sizes were exponentially distributed and that their dependence structure was captured by copulas. The copulas considered in this studies were the FGM copula, Gaussian copula and Student-t copula.

Using the martingale method and the PDMP technique, we derived the joint Laplace transform of the distribution of the vector $\left(\Psi_t^{(1)}, \Psi_t^{(2)}, X_t^{(1)}, X_t^{(2)}, t\right)$. The expression was then used to arrive at the defaultable and default-free bond price formulae under the jump diffusion model.

We then examined the bond terms structure assuming dependence structure of the jump sizes were captured by the three copulas. The results indicated that among the 3 copulas, modelling the bond price under the Student-t copula showed the widest range between both ends of the dependence parameters, i.e. $\theta = -0.95$ and $\theta = 0.95$, suggesting that it could be used to

TABLE 4.7: Calibrated parameters for zero coupon bond issued by Eskom Ltd.

Issuer	Eskom ($60.853, 8.138%, 6.165years, Baa3)		
Parameters	Gaussian	FGM	t-copula
$c^{(1)}a^{(1)}$	-0.056914	-1.745277	-3.318189
$c^{(1)}b^{(1)}$	0.35557	6.015507	11.09728
$X^{(1)}$	0.717708	0.000875	0.345322
$c^{(2)}a^{(2)}$	-2.011638	-0.05307	-4.730152
$c^{(2)}b^{(2)}$	0.600572	0.402295	2.22758
$X^{(2)}$	0.037134	0.026512	0.031688
$\mu^{(1)}$	2.957392	1.0000001	2.18082
$\mu^{(2)}$	2.160714	5.264062	2.86555
θ	0.613244	0.436324	0.203793
ρ	2.003514	2.460673	2.00001
$\sigma^{(1)}$	0.055179	0.223754	0.206944
$\sigma^{(2)}$	0.054716	0.5	0.219721
ϕ_1	1.053439	NA	$4.25 * 10^{-05}$
ϕ_2	0.010771	NA	1.95177
DOF	NA	NA	3.50542
Error	1.420084	0.037692	0.0250109
Implied Price	$59.4329	$60.8153	$60.8780
Implied Yield	8.8063%	8.4013%	8.3832%

represent riskier environment.

This was then followed by calibrations of the model to the market price of the zero coupon bond issued by three corporations: Microsoft Inc., NAB as well as Eskom Holdings. The one-day calibrations to a zero coupon bond showed that the Student-t copula provided a good fit with the lowest error for all the three bonds considered, as opposed to the other two copulas. Thence, we calibrated the model to the Microsoft Inc. zero coupon bond for an extended period of one year and found that the model showed a good fit for the date chosen, with 97% of the calibrations returning an error of less than 5%.

It would be of interest to calibrate the model to the bonds with various maturities issued by a corporation to examine the dependency between its defaultability and short rate of interest, which we leave for further research. We can also consider calibration of the model to sovereign bonds as the next objective of further research to have a better insight on the defaultability of a government.

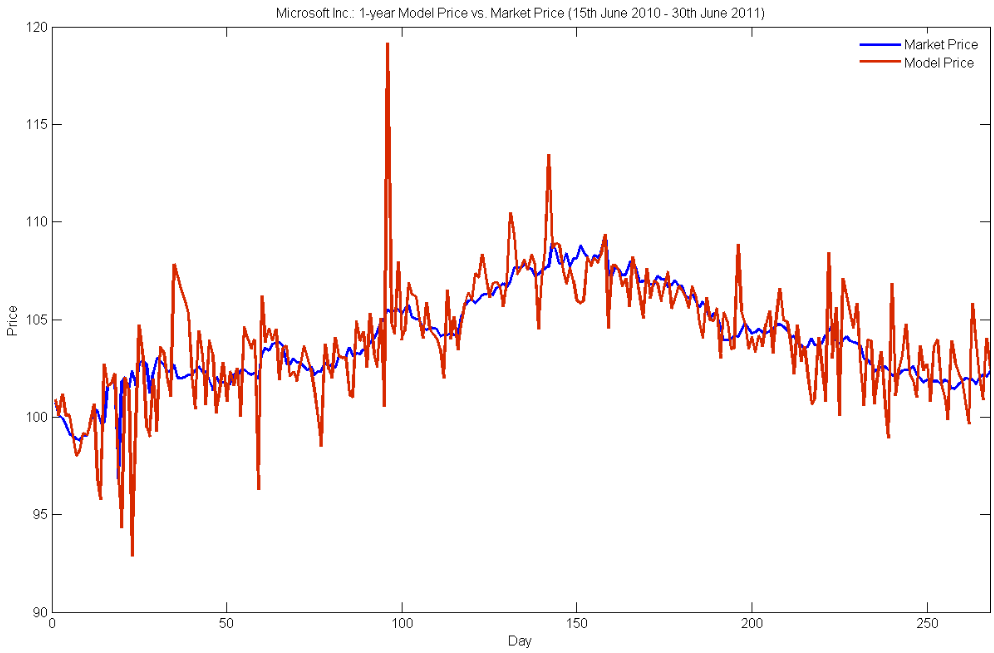

FIGURE 4.9: Model Price (red) vs. Market Price (blue)

TABLE 4.8: Summary statistics of calibrated parameters for calibration period 22 June 2010 to 30 June 2011.

Issuer	Microsoft Inc.(Aaa)			
Parameters	Mean	Std Deviation	Min	Max
$c^{(1)}a^{(1)}$	-2.091164	2.472539	-20.249450	0.101236
$c^{(1)}b^{(1)}$	14.428056	122.651762	0.011320	17.155334
$X^{(1)}$	0.215246	0.251552	0	1
$c^{(2)}a^{(2)}$	-7.013011	9.655850	-95.881207	-0.019032
$c^{(2)}b^{(2)}$	4.100117	42.740459	0.027584	4.333827
$X^{(2)}$	0.022493	0.017136	0	0.05
$\mu^{(1)}$	1.504224	1.152661	$1.32*10^{-09}$	5.63567
$\mu^{(2)}$	1.230995	1.108728	$6.76*10^{-09}$	4.52966
θ	0.003020	0.693922	-1	1
ρ	2.121745	0.225438	2	3.50457
$\sigma^{(1)}$	0.336919	0.114377	0.0964791	0.5
$\sigma^{(2)}$	0.338958	0.134567	0.047618	0.5
ϕ_1	1.813828	1.598807	$1.14*10^{-08}$	7.58983
ϕ_2	1.337496	1.442467	$5.43*10^{-09}$	10.0689
DOF	2.798754	0.863816	2	6.3169
Error	1.280941	1.538154	$2.515*10^{-12}$	13.6572
Rel Error	1.2357%	1.483%	0.000%	12.946%

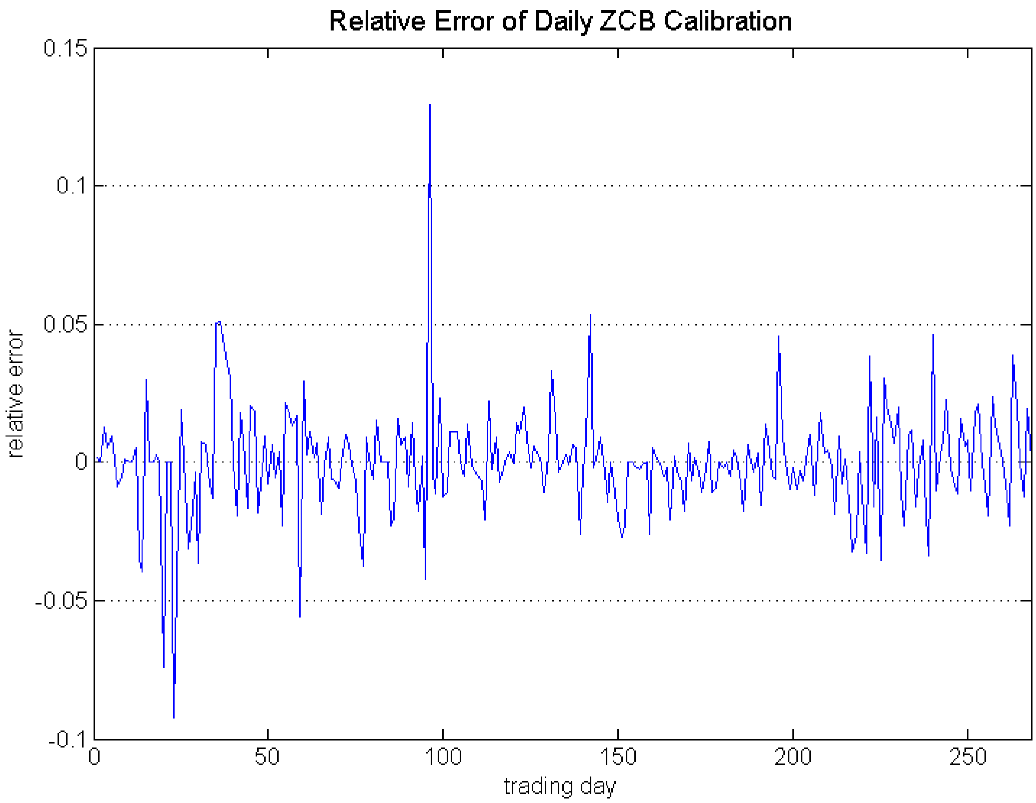

FIGURE 4.10: Jump Diffusion Model with Student-t copula dependence structure: Relative Error

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5
Conclusion

This thesis addressed the topic of copula modelling in insurance and credit risk. To do so, it devoted Chapter 2 to discuss the use of copulas in the actuarial field of study, while Chapters 3 and 4 examined the use of copula in mathematical finance. This concluding section aims to summarise and highlight the contributions of each research paper.

Four copulas were considered in the framework provided: the FGM, Gaussian, Student-t and Gumbel copulas. While there has been extensive research in the insurance and finance area using the FGM, Gumbel and Gaussian copulas, to the best of the researchers' knowledge, the study of the Student-t copula in CDS pricing and bond pricing under the jump diffusion approach as well as the study of Gaussian copula in the classical actuarial risk theory have not previously been undertaken extensively.

The article in chapter 2 titled *Neumann Series on the Recursive Moments of Copula-Dependent Aggregate Discounted Claims* employed copulas to capture the dependence structure between the inter-claim arrival time and claim sizes in classical actuarial risk theory. To do so, the expression for the m^{th} order moment proposed in [1] and [4] was represented in the form of the Volterra integral equation (VIE) of the second kind, which is widely used in renewal theory, demographics, electromagnetism and viscoelasticity. The main contribution of this article, which was the Neumann series expression for this recursive equation was derived using the Picard method of successive approximations. Based on the expression, the first two moments of the aggregate discounted claims was computed. For the dependence structure between the inter-claim arrival time and claim sizes, an FGM, Gaussian and Gumbel copula were employed together with exponential marginal distributions. The values of moments of the aggregate discounted claims were shown, as well as the loaded premium for each copula used in this study. It would have been interesting to derive the expression for the m^{th}-moment using other joint pdfs between the claim sizes and the inter-claim arrival time, such as the Weibull distribution, as in [5]. Other copulas with different claim size distributions for X may be considered in the proposed approach, which could be explored in future research. The Monte Carlo simulation and other numerical methods can also be considered to solve the VIE (such as Runge–Kutta and the collocation methods), as the next objective of further research to deal with the computation of higher moments. This article has been published in the special issue of the journal *Risks: Application of Stochastic Processes in Insurance*.

In the second article titled *A multivariate jump diffusion process for counterparty risk in CDS rates*, the multivariate jump diffusion process examined has been used in modeling counterparty risk in CDS rates. Under this process, the jumps (i.e. magnitude of contribution of primary events to default intensities) were assumed to occur simultaneously and their sizes are dependent. A homogeneous Poisson process was used as a counting process to account for simultaneous event jumps in default intensities. An FGM copula, Gaussian copula and Student-t copula, together with exponential margins were used to model the dependence structure between event jump sizes. The simulated paths of the jump diffusion intensity processes under the three copulas, were also illustrated with various dependence parameter values, θ. By applying copula-dependent default intensity to the multivariate Cox process, joint survival/default probability and other relevant joint probabilities were derived via the joint Laplace transforms, for which the PDMP theory and standard martingale methodology were used. The calculation of joint survival/default probability were shown, together with an application to CDS rate considering counterparty risk, whereby each counterparty's default intensity was assumed to follow the jump diffusion process. This was then followed by the sensitivity analyses of the CDS rate with respect to the parameters used in the jump diffusion model. The multivariate framework of the jump diffusion model with copula dependence structure has the potential to be applicable to a variety of problems, where multiple transition rates are involved in the realms of economics, finance and insurance that could be the object of further research.

The final paper, titled *Jump Diffusion Model with Copula Dependence Structure in Defaultable Bond Pricing*, examined the bivariate jump diffusion model whose jump sizes were dependent. Instead of financial counterparties' default intensity as in chapter 3, the variables considered were the default intensity of a bond issuer $X_t^{(1)}$ and the short rate $X_t^{(2)}$. Similar to the second article, the jump sizes were assumed to be exponentially distributed and that their dependence structure was captured by the FGM, Gaussian and Student-t copula. Using the martingale method and the PDMP technique, the joint Laplace Transform of the bivariate distribution $\left(\Psi_t^{(1)}, \Psi_t^{(2)}, X_t^{(1)}, X_t^{(2)}, t\right)$ was derived and used to obtain the defaultable and default-free bond price formulae under the jump diffusion model. The terms structure of the defaultable bond was then examined and the results indicated that among the 3 copulas, modelling the bond price under the FGM copula showed the lowest range between both ends of the dependence parameters, i.e. $\theta = \pm 0.95$. In line with [6], we also found that the bond price value under Student-t copula when $\theta = 0$ does not equal to its FGM and Gaussian counterparts, which corresponded to product copula. This was then followed by calibrations of zero coupon bond issued by three corporations, which were Microsoft Inc., NAB and Eskom Holdings. Our one-day calibration to each of the zero coupon bond data showed that the Student-t copula provided a good fit with the lowest error for all the three bonds considered, as opposed to the other two copulas. Thence, we calibrate the model to Microsoft Inc. zero coupon bond for an extended period of one year and found that the model showed a good fit for the period chosen, with 97% of the calibrations returning error of less than 5%. In short, this chapter contributed to the bivariate copula dependence structure from the perspective of bond price calibration.

Due to time constraints, this research did not extensively examine the computational aspects of the formula. By simply performing raw computation of the sophisticated explicit form of solution presented in all three studies, the accuracy of the numerical results may have been jeopardised. This is especially true as θ approached the tail side of the elliptical and Gumbel copulas. This can result from issues such as the slow convergence of the numerical integration, highly oscillatory integrand or working precision being too small. Hence, in the computation using Mathematica, the Global Adaptive numerical integration strategy was employed, in order to balance between the accuracy and the amount of time available. It is also noted that the results from Mathematica could have been different if this study had been computed using different software, such as MATLAB or R. This issue presents another of the many directions that may be taken in further research.

A

Derivation of the joint Laplace transform of integrated multivariate processes

1. We assume that each obligor/macroeconomic variable has default intensity process with the following dynamic:

$$dX_t^{(i)} = c^{(i)}\left(b^{(i)} + a^{(i)}X_t^{(i)}\right)dt + \sigma^{(i)}\sqrt{X_t^{(i)}}dW_t^{(i)} + dC_t^{(i)}$$

where $i = 1, 2, \ldots, n, r$ is the unique obligor of a particular financial contract and r is the short rate process governing the financial environment. $C_t^{(i)}$ is a compound Poisson process given by

$$C_t^{(i)} = \sum_{j=1}^{M_t^{(i)}} Y_j^{(i)},$$

where $M_t^{(i)} \sim Po\left(\rho^{(i)}t\right)$ is a Poisson process for each obligor and $Y^{(i)}$ is a random variable representing the jump size for each obligor i. In this study, we assume $M_t^{(i)} = M_t \sim Po(\rho t)$ for all i.

2. We also set $\Psi_t^{(i)} = \int\limits_0^t X_s^{(i)} ds$ and we try to find the generator of the process $\left(\Psi_t^{(1)}, \ldots, \Psi_t^{(n)}, X_t^{(1)}, \ldots, X_t^{(n)}, t\right)$ acting on a function

$$f\left(\Psi_t^{(1)}, \ldots, \Psi_t^{(n)}, X_t^{(1)}, \ldots, X_t^{(n)}, t\right) = \exp\left\{B(t) - \sum_{i=1}^{n}\gamma^{(i)}\Psi^{(i)} - \sum_{i=1}^{n}A_t^{(i)}x^{(i)}\right\}$$

The generator of the process is given by

$$
\begin{aligned}
& \mathscr{A} f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},X_t^{(1)},\ldots,X_t^{(n)},t\right) \\
= \; & \frac{\partial f}{\partial t}+\sum_{i=1}^{n}\frac{\partial f}{\partial \Psi^{(i)}}x^{(i)}+\sum_{i=1}^{n}c^{(i)}\left(b^{(i)}+a^{(i)}X_t^{(i)}\right)\frac{\partial f}{\partial x^{(i)}}+\frac{1}{2}\sum_{i=1}^{n}\left(\sigma^{(i)}\sqrt{x^{(i)}}\right)^2\frac{\partial^2 f}{\partial x^{(i)^2}} \\
& +\rho\left[\begin{array}{c} \int_0^\infty \overset{n}{\cdots}\int_0^\infty f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},x_t^{(1)}+y_t^{(1)},\ldots,x_t^{(n)}+y_t^{(1)},t\right) \\ \times dC\left(F_1\left(y^{(1)}\right),\cdots,F_n\left(y^{(n)}\right)\right)-f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},X_t^{(1)},\ldots,X_t^{(n)},t\right)\end{array}\right]
\end{aligned}
$$

3. We first find the partial derivatives of the function f, that is:

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \left[B'(t)-\sum_{i=1}^{n}A_t^{(i)'}x^{(i)}\right]f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},X_t^{(1)},\ldots,X_t^{(n)},t\right) \\
\frac{\partial f}{\partial x^{(i)}} &= -A_t^{(i)}f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},X_t^{(1)},\ldots,X_t^{(n)},t\right) \\
\frac{\partial f}{\partial \Psi^{(i)}} &= -\gamma^{(i)}f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},X_t^{(1)},\ldots,X_t^{(n)},t\right) \\
\frac{\partial^2 f}{\partial x^{(i)^2}} &= \left(A_t^{(i)}\right)^2 f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},X_t^{(1)},\ldots,X_t^{(n)},t\right)
\end{aligned}
$$

where

$$
f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},x_t^{(1)}+y_t^{(1)},\ldots,x_t^{(n)}+y_t^{(1)},t\right)=exp\left\{B(t)-\sum_{i=1}^{n}\gamma^{(i)}\Psi^{(i)}-\sum_{i=1}^{n}A_t^{(i)}\left(x^{(i)}+y^{(i)}\right)\right\}
$$

4. We substitute the partial derivative of the function f into the above equation and find

$$
\begin{aligned}
& \mathscr{A} f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},X_t^{(1)},\ldots,X_t^{(n)},t\right) \\
= \; & f\left(\Psi_t^{(1)},\ldots,\Psi_t^{(n)},X_t^{(1)},\ldots,X_t^{(n)},t\right)\times \\
& \left(B'(t)-\sum_{i=1}^{n}A_t^{(i)'}x^{(i)}-\sum_{i=1}^{n}\gamma^{(i)}x^{(i)}-\sum_{i=1}^{n}A_t^{(i)}c^{(i)}\left(b^{(i)}+a^{(i)}X_t^{(i)}\right)+\frac{1}{2}\sum_{i=1}^{n}\left(A_t^{(i)}\sigma^{(i)}\sqrt{x^{(i)}}\right)^2\right) \\
& +\rho\left[\begin{array}{c} \int_0^\infty \overset{n}{\cdots}\int_0^\infty \exp\left\{B(s)-\sum\limits_{i=1}^{n}\gamma^{(i)}\Psi^{(i)}-\sum\limits_{i=1}^{n}A_s^{(i)}\left(x^{(i)}+y^{(i)}\right)\right\} \\ \times dC\left(F_1\left(y^{(1)}\right),\cdots,F_n\left(y^{(n)}\right)\right)-\exp\left\{B(t)-\sum\limits_{i=1}^{n}\gamma^{(i)}\Psi^{(i)}-\sum\limits_{i=1}^{n}A_t^{(i)}x^{(i)}\right\}\end{array}\right]
\end{aligned}
$$

In order for the equation f to be a martingale, we equate the generator to 0 and obtain

$$
\begin{array}{c}
B'(t)-\sum\limits_{i=1}^{n}A_t^{(i)'}x^{(i)}-\sum\limits_{i=1}^{n}\gamma^{(i)}x^{(i)}-\sum\limits_{i=1}^{n}A_t^{(i)}c^{(i)}\left(b^{(i)}+a^{(i)}x_t^{(i)}\right)+\frac{1}{2}\sum\limits_{i=1}^{n}\left(A_t^{(i)\sigma^{(i)}}\sqrt{x^{(i)}}\right)^2 \\
+\rho\left[\begin{array}{c}\int_0^\infty \overset{n}{\cdots}\int_0^\infty \exp\left\{-\sum\limits_{i=1}^{n}A_s^{(i)}y^{(i)}\right\}dC\left(F_1\left(y^{(1)}\right),\cdots,F_n\left(y^{(n)}\right)\right) \\ -1\end{array}\right]
\end{array}
=0.
$$

5. We now gather the constant term and the $x^{(i)}$'s to find the explicit form of each variable in the generator equation, starting with

$$-A_t^{(i)'} - \gamma^{(i)} - A_t^{(i)} c^{(i)} a^{(i)} + \frac{1}{2}\left(A_t^{(i)} \sigma^{(i)}\right)^2 = 0$$

$$A_t^{(i)'} = \frac{1}{2}\left(A_t^{(i)} \sigma^{(i)}\right)^2 - \gamma^{(i)} - A_t^{(i)} c^{(i)} a^{(i)}$$

$$2A_t^{(i)'} = \left(A_t^{(i)} \sigma^{(i)}\right)^2 - 2\gamma^{(i)} - 2A_t^{(i)} c^{(i)} a^{(i)}$$

$$\frac{2A_t^{(i)'}}{\left(A_t^{(i)} \sigma^{(i)}\right)^2 - 2\gamma^{(i)} - 2A_t^{(i)} c^{(i)} a^{(i)}} = 1$$

$$\frac{A_t^{(i)'}}{\left(\sigma^{(i)} A_t^{(i)}\right)^2 - 2c^{(i)} a^{(i)} A_t^{(i)} - 2\gamma^{(i)}} = \frac{1}{2}$$

6. By adding and subtracting the same constant $\left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right)^2$, we obtain

$$\frac{A_t^{(i)'}}{\left(\sigma^{(i)} A_t^{(i)}\right)^2 - 2c^{(i)} a^{(i)} A_t^{(i)} + \left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right)^2 - \left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right)^2 - 2\gamma^{(i)}} = \frac{1}{2}$$

and we can modify the above equation by completing the square to obtain the following

$$\frac{A_t^{(i)'}}{\left[\left(\sigma^{(i)} A_t^{(i)}\right) - \left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right)\right]^2 - \left[\left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}\right]} = \frac{1}{2}$$

$$\begin{aligned} \Rightarrow \quad & \frac{A_t^{(i)'}}{\left[\left(\sigma^{(i)} A_t^{(i)}\right) - \left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right) - \sqrt{\left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right]} \\ & \times \frac{1}{\left[\left(\sigma^{(i)} A_t^{(i)}\right) - \left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right) + \sqrt{\left(\frac{c^{(i)} a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right]} \\ = \quad & \frac{1}{2}. \end{aligned}$$

7. By partial fraction, we have

$$\frac{y}{(a-b)(a+b)} = \frac{\frac{y}{2b}}{(a-b)} - \frac{\frac{y}{2b}}{(a+b)}$$

Hence, the equation can be written as

$$
\begin{aligned}
\frac{1}{2} \quad = \quad & \frac{A_t^{(i)'}}{2\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}} \times \\
& \left[\begin{array}{c}
\frac{1}{\left(\sigma^{(i)}A_t^{(i)}\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt[2]{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}} \\
-\frac{1}{\left(\sigma^{(i)}A_t^{(i)}\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)+\sqrt[2]{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}}
\end{array}\right]
\end{aligned}
$$

$$
\left[\begin{array}{c}
\frac{A_t^{(i)'}}{\left(\sigma^{(i)}A_t^{(i)}\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}} \\
-\frac{A_t^{(i)'}}{\left(\sigma^{(i)}A_t^{(i)}\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)+\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}}
\end{array}\right] = \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}.
$$

8. We perform integration on both sides with respect to s

$$
\begin{aligned}
& \int_0^t \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}ds \\
= \; & \int_0^t \frac{A^{(i)'}(s)\,ds}{\left(\sigma^{(i)}A^{(i)}(s)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}} \\
& -\int_0^t \frac{A^{(i)'}(s)\,ds}{\left(\sigma^{(i)}A^{(i)}(s)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)+\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}}
\end{aligned}
$$

and obtain

$$
\begin{aligned}
& \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}t-K^{(i)} \\
= \quad & \frac{1}{\sigma^{(i)}}\left\{\begin{array}{c}
\ln\left[\left(\sigma^{(i)}A^{(i)}(t)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}\right] \\
-\ln\left[\left(\sigma^{(i)}A^{(i)}(t)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)+\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}\right]
\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}t-K^{(i)} \\
= \;& \ln\left[\frac{\left(\sigma^{(i)}A^{(i)}(t)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}}{\left(\sigma^{(i)}A^{(i)}(t)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)+\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}}\right]
\end{aligned}
$$

giving us

$$
\exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}t-K^{(i)}\right\}=\frac{\left(\sigma^{(i)}A^{(i)}(t)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}}{\left(\sigma^{(i)}A^{(i)}(t)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)+\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}}
$$

$$
\begin{aligned}
& \left(\sigma^{(i)}A^{(i)}(t)\right)-\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}} \\
= \;& \left(\sigma^{(i)}A^{(i)}(t)\right)\exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}t-K^{(i)}\right\} \\
& -\left(\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}\right)\exp\left\{\sigma^{(i)}\sqrt[2]{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}t-K^{(i)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \left(\sigma^{(i)}A^{(i)}(t)\right)\left(1-\exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}t-K^{(i)}\right\}\right) \\
= \;& \left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)+\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}\right] \\
& -\left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)-\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}\right]\exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2+2\gamma^{(i)}}t-K^{(i)}\right\}.
\end{aligned}
$$

9. Rearranging the above equation, we obtain, for each obligor i, the expression

$$A^{(i)}(t) = \frac{\left[\sqrt{\left(c^{(i)}a^{(i)}\right)^2 + 2\left(\sigma^{(i)}\right)^2\gamma^{(i)}} + c^{(i)}a^{(i)}\right]}{\left(\sigma^{(i)}\right)^2\left(1 - \exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}t - K^{(i)}\right\}\right)} + \frac{\left[\sqrt{\left(c^{(i)}a^{(i)}\right)^2 + 2\left(\sigma^{(i)}\right)^2\gamma^{(i)}} - c^{(i)}a^{(i)}\right]\exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}t - K^{(i)}\right\}}{\left(\sigma^{(i)}\right)^2\left(1 - \exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}t - K^{(i)}\right\}\right)}.$$

10. Let $A^{(i)}(T) = \alpha^{(i)}$. The above expression, evaluated at time T, therefore becomes

$$\frac{\left[\begin{array}{c}\left[\sqrt{\left(c^{(i)}a^{(i)}\right)^2 + 2\left(\sigma^{(i)}\right)^2\gamma^{(i)}} + c^{(i)}a^{(i)}\right] \\ + \left[\sqrt{\left(c^{(i)}a^{(i)}\right)^2 + 2\left(\sigma^{(i)}\right)^2\gamma^{(i)}} - c^{(i)}a^{(i)}\right]\exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}T - K^{(i)}\right\}\end{array}\right]}{\left(\sigma^{(i)}\right)^2\left(1 - \exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}T - K^{(i)}\right\}\right)} = \alpha^{(i)}$$

$$\begin{aligned}
&\Rightarrow \left[\begin{array}{c}\left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right) + \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right] - \left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right) - \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right] \\ \exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}T - K^{(i)}\right\}\end{array}\right] \\
&= \alpha^{(i)}\sigma^{(i)}\left(1 - \exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}T - K^{(i)}\right\}\right)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left\{\alpha^{(i)}\sigma^{(i)} - \left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right) - \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right]\right\} \\
&\qquad \times \exp\left\{\sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}T - K^{(i)}\right\} \\
&= \alpha^{(i)}\sigma^{(i)} - \left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right) + \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right]
\end{aligned}$$

We therefore get

$$\exp\left\{K^{(i)}\right\} = \frac{\exp\left\{\sigma^{(i)}\left[\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}T\right]\right\}\left\{\alpha^{(i)}\sigma^{(i)} - \left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right) - \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right]\right\}}{\alpha^{(i)}\sigma^{(i)} - \left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right) + \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right]}$$

and obtain

$$\begin{aligned} K^{(i)} &= \sigma^{(i)}\sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}T + \ln\frac{\alpha^{(i)}\sigma^{(i)} - \left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right) - \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right]}{\alpha^{(i)}\sigma^{(i)} - \left[\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right) + \sqrt[2]{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}}\right]} \\ &= D^{(i)}T + \ln\frac{\left[\alpha^{(i)}\sigma^{(i)^2} - \left[c^{(i)}a^{(i)} - D^{(i)}\right]\right]}{\left[\alpha^{(i)}\sigma^{(i)^2} - \left[c^{(i)}a^{(i)} + D^{(i)}\right]\right]} \end{aligned}$$

where

$$D^{(i)} = \sqrt{\left(c^{(i)}a^{(i)}\right)^2 + 2\left(\sigma^{(i)}\right)^2\gamma^{(i)}}. \tag{A.1}$$

11. With (A.1), eventually, we arrive at the full expression of $A^{(i)}(t)$, as a function of the time to maturity $T-t$, given by

$$\left[\left[D^{(i)} + c^{(i)}a^{(i)}\right] + \left[D^{(i)} - \left(c^{(i)}a^{(i)}\right)\right]e^{-D^{(i)}(T-t)}\frac{\alpha^{(i)}\sigma^{(i)} - \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}} - \left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)}{\alpha^{(i)}\sigma^{(i)} + \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}} - \left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)}\right]$$

$$\div\left(\sigma^{(i)}\right)^2\left(1 - \frac{\alpha^{(i)}\sigma^{(i)} - \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}} - \left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)}{\alpha^{(i)}\sigma^{(i)} + \sqrt{\left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)^2 + 2\gamma^{(i)}} - \left(\frac{c^{(i)}a^{(i)}}{\sigma^{(i)}}\right)}e^{-D^{(i)}(T-t)}\right)$$

$$\Rightarrow A^{(i)}(t) = \frac{\left[\left[D^{(i)} + c^{(i)}a^{(i)}\right] + \left[D^{(i)} - \left(c^{(i)}a^{(i)}\right)\right]e^{-D^{(i)}(T-t)}\frac{\alpha^{(i)}\sigma^{(i)^2} - D^{(i)} - c^{(i)}a^{(i)}}{\alpha^{(i)}\sigma^{(i)^2} + D^{(i)} - c^{(i)}a^{(i)}}\right]}{\left(\sigma^{(i)}\right)^2\left(1 - \frac{\alpha^{(i)}\sigma^{(i)^2} - D^{(i)} - c^{(i)}a^{(i)}}{\alpha^{(i)}\sigma^{(i)^2} + D^{(i)} - c^{(i)}a^{(i)}}e^{-D^{(i)}(T-t)}\right)}$$

12. We now try to find the expression for $B(t)$ by gathering the constant terms and equate the expression to 0.

$$0 = B'(t) + \rho\left[\int_0^\infty \overset{n}{\cdots}\int_0^\infty \exp\left\{-\sum_{i=1}^n A^{(i)}(s)y^{(i)}\right\}dC\left(F_1\left(y^{(1)}\right),\cdots,F_n\left(y^{(n)}\right)\right) - 1\right] - \sum_{i=1}^n A_t^{(i)}c^{(i)}b^{(i)}$$

giving us

$$B'(t)=\sum_{i=1}^{n}A_t^{(i)}c^{(i)}b^{(i)}+\rho\left[1-\int_0^{\infty}\overset{n}{\cdots}\int_0^{\infty}\exp\left\{-\sum_{i=1}^{n}A^{(i)}(t)\,y^{(i)}\right\}dC\left(F_1\left(y^{(1)}\right),\cdots,F_n\left(y^{(n)}\right)\right)\right].$$

We integrate both sides to obtain the expression for $B(t)$.

$$B(t)=\sum_{i=1}^{n}c^{(i)}b^{(i)}\int_0^t A_s^{(i)}ds+\int_0^t\rho\left[1-\hat{c}\left(A^{(1)}(s),\cdots,A^{(n)}(s),s\right)\right]ds$$

where

$$\hat{c}\left(A^{(1)}(s),\cdots,A^{(n)}(s),s\right)=\int_0^{\infty}\overset{n}{\cdots}\int_0^{\infty}\exp\left\{-\sum_{i=1}^{n}A^{(i)}(s)\,y^{(i)}\right\}dC\left(F_1\left(y^{(1)}\right),\cdots,F_n\left(y^{(n)}\right)\right).$$

13. For the second part, we use Lemma 2.1 in Ma and Kim (2010) and use our previous notation for $A^{(i)}(T)=\alpha^{(i)}$ for all i whereby $\alpha^{(i)}$ is a constant. Note that since the process f is a martingale,

$$\begin{aligned}&\mathrm{E}\left[\exp\left\{B(T)-\sum_{i=1}^{n}\gamma^{(i)}\Psi_T^{(i)}-\sum_{i=1}^{n}\alpha^{(i)}x_T^{(i)}\right\}\mid X_0^{(1)},\ldots,X_0^{(n)}\right]\\ =\;&\exp\left\{B(0)-\sum_{i=1}^{n}\gamma^{(i)}\Psi_0^{(i)}-\sum_{i=1}^{n}A^{(i)}(0)\,x_0^{(i)}\right\}.\end{aligned}$$

And since $\Psi_0^{(i)}=0$ and $B(0)=0$, the above equation becomes

$$\mathrm{E}\left[\exp\left\{-\sum_{i=1}^{n}\gamma^{(i)}\Psi_T^{(i)}-\sum_{i=1}^{n}\alpha^{(i)}x_T^{(i)}\right\}\mid X_0^{(1)},\ldots,X_0^{(n)}\right]=\exp\left\{-B(T)-\sum_{i=1}^{n}A^{(i)}(0)\,x_0^{(i)}\right\}\text{(A.2)}$$

14. Now, the equation (A.2) is an important equation as the study will be based on this equation and its variables modifications. We will now consider the cases when each variable $\alpha^{(i)}$ and $\gamma^{(i)}$ equals to 0. When $\alpha^{(i)}=0$ for all i, then we obtain

$$A^{(i)}(t)=\frac{2\gamma^{(i)}\left[1-\exp\left(-(T-t)\,D^{(i)}\right)\right]}{D^{(i)}-\left(c^{(i)}a^{(i)}\right)+\left(D^{(i)}+\left(c^{(i)}a^{(i)}\right)\right)\exp\left(-(T-t)\,D^{(i)}\right)}$$

and

$$B(t)=\sum_{i=1}^{n}c^{(i)}b^{(i)}\int_0^t A^{(i)}(s)\,ds+\int_0^t\rho\left[1-\hat{c}\left(A^{(1)}(s),\cdots,A^{(n)}(s),s\right)\right]ds.$$

15. And when $\gamma^{(i)}=0$ for all i, then the expressions in item 14 become

$$A^{(i)}(t)=\frac{2\alpha^{(i)}c^{(i)}a^{(i)}}{\alpha^{(i)}\left(\sigma^{(i)}\right)^2\left[1-\exp\left(-(T-t)\,c^{(i)}a^{(i)}\right)\right]+\left(2\left(c^{(i)}a^{(i)}\right)\right)\exp\left(-(T-t)\,c^{(i)}a^{(i)}\right)}$$

with

$$B(t)=\sum_{i=1}^{n}c^{(i)}b^{(i)}\int_0^t A^{(i)}(s)\,ds+\int_0^t\rho\left[1-\hat{c}\left(A^{(1)}(s),\cdots,A^{(n)}(s),s\right)\right]ds.$$

16. We can use the above equations to find the Laplace transform of the distribution of the vectors $\left(X_T^{(i)}\right)$ and vectors $\left(\Psi^{(i)}\right)$ and for all i at time T. The Laplace transforms of the distribution of the vectors $\left(x_T^{(i)}\right)$ is given by

$$\mathrm{E}\left[\exp\left\{-\sum_{i=1}^{n}\alpha^{(i)}X_T^{(i)}\right\}\mid X_0^{(1)},\ldots,X_0^{(n)}\right]=\exp\left\{-B(T)-\sum_{i=1}^{n}A^{(i)}(T)\,x_0^{(i)}\right\}$$

with $A^{(i)}(T)$ and $B(T)$ now become

$$A^{(i)}(T)=\frac{2a^{(i)}c^{(i)}\alpha^{(i)}}{\alpha^{(i)}\left(\sigma^{(i)}\right)^2\left(1-e^{-a^{(i)}c^{(i)}\left(\sigma^{(i)}\right)^2T}\right)+2a^{(i)}c^{(i)}e^{-a^{(i)}c^{(i)}\left(\sigma^{(i)}\right)^2}}$$

$$B(T)=\sum_{i=1}^{n}c^{(i)}b^{(i)}\int_0^T A^{(i)}(s)\,ds+\int_0^T\rho\left[1-\hat{c}\left(A^{(1)}(s),\cdots,A^{(n)}(s),s\right)\right]ds.$$

17. The Laplace transform of the distribution of the vectors $\left(\Psi^{(i)}\right)$ for all i at time T is obtained by setting $A^{(i)}(T)=\alpha^{(i)}=0$.

$$\mathrm{E}\left[\exp\left\{-\sum_{i=1}^{n}\gamma^{(i)}\Psi_T^{(i)}\right\}\mid X_0^{(1)},\ldots,X_0^{(n)}\right]=\exp\left\{-B(T)-\sum_{i=1}^{n}A^{(i)}(T)\,x_0^{(i)}\right\},$$

where $B(T)$ and $A^{(i)}(T)$ are now given as the following:

$$A^{(i)}(T)=\frac{\left[D^{(i)}+c^{(i)}a^{(i)}\right]+\left[D^{(i)}-\left(c^{(i)}a^{(i)}\right)\right]\frac{\left[-D^{(i)}-\left(c^{(i)}a^{(i)}\right)\right]}{D^{(i)}-\left(c^{(i)}a^{(i)}\right)}\exp\left\{-TD^{(i)}\right\}}{\left(\sigma^{(i)}\right)^2\left(1-\frac{-D^{(i)}-\left(c^{(i)}a^{(i)}\right)}{D^{(i)}-\left(c^{(i)}a^{(i)}\right)}\exp\left\{-TD^{(i)}\right\}\right)}$$

$$B(T)=\sum_{i=1}^{n}c^{(i)}b^{(i)}\int_0^T A^{(i)}(s)\,ds+\int_0^T\rho\left[1-\hat{c}\left(A^{(1)}(s),\cdots,A^{(n)}(s),s\right)\right]ds$$

B
Programming Code

This appendix section contains the Mathematica & MATLAB code developed to calculate the expressions in this thesis.

B.1 Simulation of Jump Diffusion processes

```
% dr(t)=a[c-b*r(t)]dt+sigma1*sqrt[r(t)]dW(t)+dZ(t)
% simulates a CIR + compound poisson process with an exponential jump size
% distribution.
% r1(t) and r2(t) are dependent variables, captured by 3 types of copula

% Siti N Mohd Ramli 28.3.14
%% Define variables
T=5; % time horizon
timesteps=T*260;
lambda=4; % mean of poisson distribution
mu1=200; mu2=100; % mean  of jump size distribution
var1=1/mu1^2; % standard deviation of jump size distribution
var2 = 1/mu2^2;
sigma1 = 0.25;sigma2 = 0.5;% diffusion rate for counterparty1&2

b1 = 0.5;b2 = 0.3; %  drift for counterparty 1&2
a1 = 1;c1 = 0.1;
a2 = 1;c2 = 0.5;

rG1(1) = 1.5;rG2(1) = 2;rt1(1) = 1.5;rt2(1) = 2;rF1(1) = 1.5;rF2(1) = 2;% Initial intensitylevel
```

$\text{rGmin1}(1) = 1.5; \text{rGmin2}(1) = 2; \text{rtmin1}(1) = 1.5; \text{rtmin2}(1) = 2; \text{rFmin1}(1) = 1.5; \text{rFmin2}(1) = 2;$
$\text{rG01}(1) = 1.5; \text{rG02}(1) = 2; \text{rt01}(1) = 1.5; \text{rt02}(1) = 2; \text{rF01}(1) = 1.5; \text{rF02}(1) = 2;$

%%Define copula components &generate student $-t+$ Gaussian copula variables
$\text{rho1} = 0.95; \text{rhomin1} = -0.95; \text{rho0} = 0; \text{nu} = 3;$
$N = \text{poissrnd}(\text{lambda} * T)$; % number of jumps
$u = \text{rand}(N, 1) * T$; % vector of jump times.We only need 1 vector as they jump simultaneously

$\text{copuniformt1} = \text{copularnd}('t', [1\text{rho1}; \text{rho11}], \text{nu}, N);$
$\text{copuniformG1} = \text{copularnd}('\text{gaussian}', \text{rho1}, N);$
$\text{copuniformtmin1} = \text{copularnd}('t', [1\text{rhomin1}; \text{rhomin11}], \text{nu}, N);$
$\text{copuniformGmin1} = \text{copularnd}('\text{gaussian}', \text{rhomin1}, N);$
$\text{copuniformt0} = \text{copularnd}('t', [1\text{rho0}; \text{rho01}], \text{nu}, N);$
$\text{copuniformG0} = \text{copularnd}('\text{gaussian}', \text{rho0}, N);$

%%Generate FGM variables
$\text{U1} = \text{rand}(N, 2); \text{U10} = \text{rand}(N, 2);$
$A = 1 + \text{rho1}. * (1 - \text{U1}(:, 1)); B = \text{sqrt}(A.\hat{}2 - 4. * (A + 1). * \text{U10}(:, 2));$
$\text{U2} = 2. * \text{U10}(:, 2)./(B + A);$

$C = 1 + \text{rhomin1}. * (1 - \text{U1}(:, 1)); D = \text{sqrt}(C.\hat{}2 - 4. * (C - 1). * \text{U10}(:, 2));$
$\text{U3} = 2. * \text{U10}(:, 2)./(C + D);$

$E = 1 + \text{rho0}. * (1 - \text{U1}(:, 1)); F = \text{sqrt}(E.\hat{}2 - 4. * (E - 1). * \text{U10}(:, 2));$
$\text{U4} = 2. * \text{U10}(:, 2)./(F + E);$

%%Generate exponentially distributed RVs from the said copula
$\text{YG1} = \text{expinv}(\text{copuniformG1}(:, 1), 1/\text{mu1});$
$\text{YG2} = \text{expinv}(\text{copuniformG1}(:, 2), 1/\text{mu2});$
$\text{Yt1} = \text{expinv}(\text{copuniformt1}(:, 1), 1/\text{mu1});$
$\text{Yt2} = \text{expinv}(\text{copuniformt1}(:, 2), 1/\text{mu2});$
$\text{YF1} = \text{expinv}(\text{U1}(:, 1), 1/\text{mu1});$
$\text{YF2} = \text{expinv}(\text{U2}, 1/\text{mu2});$

$\text{YGmin1} = \text{expinv}(\text{copuniformGmin1}(:, 1), 1/\text{mu1});$
$\text{YGmin2} = \text{expinv}(\text{copuniformGmin1}(:, 2), 1/\text{mu2});$
$\text{Ytmin1} = \text{expinv}(\text{copuniformtmin1}(:, 1), 1/\text{mu1});$
$\text{Ytmin2} = \text{expinv}(\text{copuniformtmin1}(:, 2), 1/\text{mu2});$
$\text{YFmin1} = \text{expinv}(\text{U1}(:, 1), 1/\text{mu1});$
$\text{YFmin2} = \text{expinv}(\text{U3}, 1/\text{mu2});$

$\text{YG01} = \text{expinv}(\text{copuniformG0}(:, 1), 1/\text{mu1});$
$\text{YG02} = \text{expinv}(\text{copuniformG0}(:, 2), 1/\text{mu2});$

$\mathrm{Yt01} = \mathrm{expinv}(\mathrm{copuniformt0}(:,1), 1/\mathrm{mu1});$
$\mathrm{Yt02} = \mathrm{expinv}(\mathrm{copuniformt0}(:,2), 1/\mathrm{mu2});$
$\mathrm{YF01} = \mathrm{expinv}(\mathrm{U1}(:,1), 1/\mathrm{mu1});$
$\mathrm{YF02} = \mathrm{expinv}(\mathrm{U4}, 1/\mathrm{mu2});$

$\mathrm{W1} = \mathrm{randn}(\mathrm{timesteps}, 1); \mathrm{W2} = \mathrm{randn}(\mathrm{timesteps}, 1);$
$\mathrm{dt} = T/\mathrm{timesteps};$

%%Define the Euler approximation of the SDE with jumps
for$j = 1 : \mathrm{timesteps}$
$t = j/\mathrm{timesteps} * T;$
$\mathrm{pathG1}(j) = 0; \mathrm{pathG2}(j) = 0; \mathrm{patht1}(j) = 0;$
$\mathrm{patht2}(j) = 0; \mathrm{pathF1}(j) = 0; \mathrm{pathF2}(j) = 0;$
$\mathrm{pathGmin1}(j) = 0; \mathrm{pathGmin2}(j) = 0; \mathrm{pathtmin1}(j) = 0;$
$\mathrm{pathtmin2}(j) = 0; \mathrm{pathFmin1}(j) = 0; \mathrm{pathFmin2}(j) = 0;$
$\mathrm{pathG01}(j) = 0; \mathrm{pathG02}(j) = 0; \mathrm{patht01}(j) = 0;$
$\mathrm{patht02}(j) = 0; \mathrm{pathF01}(j) = 0; \mathrm{pathF02}(j) = 0;$
for$k = 1 : N$
if$u(k) < t$
%for rho = 1
$\mathrm{pathG1}(j) = \mathrm{pathG1}(j) + \mathrm{YG1}(k);$
$\mathrm{pathG2}(j) = \mathrm{pathG2}(j) + \mathrm{YG2}(k);$
$\mathrm{patht1}(j) = \mathrm{patht1}(j) + \mathrm{Yt1}(k);$
$\mathrm{patht2}(j) = \mathrm{patht2}(j) + \mathrm{Yt2}(k);$
$\mathrm{pathF1}(j) = \mathrm{pathF1}(j) + \mathrm{YF1}(k);$
$\mathrm{pathF2}(j) = \mathrm{pathF2}(j) + \mathrm{YF2}(k);$

%for rho $= -1$
$\mathrm{pathGmin1}(j) = \mathrm{pathGmin1}(j) + \mathrm{YGmin1}(k);$
$\mathrm{pathGmin2}(j) = \mathrm{pathGmin2}(j) + \mathrm{YGmin2}(k);$
$\mathrm{pathtmin1}(j) = \mathrm{pathtmin1}(j) + \mathrm{Ytmin1}(k);$
$\mathrm{pathtmin2}(j) = \mathrm{pathtmin2}(j) + \mathrm{Ytmin2}(k);$
$\mathrm{pathFmin1}(j) = \mathrm{pathFmin1}(j) + \mathrm{YFmin1}(k);$
$\mathrm{pathFmin2}(j) = \mathrm{pathFmin2}(j) + \mathrm{YFmin2}(k);$

%for rho = 0
$\mathrm{pathG01}(j) = \mathrm{pathG01}(j) + \mathrm{YG01}(k);$
$\mathrm{pathG02}(j) = \mathrm{pathG02}(j) + \mathrm{YG02}(k);$
$\mathrm{patht01}(j) = \mathrm{patht01}(j) + \mathrm{Yt01}(k);$
$\mathrm{patht02}(j) = \mathrm{patht02}(j) + \mathrm{Yt02}(k);$
$\mathrm{pathF01}(j) = \mathrm{pathF01}(j) + \mathrm{YF01}(k);$
$\mathrm{pathF02}(j) = \mathrm{pathF02}(j) + \mathrm{YF02}(k);$
end
end

```
if j > 1
%for rho = 1
rG1(j) = rG1(j-1) + a1*(c1 - b1*rG1(j-1))*dt + sigma1*sqrt(rG1(j-1))
*W1(j)/sqrt(timesteps) + (pathG1(j) - pathG1(j-1) + b1*dt);
rG2(j) = rG2(j-1) + a2*(c2 - b2*rG2(j-1))*dt + sigma2*sqrt(rG2(j-1))
*W2(j)/sqrt(timesteps) + (pathG2(j) - pathG2(j-1) + b2*dt);
rt1(j) = rt1(j-1) + a1*(c1 - b1*rt1(j-1))*dt + sigma1*sqrt(rt1(j-1))
*W1(j)/sqrt(timesteps) + (patht1(j) - patht1(j-1) + b1*dt);
rt2(j) = rt2(j-1) + a2*(c2 - b2*rt2(j-1))*dt + sigma2*sqrt(rt2(j-1))
*W2(j)/sqrt(timesteps) + (patht2(j) - patht2(j-1) + b2*dt);
rF1(j) = rF1(j-1) + a1*(c1 - b1*rF1(j-1))*dt + sigma1*sqrt(rF1(j-1))
*W1(j)/sqrt(timesteps) + (pathF1(j) - pathF1(j-1) + b1*dt);
rF2(j) = rF2(j-1) + a2*(c2 - b2*rF2(j-1))*dt + sigma2*sqrt(rF2(j-1))
*W2(j)/sqrt(timesteps) + (pathF2(j) - pathF2(j-1) + b2*dt);

%for rho = -1
rGmin1(j) = rGmin1(j-1) + a1*(c1 - b1*rGmin1(j-1))*dt + sigma1*sqrt(rGmin1(j-
1))*W1(j)/sqrt(timesteps) + (pathGmin1(j) - pathGmin1(j-1) + b1*dt);
rGmin2(j) = rGmin2(j-1) + a2*(c2 - b2*rGmin2(j-1))*dt + sigma2*sqrt(rGmin2(j-
1))*W2(j)/sqrt(timesteps) + (pathGmin2(j) - pathGmin2(j-1) + b2*dt);
rtmin1(j) = rtmin1(j-1) + a1*(c1 - b1*rtmin1(j-1))*dt + sigma1*sqrt(rtmin1(j-1))*
W1(j)/sqrt(timesteps) + (pathtmin1(j) - pathtmin1(j-1) + b1*dt);
rtmin2(j) = rtmin2(j-1) + a2*(c2 - b2*rtmin2(j-1))*dt + sigma2*sqrt(rtmin2(j-1))*
W2(j)/sqrt(timesteps) + (pathtmin2(j) - pathtmin2(j-1) + b2*dt);
rFmin1(j) = rFmin1(j-1) + a1*(c1 - b1*rFmin1(j-1))*dt + sigma1*sqrt(rFmin1(j-
1))*W1(j)/sqrt(timesteps) + (pathFmin1(j) - pathFmin1(j-1) + b1*dt);
rFmin2(j) = rFmin2(j-1) + a2*(c2 - b2*rFmin2(j-1))*dt + sigma2*sqrt(rFmin2(j-
1))*W2(j)/sqrt(timesteps) + (pathFmin2(j) - pathFmin2(j-1) + b2*dt);

%for rho = 0
rG01(j) = rG01(j-1) + a1*(c1 - b1*rG01(j-1))*dt + sigma1*sqrt(rG01(j-1))
*W1(j)/sqrt(timesteps) + (pathG01(j) - pathG01(j-1) + b1*dt);
rG02(j) = rG02(j-1) + a2*(c2 - b2*rG02(j-1))*dt + sigma2*sqrt(rG02(j-1))
*W2(j)/sqrt(timesteps) + (pathG02(j) - pathG02(j-1) + b2*dt);
rt01(j) = rt01(j-1) + a1*(c1 - b1*rt01(j-1))*dt + sigma1*sqrt(rt01(j-1))
*W1(j)/sqrt(timesteps) + (patht01(j) - patht01(j-1) + b1*dt);
rt02(j) = rt02(j-1) + a2*(c2 - b2*rt02(j-1))*dt + sigma2*sqrt(rt02(j-1))
*W2(j)/sqrt(timesteps) + (patht02(j) - patht02(j-1) + b2*dt);
rF01(j) = rF01(j-1) + a1*(c1 - b1*rF01(j-1))*dt + sigma1*sqrt(rF01(j-1))
*W1(j)/sqrt(timesteps) + (pathF01(j) - pathF01(j-1) + b1*dt);
rF02(j) = rF02(j-1) + a2*(c2 - b2*rF02(j-1))*dt + sigma2*sqrt(rF02(j-1))
*W2(j)/sqrt(timesteps) + (pathF02(j) - pathF02(j-1) + b2*dt);
end
end
```

```
%%Plot
figure;
plot(rG1,'-b','MarkerSize',2)
holdon
plot(rG2,'-r','MarkerSize',2)
holdon
plot(rG01,'-m','MarkerSize',2)
holdon
plot(rG02,'-g','MarkerSize',2)
holdon
plot(rGmin1,'-c','MarkerSize',2)
holdon
plot(rGmin2,'-k','MarkerSize',2)
hleg = legend('GaussianP10.95','P20.95','GaussianP10','P20','GaussianP1-0.95','P2-0.95');
xlabel('day'); ylabel('Intensity Level');

xlim([0timesteps]);

figure;
plot(rt1,'-b','MarkerSize',2)
holdon
plot(rt2,'-r','MarkerSize',2)
holdon
plot(rt01,'-m','MarkerSize',2)
holdon
plot(rt02,'-g','MarkerSize',2)
holdon
plot(rtmin1,'-c','MarkerSize',2)
holdon
plot(rtmin2,'-k','MarkerSize',2)
hleg = legend('Student-tP10.95','P20.95','Student-tP10','P20','Student-tP1-0.95','P2-
0.95');
xlabel('day'); ylabel('Intensity Level');
xlim([0timesteps]);

figure
plot(rF1,'-b','MarkerSize',2)
holdon
plot(rF2,'-r','MarkerSize',2)
holdon
plot(rF01,'-m','MarkerSize',2)
holdon
plot(rF02,'-g','MarkerSize',2)
```

holdon
plot(rFmin1,$'-c'$,'MarkerSize',2)
holdon
plot(rFmin2,$'-k'$,'MarkerSize',2)
hleg = legend('FGMP10.95','P20.95','FGMP10','P20','FGMP1 – 0.95','P2 – 0.95');
xlabel('day'); ylabel('Intensity Level');
xlim([0 1300]);

B.2 Programming Code for Chapter 2

Define FGM copula with Weibull and Exponential margins
ClearAll[$\theta, p, l, \beta, \alpha, \delta$];
$\mathscr{F}[\theta\_, p\_, l\_, \beta\_]$ = CopulaDistribution[{"FGM", θ},
{WeibullDistribution[p, l], ExponentialDistribution[β]}];
$\mathscr{F}$E[$\theta_, \alpha_, \beta_$] = CopulaDistribution[{"Binormal", θ},
{ExponentialDistribution[α], ExponentialDistribution[β]}];

Drawing copula plot
Table[ListPlot[RandomVariate[$\mathscr{F}[\theta, p, l, \beta]$, 100],
PlotLabel → Row[{"θ = ", θ}, {"Shape = ", p}, {"Scale = ", l}, {"Time = ", β}]],
{θ, {–0.999, –0.5, 0.0000000000000001, 0.5, 0.999}}, {p, {2}}, {l, {1}}, {β, {100}}];

Table[ListPlot[RandomVariate[$\mathscr{F}$E[θ, α, β], 500],
PlotLabel → Row[{"θ = ", θ}, {"Size = ", α}, {"Time = ", β}]],
{θ, {–0.999, –0.5, 0.0000000000000001, 0.5, 0.999}}, {α, {0.0001}}, {β, {10}}]

Illustration of the PDF and CDF of the above FGM copula with $\theta = 0, \rho = 2, l = 1, \alpha = 1, \beta = 100$
Plot3D[Evaluate@PDF[$\mathscr{F}$[0, 2, 1, 100], {x, t}], {x, 0, 10}, {t, 0, 0.5}]
Plot3D[Evaluate@CDF[$\mathscr{F}$[0, 2, 1, 100], {x, t}], {x, 0.5, 2000}, {t, 0, 1}]

Solving the Volterra equation with Neumann series
ClearAll[$\theta, p, l, \beta, \alpha, \delta, T$];
$\alpha = 1; \beta = 1; \delta = 0.04; T = 5$;
FGMWeiExp[x_, s_, θ_]:=PDF[$\mathscr{F}$E[θ, α, β], {x, s}];
FGMWeiExpN[x_, s_, θ_]:=Flatten[Apply[List, FGMWeiExp[x, s, θ]][[1]]][[1]]
meanFGMrec[θ_]:=
N[NIntegrate[Exp[$-\delta * s$] $* x$FGMWeiExp[x, s, θ], {x, 0, ∞}, {s, 0, T}]+
βNIntegrate[Exp[$-\delta * (T - s + u)$] $* x *$ FGMWeiExp[x, u, θ], {s, 0, T},
{u, 0, s}, {x, 0, ∞}], 16]
Table[meanFGMrec[–0.9], {β, {1}}, {α, {0.01, 0.1, 1, 10, 15}}]

ListPlot3D[Table[meanFGMrec[–0.9], {α, 0.01, 100.01, 10}, {β, 0.001, 100.01, 10}],
AxesLabel → {"β", "α", "1st Moment"}]

Checking section - Barges (2011) 1^{st} *moment*
$\text{ClearAll}[\theta, p, l, \beta, \alpha, \delta, t];$
$\alpha = 1; \beta = 100; \delta = 0.04;$
$F_X = 1 - \text{Exp}[-\alpha x];$
$\text{EX} = \text{NIntegrate}[\text{Exp}[-2\alpha x], \{x, 0, \infty\}];$
$\mu_B[\theta\_, \text{t}\_]\text{:=}\beta \frac{1}{\alpha}\left(\frac{1-\text{Exp}[-\delta t]}{\delta}\right) + \theta\beta\left(\text{EX} - \frac{1}{\alpha}\right)\left(\frac{1-\text{Exp}[-(2\beta+\delta)t]}{2\beta+\delta}\right);$
$\mu_B[-0.9, 5]$

Calculate 2^{nd} *moment*
$\text{ClearAll}[\theta, \alpha, \beta, \delta, T];$
$\mathscr{D}[\theta\_, \alpha\_, \beta\_]\text{:=CopulaDistribution}[\{\text{“GumbelHougaard”}, \theta\},$
$\{\text{ExponentialDistribution}[\alpha], \text{ExponentialDistribution}[\beta]\}];$
$\alpha = 10; \delta = 0.04; \beta = 1; \theta = 0.9; T = 5;$
$\text{GumWeibExpg}[\theta\_, \text{x}\_, \text{s}\_]\text{:=PDF}[\mathscr{D}[\theta, \alpha, \beta], \{x, s\}];$
$\text{Assuming}[u \geq 0\&\&x \geq 0\&\&s \geq 0\&\&h \geq 0\&\&y \geq 0\&\&\tau \geq 0,$
$\text{NIntegrate}\left[\text{Exp}[-2\delta s]x^2\text{GumWeibExpg}[\theta, x, s],\right.$
$\left\{x, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\}, \{s, 0, T\},$
$\text{Method} \to \{\text{GlobalAdaptive}, \text{MaxErrorIncreases} \to 15000\}]+$
$\beta\text{NIntegrate}\left[\text{Exp}[-2\delta(T-s) - 2\delta\tau] * x^2 * \text{GumWeibExpg}[\theta, x, \tau], \{s, 0, T\},\right.$
$\{\tau, 0, s\}, \left\{x, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\},$
$\text{Method} \to \{\text{GlobalAdaptive}, \text{MaxErrorIncreases} \to 15000\}]+$
$2\text{NIntegrate}[\text{Exp}[-2\delta s - \delta\tau]xh\text{GumWeibExpg}[\theta, x, s]\text{GumWeibExpg}[\theta, h, \tau],$
$\{s, 0, T\}, \{\tau, 0, T-s\}, \left\{x, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\},$
$\left\{h, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\},$
$\text{Method} \to \{\text{GlobalAdaptive}, \text{MaxErrorIncreases} \to 15000\}]+$
$2\beta\text{NIntegrate}[\text{Exp}[-\delta(T-s-\tau+u) - 2\delta s]xh\text{GumWeibExpg}[\theta, x, s]\text{GumWeibExpg}[\theta, h, u],$
$\{s, 0, T\}, \left\{x, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\}, \{\tau, 0, T-s\},$
$\{u, 0, \tau\}, \left\{h, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\},$
$\text{Method} \to \{\text{GlobalAdaptive}, \text{MaxErrorIncreases} \to 15000\}]+$
$2\beta\text{NIntegrate}[\text{Exp}[-2\delta(T-s) - 2\delta\tau - \delta y]xh\text{GumWeibExpg}[\theta, x, \tau]\text{GumWeibExpg}[\theta, h, y],$
$\{s, 0, T\}, \{\tau, 0, s\}, \{y, 0, s-\tau\},$
$\left\{h, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\},$
$\left\{x, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\},$
$\text{Method} \to \{\text{GlobalAdaptive}, \text{MaxErrorIncreases} \to 15000\}]+$
$2\beta^2\text{NIntegrate}[\text{Exp}[-2\delta(T-s) - 2\delta\tau - \delta(s-\tau-y+u)]xh\text{GumWeibExpg}[\theta, x, \tau]$
$\text{GumWeibExpg}[\theta, h, u], \{s, 0, T\}, \{\tau, 0, s\}, \{y, 0, s-\tau\}, \{u, 0, y\},$
$\left\{h, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\},$
$\left\{x, \frac{-\text{Log}[0.999999999999]}{\alpha}, \frac{-\text{Log}[0.000000000001]}{\alpha}\right\},$

Method $\rightarrow$ {GlobalAdaptive, MaxErrorIncreases $\rightarrow$ 15000}]]

2^{nd} moment checking tool - Barges FGM
$\theta = -0.9; \alpha = 0.01; \delta = 0.04; \beta = 1; T = 5;$
$\mathrm{X2} = \int_0^\infty 2x\mathrm{Exp}[-\alpha x]^2 dx; \mathrm{X1} = \int_0^\infty \mathrm{Exp}[-\alpha x]^2\, dx;$
$\beta \frac{2}{\alpha^2} * \frac{1-\mathrm{Exp}[-2\delta T]}{2\delta} + \theta\beta \left(\mathrm{X2} - \frac{2}{\alpha^2}\right)\left(\frac{1-\mathrm{Exp}[-2T(\delta+\beta)]}{2\beta+\delta}\right) + 2\beta^2 * \frac{1}{\alpha^2}\left(\frac{1-\mathrm{Exp}[-\delta T]}{\sqrt{2}\delta}\right)^2 +$
$2\theta^2\beta^2\left(\mathrm{X1} - \frac{1}{\alpha}\right)^2\left(\frac{1}{2(\delta+\beta)(\delta+2\beta)} - \frac{\mathrm{Exp}[-(\delta+2\beta)T]}{\delta(\delta+2\beta)} + \frac{\mathrm{Exp}[-2(\delta+\beta)T]}{2\delta(\delta+\beta)}\right) +$
$2\theta\frac{\beta^2}{\alpha}\left(\mathrm{X1} - \frac{1}{\alpha}\right)$
$\left(\frac{1}{2\delta(\delta+2\beta)} - \frac{\mathrm{Exp}[-(\delta+2\beta)T]}{(\delta+2\beta)(\delta-2\beta)} + \frac{\mathrm{Exp}[-2\delta T]}{2\delta(\delta-2\beta)} + \frac{1}{\delta(\delta+2\beta)} - \frac{\mathrm{Exp}[-\delta T]}{\delta(\delta+2\beta)} + \frac{\mathrm{Exp}[-2(\delta+\beta)T]}{2(\delta+\beta)(\delta+2\beta)}\right)$

B.3 Programming Code for Chapter 3

Define A_b and A_r
$T = 1$;$\rho = 4$; =3; sr=0.0023; $\pi s = 0.5$;
$c_r = 0.3; a_r = -1; b_r = 0; \sigma_r = 0.12; \gamma_r = 1; r_0 = 0.4; \beta = 5;$
$c_b = 0.2; a_b = -1; b_b = 0; \sigma_b = 0.09; \gamma_b = 1; b_0 = 0.05; \lambda = 7;$
$c_l = 0.5; a_l = -1; b_l = 0; \sigma_l = 0.1; \gamma_l = 1; l_0 = 0.0361; \alpha = 10;$

compr=$\sqrt{(c_r a_r)^2 + 2(\sigma_r\gamma_r)^2}$;
compb = $\sqrt{(c_b a_b)^2 + 2(\sigma_b\gamma_b)^2}$;
compl = $\sqrt{(c_l a_l)^2 + 2(\sigma_l\gamma_l)^2}$;
A_r[s_]:=$(2r_0(1-\mathrm{Exp}[-s\mathrm{compr}]))/((\mathrm{compr} - c_r a_r) + (\mathrm{compr} + c_r a_r)\mathrm{Exp}[-s\mathrm{compr}])$;
A_b[s_]:=$(2b_0(1-\mathrm{Exp}[-s\mathrm{compb}]))/((\mathrm{compb} - c_b a_b) + (\mathrm{compb} + c_b a_b)\mathrm{Exp}[-s\mathrm{compb}])$;
A_l[s_]:=$(2l_0(1-\mathrm{Exp}[-s\mathrm{compl}]))/((\mathrm{compl} - c_l a_l) + (\mathrm{compl} + c_l a_l)\mathrm{Exp}[-s\mathrm{compl}])$;
C_b[s_]:=
$\left(\left(2\mathrm{compbExp}\left[-\frac{s\mathrm{compb}+c_b a_b}{2}\right]\right)/((\mathrm{compb} - c_b a_b) + (\mathrm{compb} + c_b a_b)\mathrm{Exp}[-s\mathrm{compb}])\right)^{\frac{2c_b b_b}{\sigma_b{}^2}}$;
C_r[s_]:=
$\left(\left(2\mathrm{comprExp}\left[-\frac{s\mathrm{compr}+c_r a_r}{2}\right]\right)/((\mathrm{compr} - c_r a_r) + (\mathrm{compr} + c_r a_r)\mathrm{Exp}[-s\mathrm{compr}])\right)^{\frac{2c_r b_r}{\sigma_r{}^2}}$;
C_l[s_]:=
$\left(\left(2\mathrm{complExp}\left[-\frac{s\mathrm{compl}+c_l a_l}{2}\right]\right)/((\mathrm{compl} - c_l a_l) + (\mathrm{compl} + c_l a_l)\mathrm{Exp}[-s\mathrm{compl}])\right)^{\frac{2c_l b_l}{\sigma_l{}^2}}$;

GnEE[θ_]:=CopulaDistribution[{"FGM", θ},
{ExponentialDistribution[α], ExponentialDistribution[β]}];
GnEEB = CopulaDistribution[{"FGM", 0},
{ExponentialDistribution[λ], ExponentialDistribution[β]}];

chatB = NIntegrate [Exp [$-A_b[s]x - 0 * A_r[s]y$] PDF[GnEEB, {x, y}], {$x, 0, \infty$},
{$y, 0, \infty$}, {$s, 0, T$}]//FullSimplify;
chatB5 = NIntegrate [Exp [$-A_b[s]x - 0 * A_r[s]y$] PDF[GnEEB, {x, y}], {$x, 0, \infty$},

$\{y,0,\infty\},\left\{s,0,\frac{T}{2}\right\}]$ //FullSimplify;
chat = NIntegrate $[\text{Exp}[-A_l[s]x - A_r[s]y]\,\text{PDF}[\text{GnEE}[2],\{x,y\}],\{x,0,\infty\},$
$\{y,0,\infty\},\{s,0,T\}]$//FullSimplify;
chatS = NIntegrate $[\text{Exp}[-A_l[s]x - 0*A_r[s]y]\,\text{PDF}[\text{GnEE}[1],\{x,y\}],\{x,0,\infty\},$
$\{y,0,\infty\},\{s,0,T\}]$//FullSimplify;
chatR = NIntegrate $[\text{Exp}[-0*A_l[s]x - A_r[s]y]\,\text{PDF}[\text{GnEE}[1],\{x,y\}],\{x,0,\infty\},$
$\{y,0,\infty\},\{s,0,T\}]$//FullSimplify;

$\text{BB} = C_l[T]C_r[T]\text{Exp}[-A_l[T]l_0 - A_r[T]r_0 - \rho T + \rho\text{*chat}]$;
$\text{Bs} = C_l[T]C_r[T]\text{Exp}[-A_l[T]l_0 - A_r[T]r_0 - \rho T + \rho\text{*chatS}]$;
$\text{Bb} = C_b[T]C_r[T]\text{Exp}[-A_b[T]b_0 - A_r[T]r_0 - \rho\text{B}T + \rho\text{B*chatB}]$;
$\text{Bb5} = C_b\left[\frac{T}{2}\right]C_r\left[\frac{T}{2}\right]\text{Exp}\left[-A_b\left[\frac{T}{2}\right]b_0 - A_r\left[\frac{T}{2}\right]r_0 - \rho\text{B}\frac{T}{2} + \rho\text{B*chatB5}\right]$;

$\text{Export}\left[\text{“Gumbel.xlsx”},\text{Table}\left[\frac{2(1-\pi\text{s})(\text{Seller}[1]-\text{Both}[1])}{\text{Exp}\left[\frac{\text{rf}}{2}\right]\text{Buyer05}\left[\frac{1}{2}\right]+\text{Buyer}[1]},\right.\right.$
$\{\theta,\{-0.995,-0.99,-0.95,-0.9,-0.5,0,0.5,0.9,0.95,0.99,0.995\}\},$
$\{\text{br},\{0.01,0.05,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1,2\}\}]]$;
Export[“BothSurv.xlsx”,
$\text{Table}[\text{BB},\{\theta,\{-0.995,-0.99,-0.95,-0.9,-0.5,0,0.5,0.9,0.95,0.99,0.995\}\},$
$\{\text{br},\{0\}\}]]$;
Export[“SellerSurv.xlsx”,
Table[Seller[1] + Both[1],
$\{\theta,\{-0.995,-0.99,-0.95,-0.9,-0.5,0,0.5,0.9,0.95,0.99,0.995\}\},$
$\{\text{br},\{0\}\}]]$;
Export[“RCSurv.xlsx”,
Table[RC[1] + Both[1],
$\{\theta,\{-0.995,-0.99,-0.95,-0.9,-0.5,0,0.5,0.9,0.95,0.99,0.995\}\},\{\text{br},\{0\}\}]]$;

B.4 Programming Code for Chapter 4

Bond Price Calibration
ClearAll[cr, cb, ab, ar, br, bb, σ1, σ2, σr, σb, r0, b0, θF, θG, θT, θST,
θGm, ρ, α, β, DoF, P, t, BondPriceF, BondPriceG, BondPriceT, BondPriceStdT,
BondPriceGm]

αr = 0; αb = 0; $d = 31$; $t = 6.165$; $P = 60.853$;

BondPriceF[cr_?NumericQ, cb_?NumericQ, ab_?NumericQ, ar_?NumericQ,
br_?NumericQ, bb_?NumericQ, σr_?NumericQ, σb_?NumericQ, r0_?NumericQ,
b0_?NumericQ, θF_?NumericQ, ρ_?NumericQ, α_?NumericQ, β_?NumericQ]:=
Assuming[$x > 0$&&$y > 0$,
Abs[

$$\Bigg(2^{\frac{2\text{bbcb}}{\sigma\text{b}^2}}$$
$$\Bigg(\Bigg(e^{\frac{t}{2}*\left(-\text{abcb}-\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}\right)}\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}\Bigg)\Big/$$
$$\left(-\text{abcb}+\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}+e^{-t*\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}}\left(\text{abcb}+\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}\right)\right)\Bigg)^{\frac{2\text{bbcb}}{\sigma\text{b}^2}}\Bigg)$$
$$\Bigg(2^{\frac{2\text{brcr}}{\sigma\text{r}^2}}$$
$$\Bigg(\Bigg(e^{\frac{t}{2}*\left(-\text{arcr}-\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}\right)}\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}\Bigg)\Big/$$
$$\left(-\text{arcr}+\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}+e^{-t*\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}}\left(\text{arcr}+\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}\right)\right)\Bigg)^{\frac{2\text{brcr}}{\sigma\text{r}^2}}\Bigg)$$
$$\text{Exp}[$$
$$-\left(\left(2\left(1-e^{-t*\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}}\right)\right)\Big/\right.$$
$$\left.\left(-\text{abcb}+\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}+e^{-t*\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}}\left(\text{abcb}+\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}\right)\right)\right)\text{b0}-$$
$$\left(\frac{2\left(1-e^{-t*\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}}\right)}{-\text{arcr}+\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}+e^{-t*\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}}\left(\text{arcr}+\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}\right)}\right)\text{r0}-$$
$$\rho t+$$
$$\rho\text{NIntegrate}[$$
$$\text{Exp}\left[-\left(\left(2\left(1-e^{-t*\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}}\right)\right)/\left(-\text{abcb}+\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}+\right.\right.\right.$$
$$\left.\left.e^{-t*\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}}\left(\text{abcb}+\sqrt{\text{ab}^2\text{cb}^2+2\sigma\text{b}^2}\right)\right)\right)x-$$
$$\left(\left(2\left(1-e^{-t*\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}}\right)\right)/\left(-\text{arcr}+\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}+\right.\right.$$
$$\left.\left.\left.e^{-t*\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}}\left(\text{arcr}+\sqrt{\text{ar}^2\text{cr}^2+2\sigma\text{r}^2}\right)\right)\right)y\right]*$$
$$\text{PDF}[\text{CopulaDistribution}[\{\text{“FGM”},\theta\text{F}\},$$
$$\{\text{ExponentialDistribution}[\alpha],\text{ExponentialDistribution}[\beta]\}],\{x,y\}],$$
$$\{x,0,\infty\},\{y,0,\infty\},\{\tau,0,t\},\text{AccuracyGoal}\to 10]]-P]];$$

$$\text{FGMCal}=$$
$$\text{NMinimize}[\{\text{BondPriceF}[\text{cr},\text{cb},\text{ab},\text{ar},\text{br},\text{bb},\sigma\text{r},\sigma\text{b},\text{r0},\text{b0},\theta\text{F},\rho,\alpha,\beta],$$
$$\alpha>1\&\&\beta>1\&\&0<\sigma\text{b}\leq 0.5\&\&0<\sigma\text{r}\leq 0.5\&\&\text{cr}*\text{br}\geq 0\&\&\text{cb}*\text{bb}\geq 0\&\&$$
$$\text{ar}*\text{cr}<0\&\&\text{ab}*\text{cb}<0\&\&\text{b0}\geq 0\&\&-1\leq\theta\text{F}\leq 1\&\&2\text{cbbb}\geq\sigma\text{b}^2\&\&2*\text{cr}*\text{br}\geq\sigma\text{r}^2\&\&$$
$$0<\text{r0}\leq 0.05\&\&0<\text{b0}\leq 1\&\&\rho\geq 2\},$$
$$\{\text{cr},\text{cb},\text{ab},\text{ar},\text{br},\text{bb},\sigma\text{r},\sigma\text{b},\text{r0},\text{b0},\theta\text{F},\rho,\alpha,\beta\}]//\text{AbsoluteTiming}$$

Testing the error and obtain model price

$$\text{ClearAll}[\text{cr},\text{cb},\text{ab},\text{ar},\text{br},\text{bb},\sigma 1,\sigma 2,\sigma\text{r},\sigma\text{b},\text{r0},\text{b0},\theta\text{T},\rho,\alpha,\beta,t]$$
$$\text{ModelPrice}[\text{cr\_?NumericQ},\text{cb\_?NumericQ},\text{ab\_?NumericQ},\text{ar\_?NumericQ},$$
$$\text{br\_?NumericQ},\text{bb\_?NumericQ},\sigma\text{r\_?NumericQ},\sigma\text{b\_?NumericQ},\text{r0\_?NumericQ},$$
$$\text{b0\_?NumericQ},\theta\text{T\_?NumericQ},\rho\text{\_?NumericQ},\alpha\text{\_?NumericQ},\beta\text{\_?NumericQ},$$

$$\sigma 1\_?\mathrm{NumericQ}, \sigma 2\_?\mathrm{NumericQ}, \mathrm{DoF}\_?\mathrm{NumericQ}] :=$$
$$\Big(2^{\frac{2\mathrm{bbcb}}{\sigma\mathrm{b}^2}}$$
$$\bigg(\bigg(e^{\frac{t}{2}*\left(-\mathrm{abcb}-\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}\right)}\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}\bigg)\bigg/$$
$$\left(-\mathrm{abcb}+\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}+e^{-t*\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}}\left(\mathrm{abcb}+\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}\right)\right)\bigg)^{\frac{2\mathrm{bbcb}}{\sigma\mathrm{b}^2}}\Big)$$
$$\Big(2^{\frac{2\mathrm{brcr}}{\sigma\mathrm{r}^2}}$$
$$\bigg(\bigg(e^{\frac{t}{2}*\left(-\mathrm{arcr}-\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}\right)}\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}\bigg)\bigg/$$
$$\left(-\mathrm{arcr}+\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}+e^{-t*\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}}\left(\mathrm{arcr}+\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}\right)\right)\bigg)^{\frac{2\mathrm{brcr}}{\sigma\mathrm{r}^2}}\Big)$$
$$\mathrm{Exp}[$$
$$-\bigg(\bigg(2\left(1-e^{-t*\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}}\right)\bigg)\bigg/$$
$$\left(-\mathrm{abcb}+\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}+e^{-t*\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}}\left(\mathrm{abcb}+\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}\right)\right)\bigg)\mathrm{b0}-$$
$$\bigg(\bigg(2\left(1-e^{-t*\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}}\right)\bigg)\bigg/$$
$$\left(-\mathrm{arcr}+\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}+e^{-t*\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}}\left(\mathrm{arcr}+\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}\right)\right)\bigg)\mathrm{r0}-$$
$$\rho t+$$
$$\rho*$$
$$\mathrm{NIntegrate}[$$
$$\mathrm{Exp}[$$
$$-\bigg(\bigg(2\left(1-e^{-t*\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}}\right)\bigg)/\bigg(-\mathrm{abcb}+\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}+e^{-t*\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}}$$
$$\left(\mathrm{abcb}+\sqrt{\mathrm{ab}^2\mathrm{cb}^2+2\sigma\mathrm{b}^2}\right)\bigg)\bigg)x-$$
$$\bigg(\bigg(2\left(1-e^{-t*\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}}\right)\bigg)\bigg/$$
$$\left(-\mathrm{arcr}+\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}+e^{-t*\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}}\left(\mathrm{arcr}+\sqrt{\mathrm{ar}^2\mathrm{cr}^2+2\sigma\mathrm{r}^2}\right)\right)\bigg)y\Big]*$$
$$\mathrm{Flatten}[$$
$$\mathrm{Apply}[\mathrm{List}, \mathrm{PDF}[\mathrm{CopulaDistribution}[$$
$$\left\{\text{“MultivariateT”}, \left\{\left\{\sigma 1^2, \theta\mathrm{T}\sigma 1\sigma 2\right\}, \left\{\theta\mathrm{T}\sigma 1\sigma 2, \sigma 2^2\right\}\right\}, \mathrm{DoF}\right\},$$
$$\{\mathrm{ExponentialDistribution}[\alpha], \mathrm{ExponentialDistribution}[\beta]\}],$$
$$\{x,y\}]][[1]]][[1]], \{x,0,\infty\}, \{y,0,\infty\}, \{\tau,0,t\},$$
$$\mathrm{AccuracyGoal} \to 10]]$$

$$\mathrm{ModelPrice}[\mathrm{cr},\mathrm{cb},\mathrm{ab},\mathrm{ar},\mathrm{br},\mathrm{bb},\sigma\mathrm{r},\sigma\mathrm{b},\mathrm{r0},\mathrm{b0},\theta\mathrm{T},\rho,\alpha,\beta,\sigma 1,\sigma 2,\mathrm{DoF}]$$

C

A multivariate jump diffusion process for counterparty risk in CDS rates

C.1 CDS Rates Sensitivity Analysis

C.1.1 FGM Copula

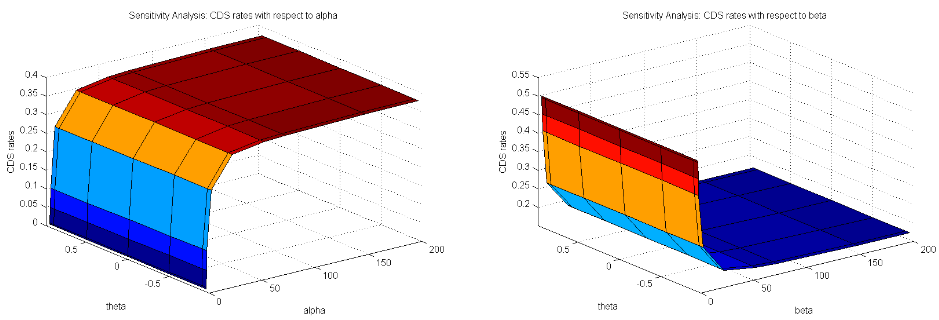


FIGURE C.1: Sensitivity of CDS rates under FGM copula with respect to seller's and RC's jump size jump size (α and β respectively)

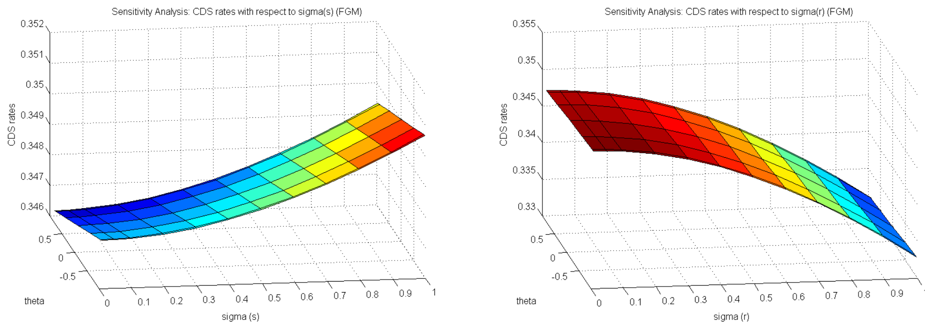


FIGURE C.2: Sensitivity of CDS rates under FGM copula with respect to seller's and RC's diffusion rates ($\sigma^{(s)}$ and $\sigma^{(r)}$ respectively)

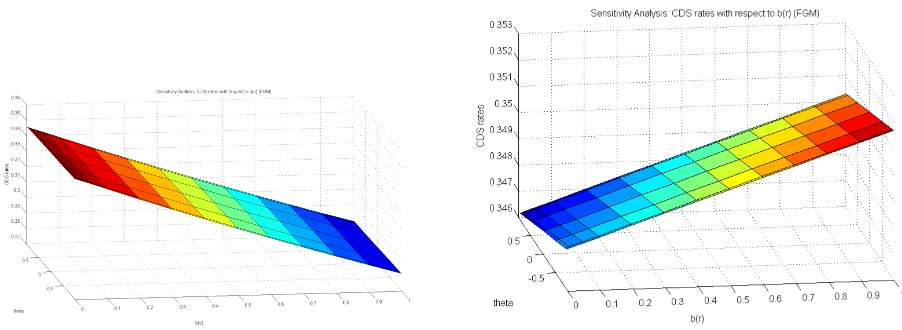


FIGURE C.3: Sensitivity of CDS rates under FGM copula with respect to seller's and RC's long term mean ($b^{(s)}$ and $b^{(r)}$ respectively)

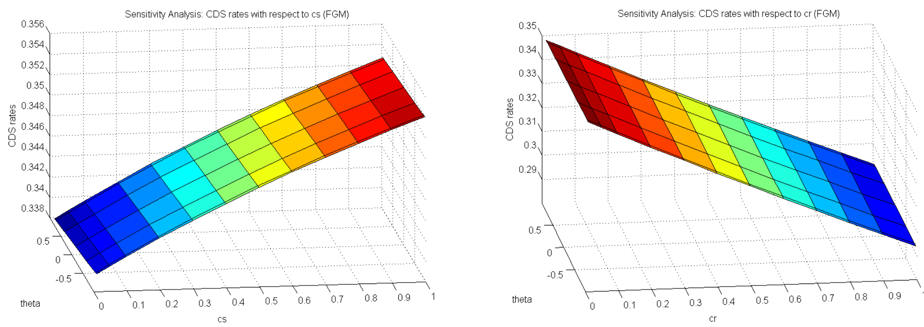


FIGURE C.4: Sensitivity of CDS rates under FGM copula with respect to seller's and RC's decay rate ($c^{(s)}$ and $c^{(r)}$ respectively)

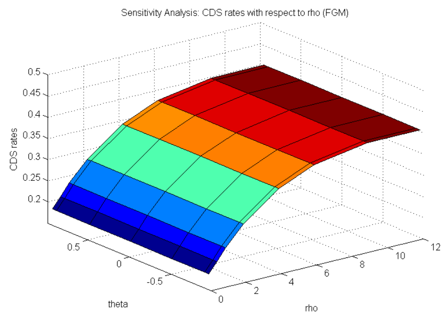


FIGURE C.5: Sensitivity of CDS rates under FGM copula with respect to frequency of yearly jump events, ρ

C.1.2 Gaussian Copula

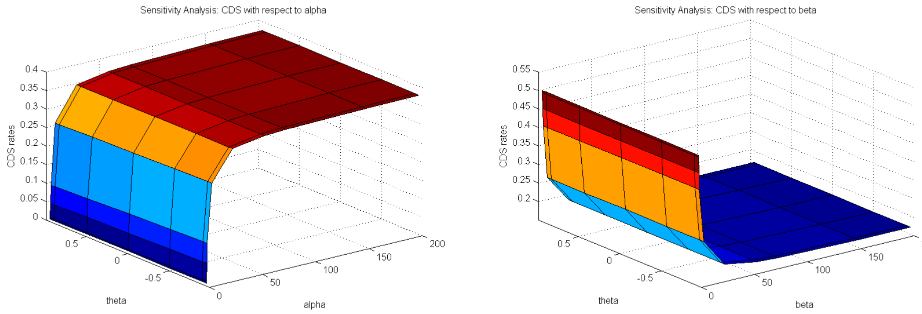


FIGURE C.6: Sensitivity of CDS rates under Gaussian copula with respect to seller's and RC's jump size jump size (α and β respectively)

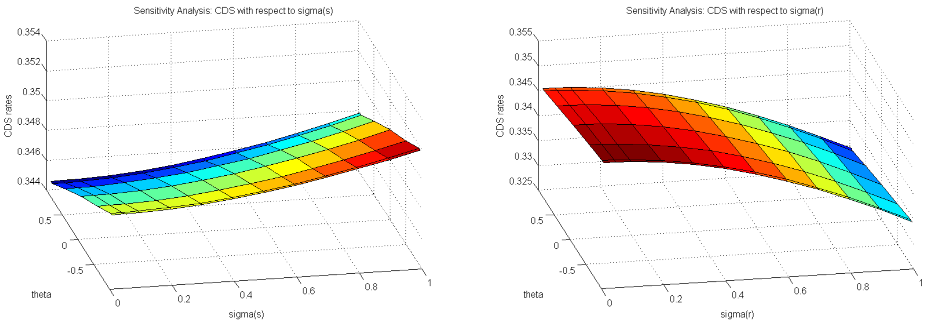


FIGURE C.7: Sensitivity of CDS rates under Gaussian copula with respect to seller's and RC's diffusion rates ($\sigma^{(s)}$ and $\sigma^{(r)}$ respectively)

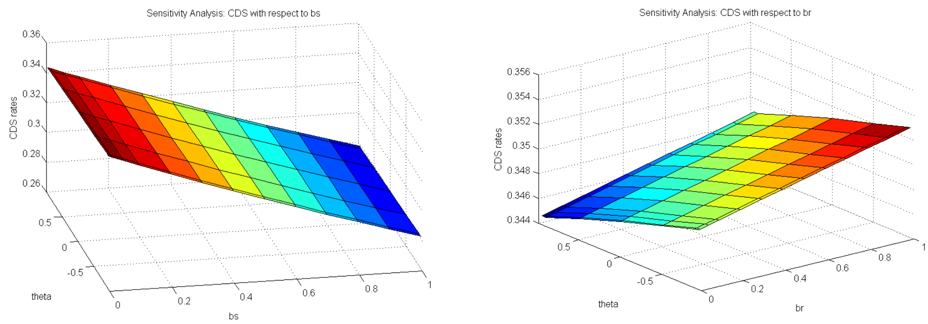


FIGURE C.8: Sensitivity of CDS rates under Gaussian copula with respect to seller's and RC's long term mean ($b^{(s)}$ and $b^{(r)}$ respectively)

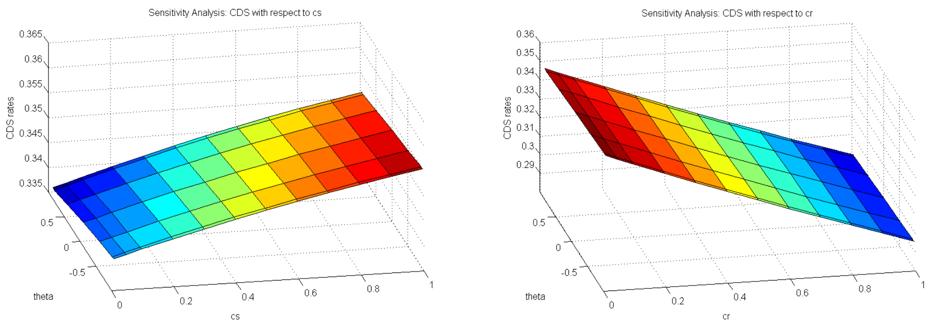


FIGURE C.9: Sensitivity of CDS rates under Gaussian copula with respect to seller's and RC's decay rate ($c^{(s)}$ and $c^{(r)}$ respectively)

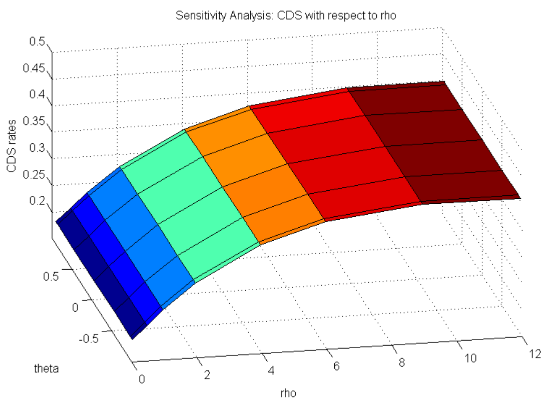


FIGURE C.10: Sensitivity of CDS rates under Gaussian copula with respect to frequency of yearly jump events, (ρ)

D

Jump diffusion model with copula dependence structure in defaultable bond pricing

D.1 Bond Price and yield as a function of tenor and θ with jump size distribution $\mu_t^{(1)} = 100$, and $\mu_t^{(2)} = 200$

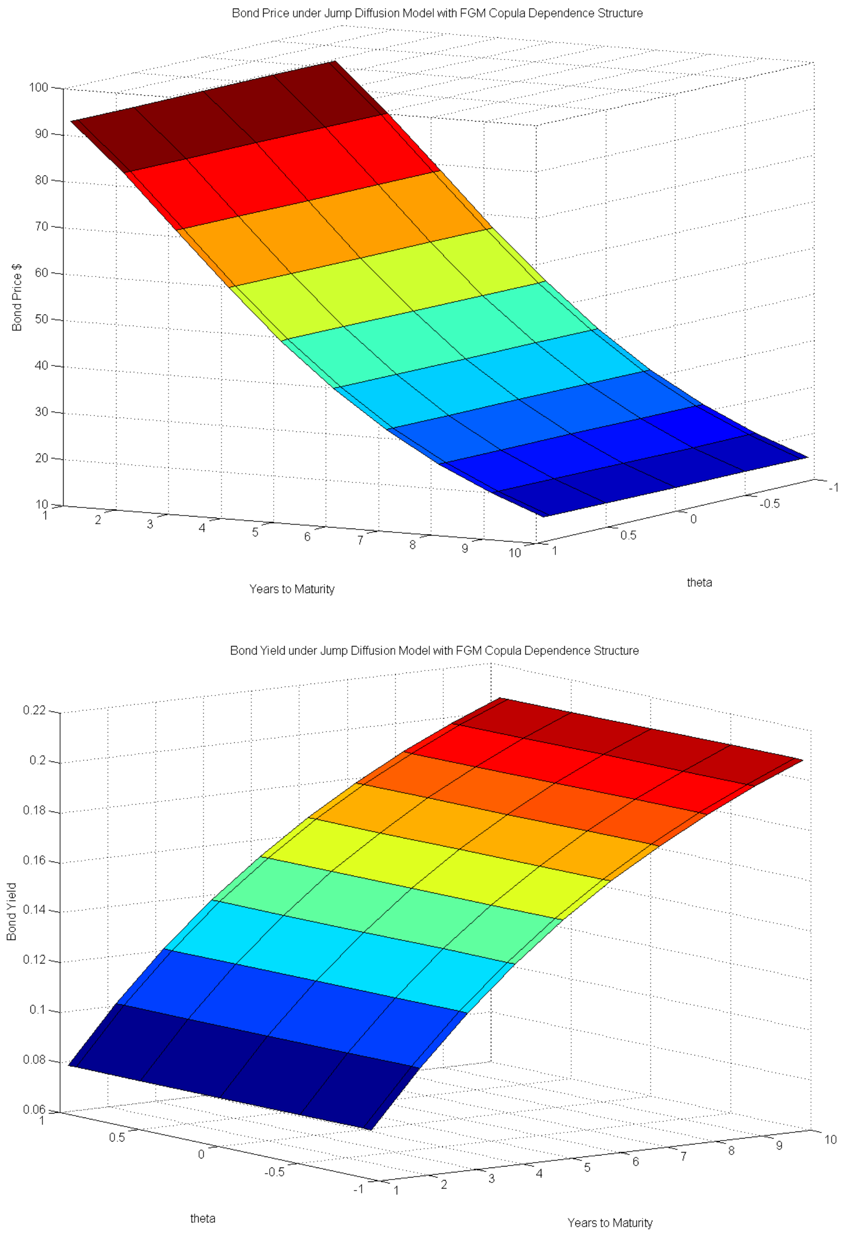
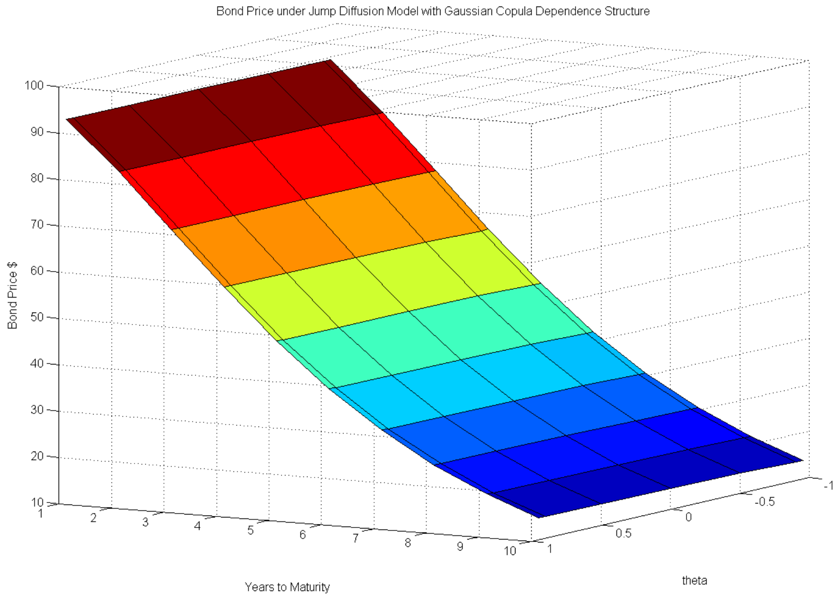


FIGURE D.1: Bond price and yield as a function of θ and tenor under the FGM copula dependence structure



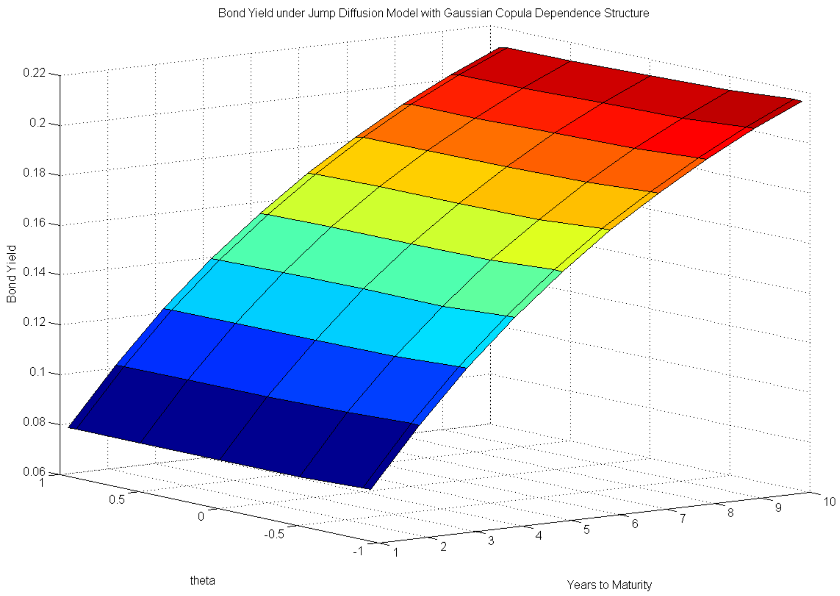


FIGURE D.2: Bond price and yield as a function of θ and tenor under the Gaussian copula dependence structure

TABLE D.1: Prices of zero coupon bond under jump diffusion model with student-t copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	89.424	75.583	61.109	47.708	36.232	26.924	19.668	14.172	10.103	7.134
-0.9	89.564	75.799	61.338	47.906	36.38	27.023	19.724	14.197	10.108	7.142
-0.5	89.578	75.880	61.524	48.201	36.758	27.443	20.149	14.599	10.446	7.391
0	90.002	76.521	62.185	48.758	37.161	27.693	20.273	14.6340	10.469	7.445
0.5	90.030	76.678	62.410	48.983	37.366	27.870	20.420	14.751	10.538	7.461
0.9	90.121	76.713	62.55	49.336	37.900	28.520	21.114	15.433	11.167	8.016
0.95	90.151	76.885	62.805	49.609	38.168	28.767	21.333	15.620	11.322	8.1418

TABLE D.2: Prices of zero coupon bond under jump diffusion model with Gaussian copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	92.529	82.015	70.281	58.598	47.798	38.312	30.283	23.673	18.343	14.114
-0.9	92.530	82.016	70.284	58.602	47.803	38.318	30.290	23.680	18.350	14.121
-0.5	92.626	82.157	70.518	58.933	48.148	38.657	30.609	23.971	18.609	14.346
0	92.628	82.243	70.577	58.934	48.216	38.745	30.708	24.073	18.710	14.441
0.5	92.631	82.255	70.608	58.986	48.221	38.787	30.788	24.180	18.831	14.558
0.9	92.633	82.270	70.645	59.050	48.310	38.854	30.828	24.199	18.833	14.568
0.95	92.634	82.284	70.680	59.110	48.394	38.956	30.942	24.316	18.949	14.668

TABLE D.3: Prices of zero coupon bond under jump diffusion model with FGM copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	92.627	82.332	70.805	59.332	48.714	39.358	31.400	24.804	19.442	15.146
-0.9	92.627	82.332	70.806	59.334	48.716	39.361	31.404	24.808	19.445	15.149
-0.5	92.627	82.336	70.814	59.349	48.736	39.386	31.431	24.837	19.474	15.177
0	92.628	82.340	70.824	59.366	48.762	39.417	31.466	24.873	19.510	15.211
0.5	92.629	82.344	70.835	59.384	48.787	39.448	31.501	24.909	19.546	15.246
0.9	92.629	82.347	70.843	59.400	48.807	39.473	31.528	24.938	19.575	15.273
0.95	92.629	82.347	70.844	59.401	48.810	39.476	31.532	24.942	19.579	15.279

TABLE D.4: Yield (in %) of zero coupon bond under jump diffusion model with student-t copula dependence structure

θ	1	2	3	4	5	6	7	8	9	10
-0.95	11.827	15.024	17.842	20.324	22.513	24.445	26.152	27.665	29.008	30.217
-0.9	11.652	14.860	17.695	20.200	22.413	24.369	26.101	27.637	29.002	30.202
-0.5	11.635	14.799	17.576	20.015	22.160	24.050	25.717	27.192	28.530	29.756
0	11.109	14.317	17.158	19.671	21.894	23.863	25.607	27.154	28.499	29.662
0.5	11.074	14.199	17.017	19.533	21.760	23.731	25.477	27.027	28.406	29.634
0.9	10.962	14.173	16.930	19.319	21.415	23.256	24.879	26.312	27.580	28.707
0.95	10.925	14.046	16.771	19.155	21.244	23.079	24.695	26.122	27.386	28.507

TABLE D.5: Yield (in %) of zero coupon bond under jump diffusion model with Gaussian copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	8.074	10.421	12.474	14.295	15.909	17.340	18.608	19.735	20.736	21.629
-0.9	8.074	10.421	12.473	14.294	15.907	17.337	18.604	19.730	20.731	21.623
-0.5	7.961	10.326	12.348	14.133	15.740	17.164	18.427	19.547	20.543	21.430
0	7.959	10.268	12.317	14.132	15.708	17.120	18.372	19.484	20.471	21.350
0.5	7.956	10.260	12.301	14.107	15.705	17.099	18.328	19.418	20.384	21.252
0.9	7.953	10.250	12.281	14.076	15.663	17.065	18.306	19.406	20.383	21.244
0.95	7.952	10.241	12.263	14.047	15.622	17.014	18.244	19.334	20.301	21.161

TABLE D.6: Yield (in %) of zero coupon bond under jump diffusion model with FGM copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	7.960	10.209	12.197	13.940	15.470	16.814	17.996	19.038	19.958	20.773
-0.9	7.960	10.209	12.196	13.939	15.469	16.812	17.994	19.036	19.956	20.771
-0.5	7.960	10.206	12.192	13.933	15.459	16.800	17.979	19.018	19.936	20.749
0	7.959	10.204	12.186	13.924	15.448	16.785	17.961	18.996	19.911	20.721
0.5	7.958	10.201	12.181	13.915	15.436	16.770	17.942	18.975	19.887	20.694
0.9	7.957	10.197	12.176	13.908	15.426	16.757	17.927	18.957	19.867	20.672
0.95	7.957	10.198	12.176	13.908	15.425	16.756	17.922	18.955	19.865	20.670

D.2 Bond Price and yield as a function of tenor and θ with jump size distribution $\mu_t^{(1)} = 5$, and $\mu_t^{(2)} = 10$

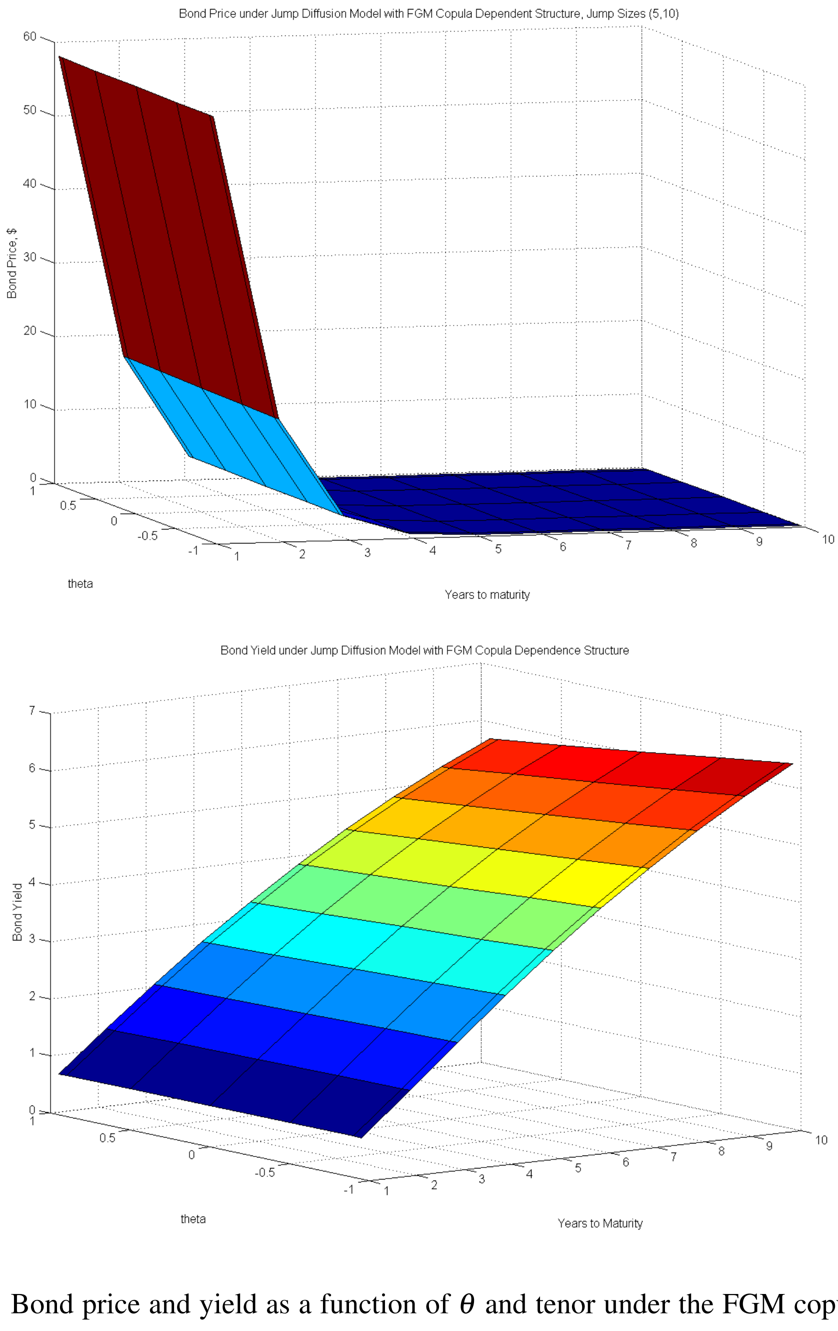


FIGURE D.3: Bond price and yield as a function of θ and tenor under the FGM copula dependence structure

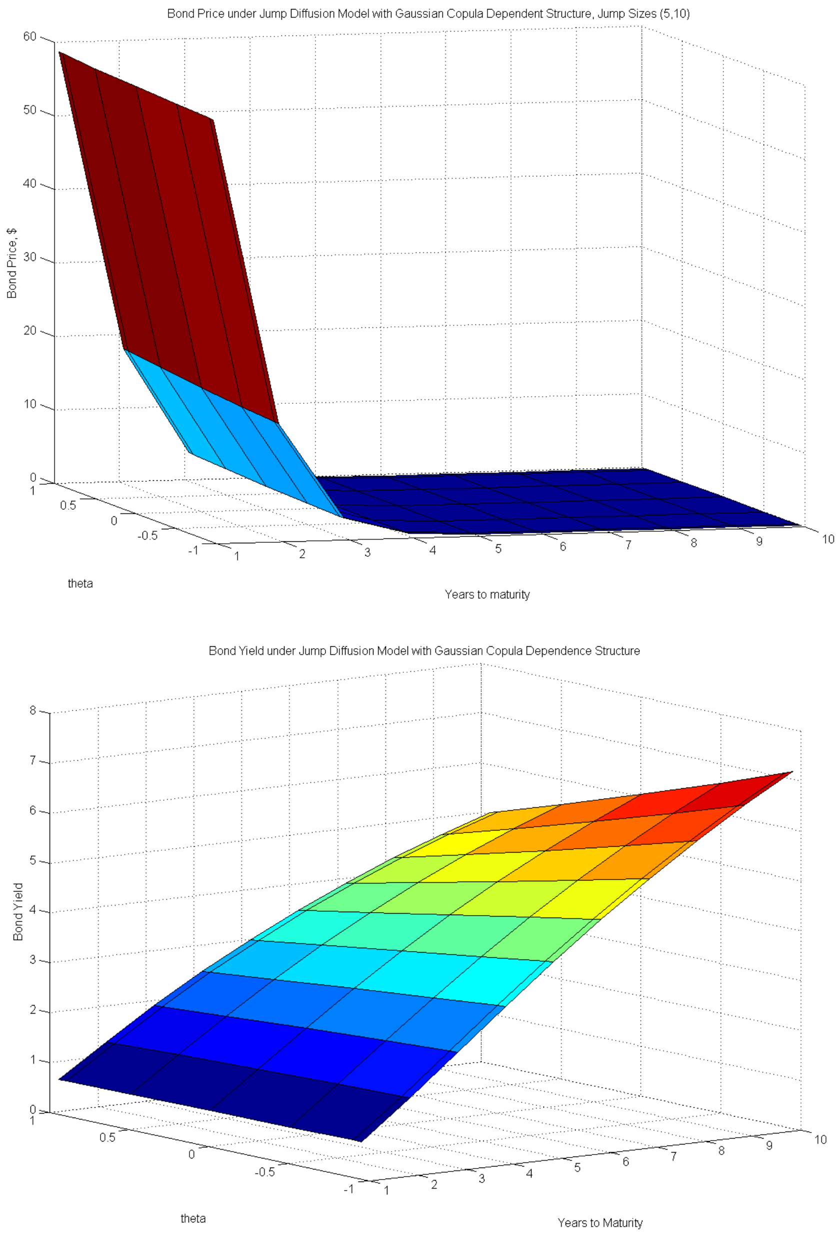


FIGURE D.4: Bond price and yield as a function of θ and tenor under the Gaussian copula dependence structure

TABLE D.7: Prices of zero coupon bond under jump diffusion model with student-t copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	51.019	11.845	1.732	0.188	0.017	0.001	8.80E-05	5.6E-06	3.4E-07	2.0E-08
-0.9	51.030	11.893	1.751	0.192	0.017	0.001	9.4E-05	6.1E-06	3.7E-07	2.2E-08
-0.5	51.275	12.348	1.928	0.228	0.0224	0.002	0.0002	1.1E-05	7.6E-07	5.1E-08
0	51.867	13.089	2.204	0.287	0.0314	0.003	0.0003	2.3E-05	1.8E-06	1.4E-07
0.5	52.791	14.085	2.578	0.372	0.046	0.005	0.0005	4.9E-05	4.4E-06	3.9E-07
0.9	53.883	15.191	3.010	0.478	0.065	0.008	0.0009	9.9E-05	1.0E-05	1.0E-06
0.95	54.065	15.368	3.080	0.496	0.069	0.009	0.001	0.0001	1.1E-05	1.2E-06

TABLE D.8: Prices of zero coupon bond under jump diffusion model with Gaussian copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	57.359	15.816	2.736	0.347	0.036	0.003	0.00024	1.7E-05	1.2E-06	7.6E-08
-0.9	57.387	15.862	2.757	0.352	0.036	0.003	0.0003	1.8E-05	1.3E-06	8.2E-08
-0.5	57.669	16.283	2.944	0.396	0.044	0.004	0.0004	2.8E-05	2.1E-06	1.5E-07
0	58.036	16.873	3.215	0.465	0.056	0.006	0.0006	4.9E-05	4.1E-06	3.3E-07
0.5	58.470	17.562	3.541	0.552	0.072	0.008	0.0009	8.7E-05	8.1E-06	7.4E-07
0.9	58.866	18.195	3.852	0.639	0.09	0.011	0.00130	0.00014	1.4E-05	1.4E-06
0.95	58.87	18.251	3.885	0.650	0.092	0.012	0.00136	0.00015	1.5E-05	1.6E-06

TABLE D.9: Prices of zero coupon bond under jump diffusion model with FGM copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	57.797	16.488	3.036	0.419	0.048	0.005	0.0004	3.4E-05	2.7E-06	2.0E-07
-0.9	57.810	16.508	3.045	0.422	0.048	0.005	0.0004	3.5E-05	2.7E-06	2.1E-07
-0.5	57.910	16.669	3.120	0.440	0.051	0.005	0.0005	4.1E-05	3.3E-06	2.5E-07
0	58.036	16.873	3.215	0.465	0.056	0.006	0.001	4.9E-05	4.1E-06	3.3E-07
0.5	58.163	17.079	3.313	0.491	0.060	0.007	0.001	5.9E-05	5.1E-06	4.3E-07
0.9	58.264	17.246	3.394	0.512	0.065	0.007	0.001	6.9E-05	6.2E-06	5.3E-07
0.95	58.277	17.267	3.404	0.515	0.065	0.007	0.001	7.0E-05	6.3E-06	5.5E-07

TABLE D.10: Yield (in %) of zero coupon bond under jump diffusion model with student-t copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	96.006	190.56	286.50	380.36	470.04	554.41	633.00	705.74	772.80	834.48
-0.9	95.963	189.97	285.05	377.81	466.23	549.27	626.50	697.90	763.65	824.08
-0.5	95.028	184.58	272.96	357.58	437.05	510.80	578.75	641.07	698.11	750.24
0	92.802	176.40	256.69	332.05	401.77	465.72	524.12	577.28	625.64	669.63
0.5	89.425	166.46	238.52	304.91	365.51	420.53	470.36	515.45	556.23	593.17
0.9	85.587	156.57	221.48	280.36	333.49	381.32	424.34	463.06	499.34	530.85
0.95	84.961	155.09	219.01	276.86	328.98	375.85	417.93	456.18	490.39	521.65

TABLE D.11: Yield (in %) of zero coupon bond under jump diffusion model with Gaussian copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	74.340	151.45	231.86	312.16	390.11	464.40	534.33	599.63	660.27	716.40
-0.9	74.255	151.09	231.02	310.69	387.91	461.40	530.51	594.97	654.80	710.13
-0.5	73.405	147.82	223.88	298.54	370.03	437.40	500.23	558.45	612.17	661.62
0	72.306	143.45	214.50	282.99	347.64	407.88	463.55	514.75	561.72	604.73
0.5	71.029	138.62	204.52	266.92	325.00	378.54	427.61	472.44	513.32	550.59
0.9	69.880	134.44	196.11	253.64	306.59	354.99	399.04	439.07	475.41	508.42
0.95	69.866	134.08	195.26	252.22	304.58	352.36	395.83	435.29	471.11	503.63

TABLE D.12: Yield (in %) of zero coupon bond under jump diffusion model with FGM copula dependence structure for years to maturity 1–10

θ	1	2	3	4	5	6	7	8	9	10
-0.95	73.019	146.27	220.54	292.99	362.01	426.79	487.01	542.67	593.91	641.00
-0.9	72.982	146.12	220.22	292.46	361.24	425.78	485.75	541.17	592.18	639.05
-0.5	72.681	144.93	217.67	288.22	355.15	417.74	475.78	529.29	578.47	623.59
0	72.306	143.45	214.50	282.99	347.64	407.88	463.55	514.75	561.72	604.73
0.5	71.931	141.97	211.36	277.83	340.26	398.20	451.58	500.55	545.37	586.35
0.9	71.632	140.80	208.88	273.75	334.44	390.59	442.19	489.43	532.59	572.00
0.95	71.595	140.65	208.57	273.25	333.72	389.64	441.02	488.05	531.01	570.23

D.3 Daily changes in calibrated parameters of Microsoft Inc ZCB price

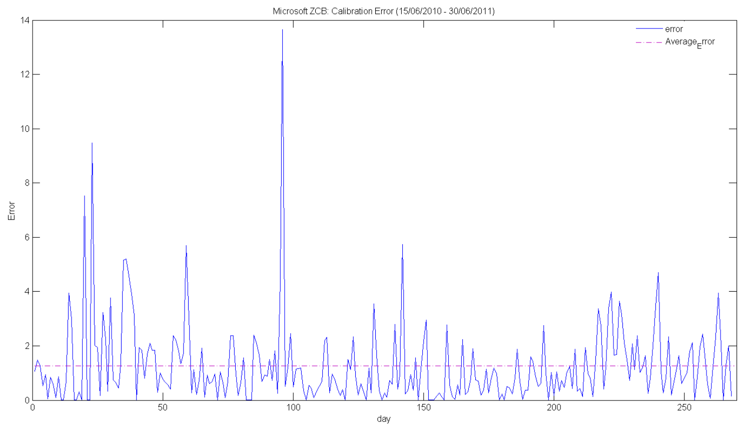


FIGURE D.5: 1-year calibrated error

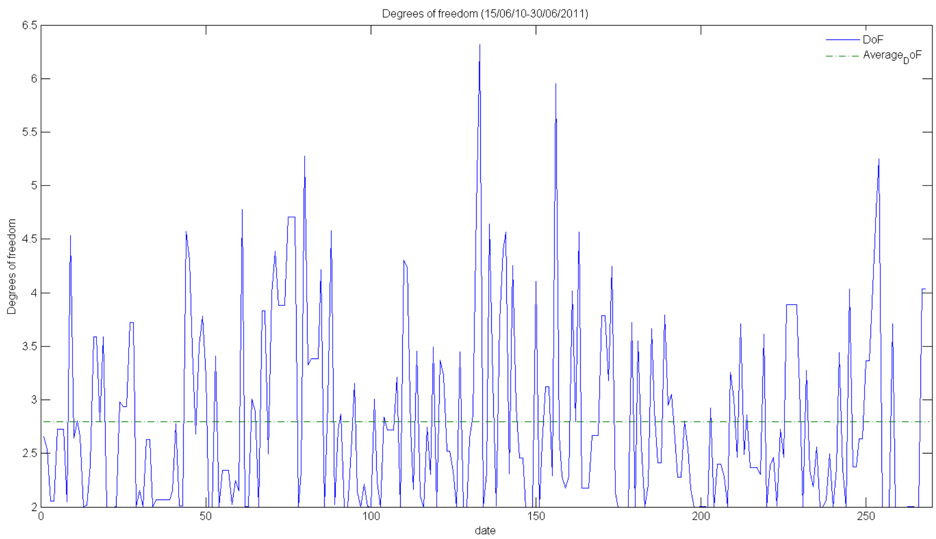


FIGURE D.6: 1-year calibrated degrees of freedom

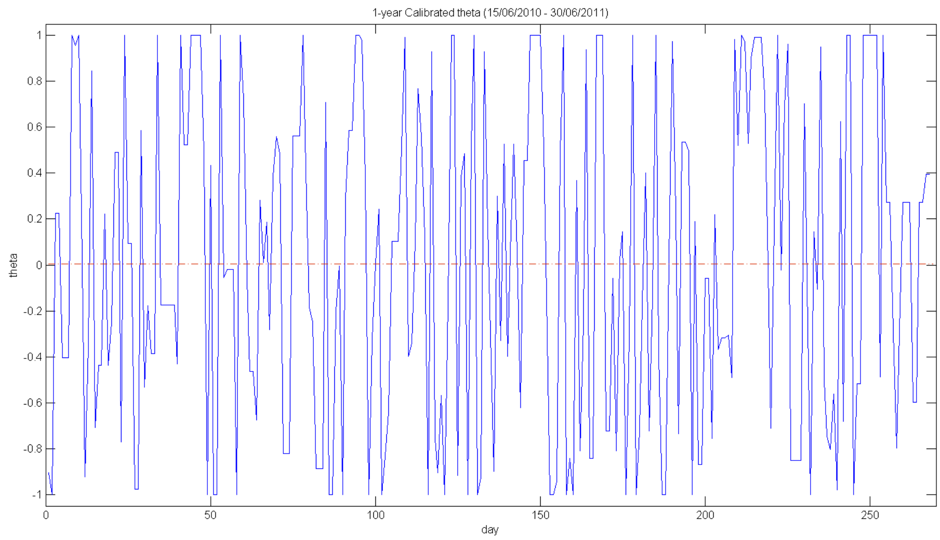


FIGURE D.7: 1-year calibrated θ

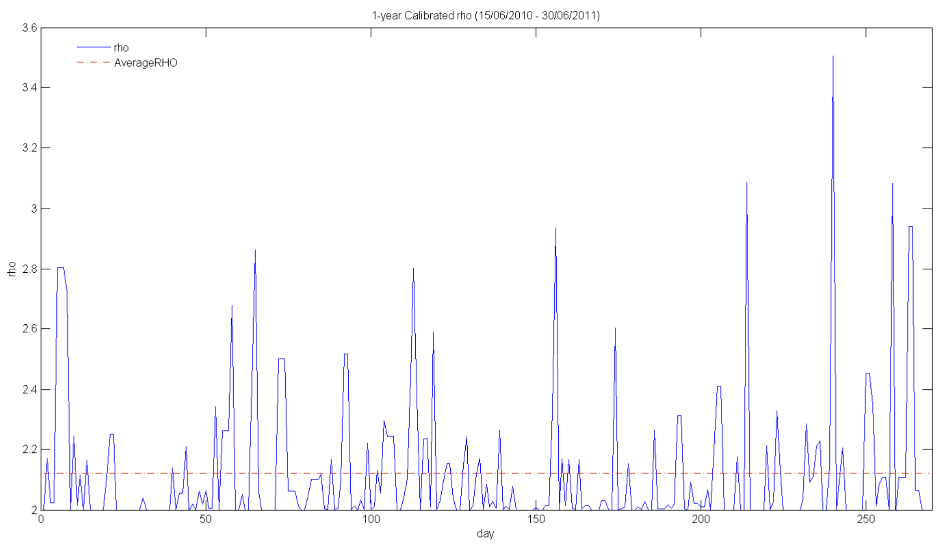


FIGURE D.8: 1-year calibrated ρ

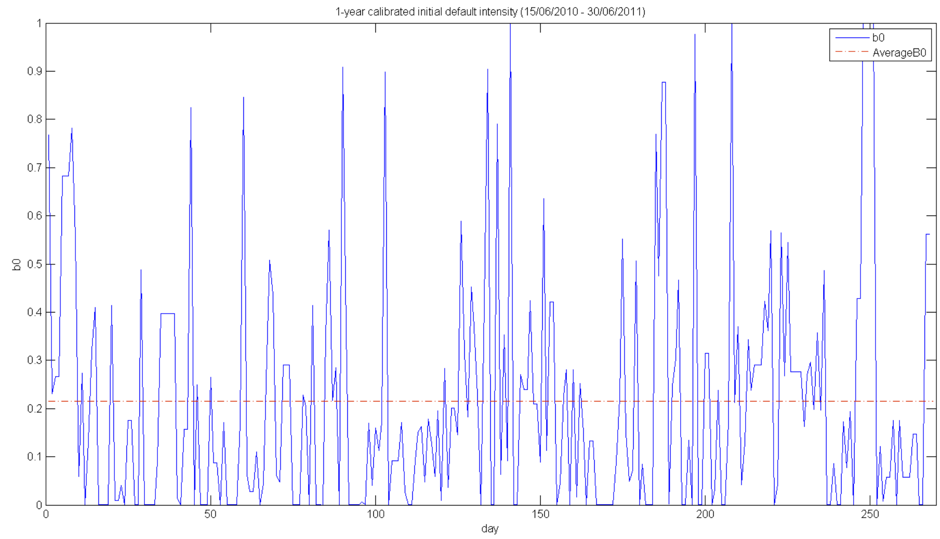


FIGURE D.9: 1-year calibrated $X_0^{(1)}$

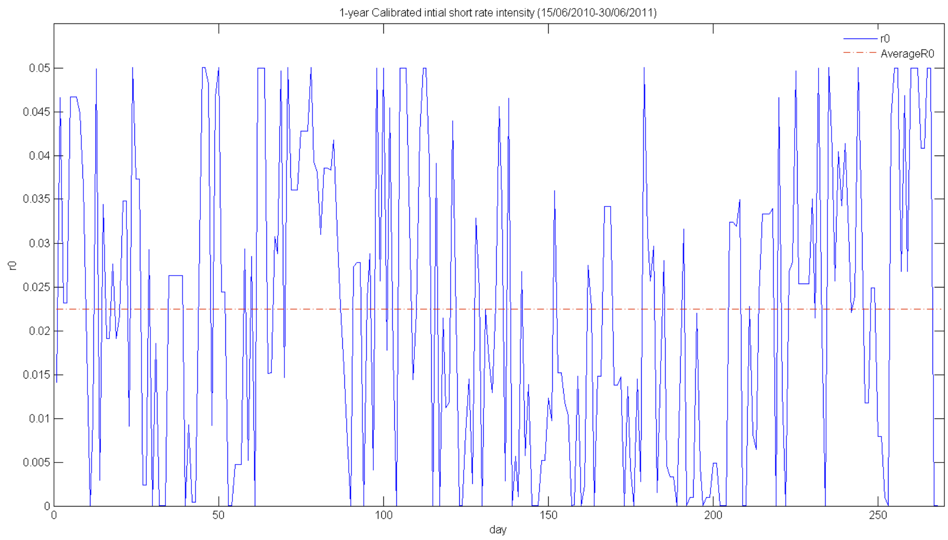


FIGURE D.10: 1-year calibrated $X_0^{(2)}$

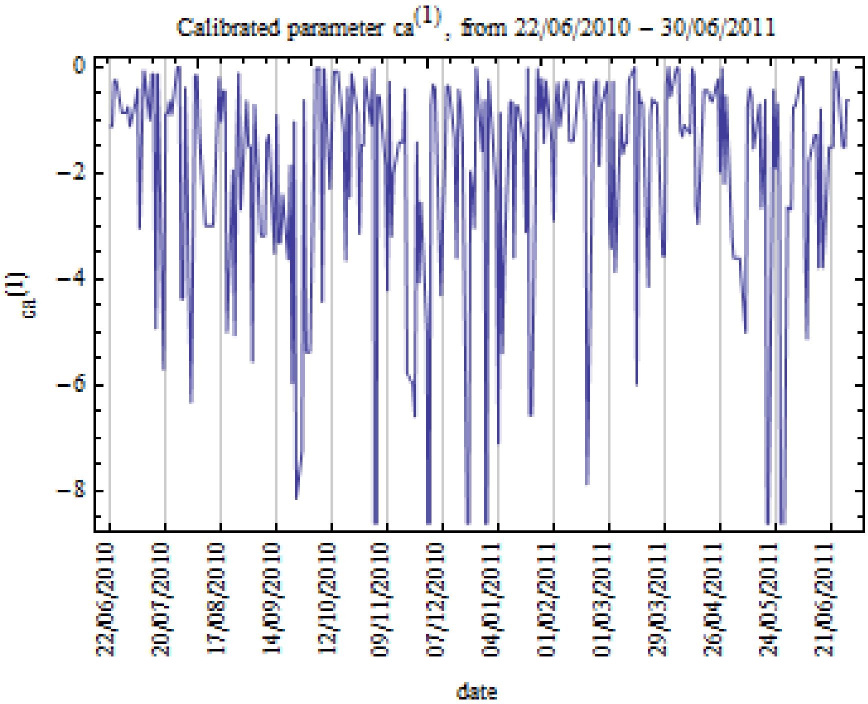


FIGURE D.11: 1-year calibrated $ca^{(1)}$ (decay rate)

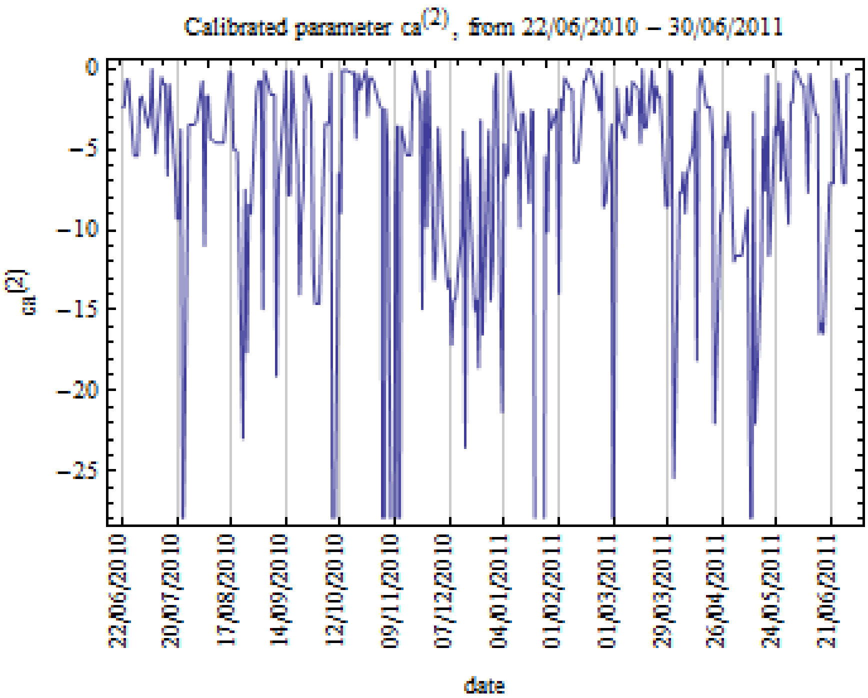


FIGURE D.12: 1-year calibrated $ca^{(2)}$ (decay rate)

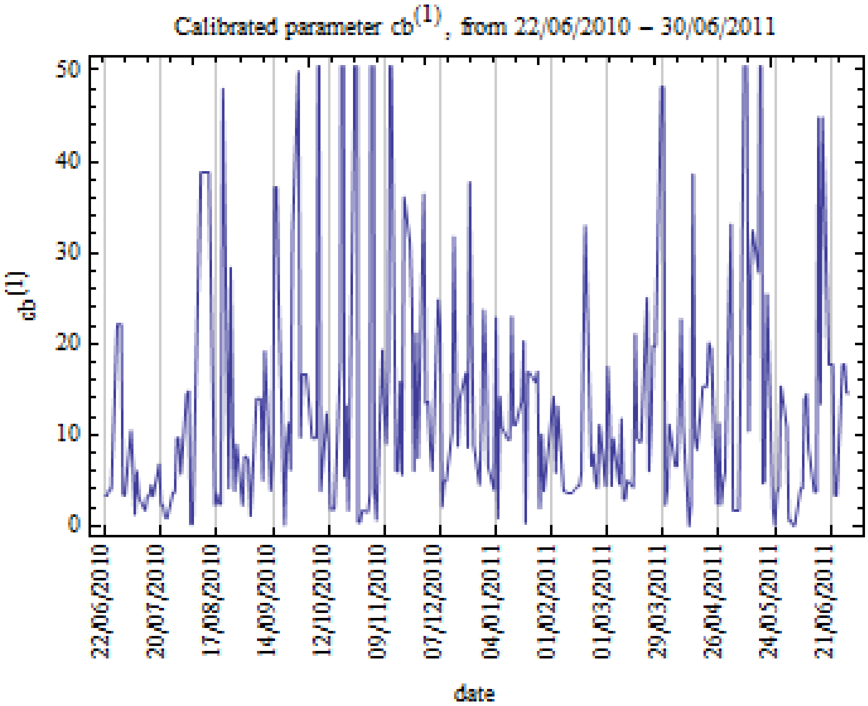


FIGURE D.13: 1-year calibrated $cb^{(1)}$ (constant reversion level)

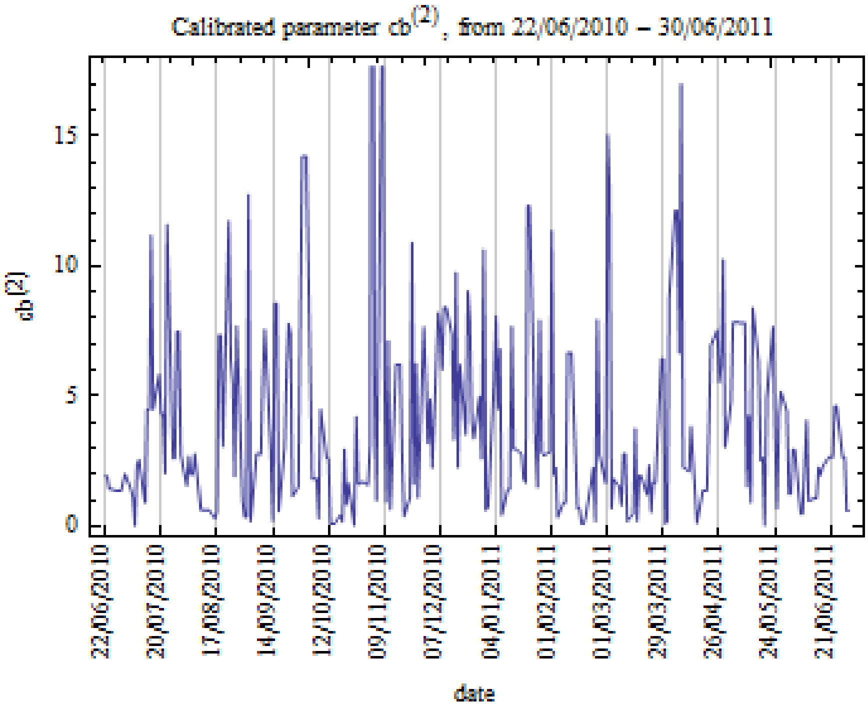


FIGURE D.14: 1-year calibrated $cb^{(2)}$ (constant reversion level)

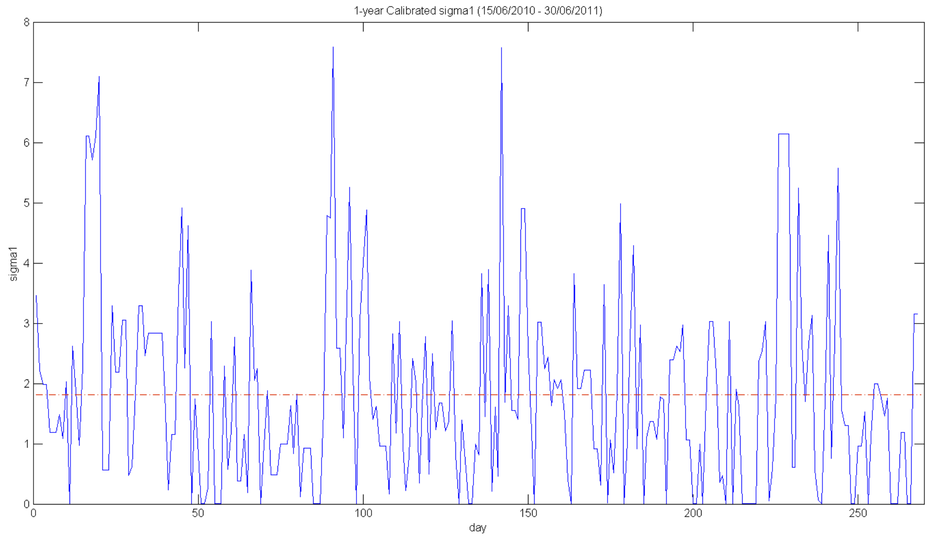


FIGURE D.15: 1-year calibrated $\phi_{(1)}$ (volatility of elliptical copula)

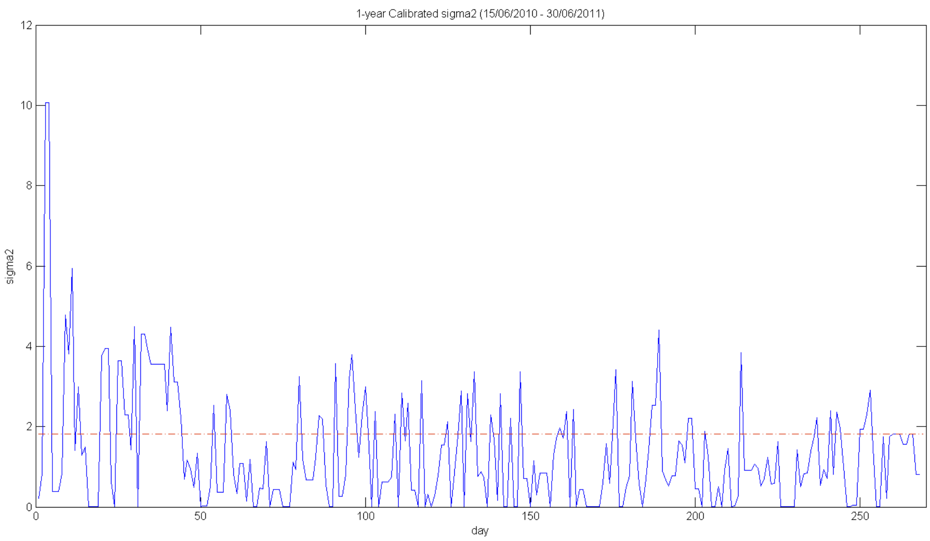


FIGURE D.16: 1-year calibrated $\phi_{(2)}$ (volatility of elliptical copula)

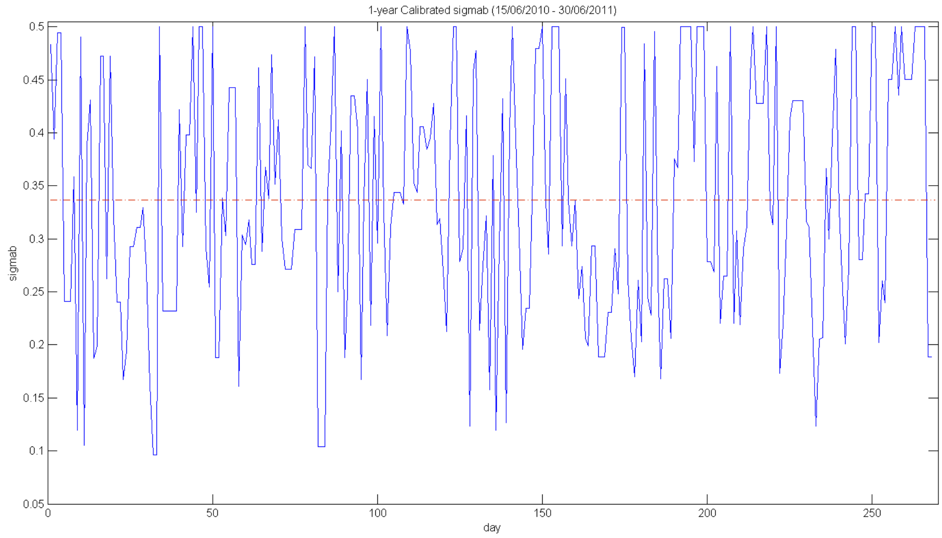


FIGURE D.17: 1-year calibrated $\sigma^{(1)}$

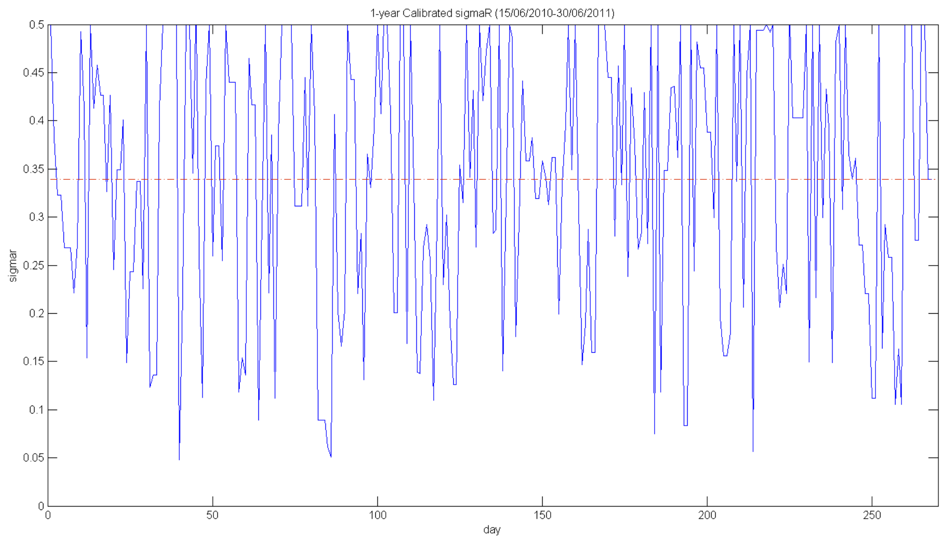


FIGURE D.18: 1-year calibrated $\sigma^{(2)}$

D.4 One-Year Microsoft Inc. Zero Coupon Bond Mkt Data and Mod. Price & Yield

Date	Mkt Price $	Mod. Price	Mkt Yield %	Mod. Yield	Rel. Error
22/6/10	100.72	100.906	2.307	-0.302	0.0018508
23/6/10	100.09	100.097	2.432	-0.033	6.89E-05
24/6/10	99.92	101.192	2.536	-0.397	0.012735
25/6/10	99.53	100.062	2.645	-0.021	0.0053492
28/6/10	99.14	100.094	2.775	-0.032	0.0096036
29/6/10	99	99.057	3.068	0.320	0.0005598
30/6/10	98.87	98.0314	3.179	0.674	0.0084975
1/7/10	98.79	98.2835	3	0.587	0.0051268
2/7/10	99.06	99.1539	2.936	0.288	0.0009253
6/7/10	99.66	99.66	2.674	0.116	1.172E-10
7/7/10	100.13	100.678	2.5	-0.230	1.203E-12
8/7/10	100.37	96.8662	2.384	1.090	0.0067928
9/7/10	99.71	95.7598	2.503	1.488	0.0396173
12/7/10	99.72	102.71	2.132	-0.910	0.0299878
13/7/10	101.55	101.55	2.01	-0.525	6.04E-13
14/7/10	101.72	101.72	1.894	-0.582	4.302E-12
15/7/10	101.9	102.209	1.887	-0.746	0.0030356
16/7/10	96.83	96.83	2.086	1.111	1.264E-09
19/7/10	101.83	94.305	1.944	2.038	0.0738976
20/7/10	102.02	102.02	1.821	-0.686	3.475E-09
21/7/10	101.41	101.41	1.976	-0.481	1.182E-07
22/7/10	102.37	92.8977	1.714	2.574	0.0925297
23/7/10	101.63	99.6171	1.729	0.133	0.0198064
26/7/10	102.77	104.718	1.615	-1.584	0.0189501
27/7/10	102.87	103.036	1.596	-1.031	0.0016183
28/7/10	102.73	99.4985	1.668	0.175	0.0314566
29/7/10	101.25	98.9932	1.643	0.352	0.0222894
30/7/10	102.31	101.974	1.725	-0.677	0.0032845
2/8/10	103.02	99.2503	1.534	0.263	0.0365918
3/8/10	102.86	103.614	1.613	-1.231	0.007306
4/8/10	102.66	103.305	1.707	-1.129	0.0062849
5/8/10	102.27	101.829	1.787	-0.632	0.0043149
6/8/10	102.37	101.041	1.762	-0.362	0.0129823
9/8/10	102.68	107.85	1.741	-2.617	0.050346
10/8/10	102	107.206	1.959	-2.415	0.051044
11/8/10	101.98	106.567	1.984	-2.212	0.0449822
12/8/10	102.02	105.932	1.96	-2.008	0.0383443
13/8/10	102.17	105.3	1.974	-1.803	0.030638
16/8/10	102.13	102.13	1.909	-0.742	1.678E-08

*Mkt. = Market; Mod. = Model; Rel. = Relative

Date	Mkt Price $	Mod. Price	Mkt Yield %	Mod. Yield	Rel. Error
17/8/10	102.38	100.434	1.776	-0.153	0.0190035
18/8/10	102.59	104.426	1.759	-1.522	0.017894
19/8/10	102.32	103.133	1.883	-1.087	0.0079468
20/8/10	102.31	100.629	1.95	-0.222	0.0164347
23/8/10	101.84	103.944	1.964	-1.367	0.0206578
24/8/10	101.36	103.207	2.055	-1.118	0.0182195
25/8/10	102.05	100.209	2.025	-0.074	0.0180376
26/8/10	101.66	101.375	2.098	-0.486	0.0028024
27/8/10	101.8	102.798	2.06	-0.981	0.0098031
30/8/10	101.59	100.819	2.172	-0.292	0.0075937
31/8/10	101.7	102.356	2.223	-0.831	0.0064533
1/9/10	102.18	101.604	2.081	-0.569	0.0056387
2/9/10	102.11	102.524	2.094	-0.892	0.0040519
3/9/10	102.412	100.045	1.954	-0.016	0.0231145
7/9/10	102.266	104.061	2.069	-1.427	0.0175495
8/9/10	102.152	103.493	2.107	-1.233	0.0131242
9/9/10	102.233	103.958	2.067	-1.394	0.0168714
10/9/10	101.972	96.2656	2.084	1.388	0.0559602
13/9/10	103.184	106.221	1.604	-2.168	0.0294679
14/9/10	103.542	103.813	1.626	-1.351	0.0026126
15/9/10	103.412	104.545	1.606	-1.604	0.0097038
16/9/10	103.708	103.91	1.529	-1.387	0.0019465
17/9/10	103.778	104.509	1.463	-1.595	0.0070417
20/9/10	103.824	101.905	1.396	-0.688	0.0184869
21/9/10	103.626	103.513	1.518	-1.256	0.0010912
22/9/10	102.716	103.64	1.736	-1.302	0.008998
23/9/10	102.714	102.1	1.784	-0.759	0.0059778
24/9/10	102.978	102.299	1.689	-0.831	0.0065955
27/9/10	102.796	101.831	1.729	-0.666	0.009385
28/9/10	102.748	102.748	1.774	-0.995	2.448E-14
29/9/10	102.586	103.638	1.844	-1.310	0.0102525
30/9/10	102.4	103.046	1.839	-1.102	0.0063084
1/10/10	102.554	102.458	1.879	-0.894	0.0009403
4/10/10	102.13	101.436	1.997	-0.527	0.0067967
5/10/10	102.322	99.9484	1.93	0.019	0.0231975
6/10/10	102.316	98.4844	1.863	0.569	0.0231988
7/10/10	102.774	103.725	1.818	-1.352	0.0092527
8/10/10	102.632	102.475	1.812	-0.906	0.0017091
12/10/10	102.536	104.1	1.724	-1.491	0.01521
13/10/10	103.132	103.132	1.542	-1.148	1.493E-08
14/10/10	103.022	103.022	1.575	-1.109	2.83E-08
15/10/10	103.042	103.042	1.468	-1.118	1.174E-08
18/10/10	103.536	101.147	1.374	-0.428	0.0230785

Date	Mkt Price $	Mod. Price	Mkt Yield %	Mod. Yield	Rel. Error
19/10/10	103.121	101.023	1.647	-0.383	0.0203455
20/10/10	103.278	104.939	1.58	-1.801	0.0160803
21/10/10	103.22	103.907	1.52	-1.436	0.0066547
22/10/10	103.438	104.364	1.528	-1.601	0.0089483
25/10/10	103.438	102.55	1.625	-0.950	0.0085851
26/10/10	103.818	105.314	1.359	-1.945	0.0144101
27/10/10	104.022	103.293	1.299	-1.223	0.0070046
28/10/10	104.358	102.531	1.218	-0.946	0.0175096
29/10/10	104.834	105.076	1.088	-1.867	0.0023064
1/11/10	104.998	100.564	0.974	-0.214	0.0422288
2/11/10	105.496	119.153	0.822	-6.478	0.129457
3/11/10	105.348	104.846	0.961	-1.794	0.004764
4/11/10	105.45	104.256	0.923	-1.584	0.0113215
5/11/10	105.53	107.982	1.028	-2.901	0.0232396
8/11/10	105.252	103.951	0.977	-1.479	0.004555
9/11/10	105.575	104.432	0.85	-1.656	0.0108303
10/11/10	105.719	106.878	0.819	-2.531	0.0109611
12/11/10	105.008	106.154	1.08	-2.280	0.0033631
15/11/10	104.974	104.991	1.057	-1.869	0.0001577
16/11/10	104.564	104.026	1.236	-1.519	0.0051499
17/11/10	104.442	105.894	1.346	-2.199	0.0043668
18/11/10	104.574	104.482	1.209	-1.690	0.0008765
19/11/10	104.537	104.214	1.301	-1.593	0.0030871
22/11/10	104.5	104.001	1.283	-1.520	0.0047772
23/11/10	104.142	103.448	1.485	-1.316	0.0066615
24/11/10	104.2	102.01	1.427	-0.776	0.0210166
26/11/10	104.261	103.996	1.427	-1.524	0.0025448
29/11/10	104.195	105.158	1.387	-1.959	0.0092396
30/11/10	104.212	103.46	1.428	-1.330	0.0072167
1/12/10	105.142	104.73	1.103	-1.805	0.003922
2/12/10	105.67	105.83	0.78	-2.211	0.0015185
3/12/10	105.955	106.35	0.725	-2.403	0.0037318
6/12/10	105.986	105.969	0.857	-2.271	0.0001638
7/12/10	105.841	107.347	0.806	-2.774	0.0142269
8/12/10	106.018	107.149	0.684	-2.705	0.0106693
9/12/10	106.248	108.364	0.755	-3.143	0.022027
10/12/10	106.296	107.202	0.667	-2.730	0.0084242
13/12/10	106.324	106.137	0.754	-2.350	0.0017629
14/12/10	106.238	106.846	0.619	-2.613	0.0057222
15/12/10	106.601	106.908	0.532	-2.638	0.0028813
16/12/10	106.676	106.661	0.475	-2.551	0.0001384
17/12/10	106.844	105.666	0.508	-2.186	0.0110289
20/12/10	106.66	106.928	0.473	-2.660	0.0025122
21/12/10	106.918	110.465	0.371	-3.930	0.0331716
22/12/10	107.649	109.319	0.276	-3.530	0.015509
23/12/10	107.66	107.31	0.242	-2.808	0.0032545

Date	Mkt Price $	Mod. Price	Mkt Yield %	Mod. Yield	Rel. Error
27/12/10	107.814	108.088	0.313	-3.105	0.0025394
28/12/10	107.644	107.54	0.33	-2.908	0.0009636
29/12/10	107.596	108.315	0.36	-3.194	0.0066794
30/12/10	107.201	107.797	0.326	-3.009	0.0055625
31/12/10	107.299	104.509	0.372	-1.781	0.0260058
3/1/11	107.474	107.071	-0.069	-2.754	0.0037538
4/1/11	107.629	108.55	-0.092	-3.301	0.0085534
5/1/11	107.705	113.451	-0.053	-5.038	0.0533501
6/1/11	108.88	108.661	-0.427	-3.349	0.00201
7/1/11	108.53	108.896	-0.314	-3.439	0.0033721
10/1/11	107.86	108.814	-0.184	-3.420	0.0088458
11/1/11	107.86	107.483	-0.137	-2.932	0.0034913
12/1/11	108.39	106.825	-0.356	-2.689	0.0144429
13/1/11	107.65	107.65	-0.157	-3.001	6.265E-09
14/1/11	108.12	106.969	-0.209	-2.749	0.0106462
18/1/11	108.8	105.84	-0.371	-2.332	0.0272017
19/1/11	108.4	105.97	-0.268	-2.384	0.0304703
20/1/11	108.2	108.2	-0.203	-3.230	3.694E-08
21/1/11	107.74	107.74	-0.024	-3.062	5.383E-09
24/1/11	108.3	108.141	-0.201	-3.223	0.0014647
25/1/11	108.17	107.907	-0.189	-3.139	0.0024293
26/1/11	108.45	108.334	-0.3	-3.303	0.00107
27/1/11	109.36	109.36	-0.337	-3.688	1.597E-09
28/1/11	107.325	104.544	0.004	-1.851	0.0259077
31/1/11	107.187	107.762	0.362	-3.105	0.005366
1/2/11	107.637	107.765	-0.02	-3.110	0.0011889
2/2/11	107.525	107.503	0.005	-3.014	0.0002078
3/2/11	107.262	106.692	0.155	-2.706	0.0053105
4/2/11	107.287	107.093	0.357	-2.863	0.0018113
7/2/11	107.887	105.65	-0.103	-2.311	0.0207329
8/2/11	107.987	108.205	-0.15	-3.303	0.002015
9/2/11	107.537	107.22	-0.04	-2.929	0.0029516
10/2/11	106.918	106.132	0.148	-2.509	0.0073539
11/2/11	106.95	105.055	0.267	-2.086	0.0177168
14/2/11	106.875	107.63	0.552	-3.105	0.0070676
15/2/11	106.793	106.085	0.409	-2.505	0.0066259
16/2/11	106.793	106.989	0.253	-2.863	0.0018349
17/2/11	107.231	106.834	0.153	-2.805	0.0037039
18/2/11	107.062	105.919	0.215	-2.448	0.0106768
22/2/11	106.662	107.468	0.441	-3.070	0.0075531
23/2/11	106.75	105.568	0.415	-2.322	0.0110751
24/2/11	106.998	106.029	0.331	-2.509	0.0090546
25/2/11	106.725	106.725	0.442	-2.788	2.774E-09
28/2/11	106.7	106.474	0.351	-2.698	0.0021168
1/3/11	106.193	106.193	0.593	-2.589	1.034E-08
2/3/11	106.075	105.564	0.643	-2.339	0.0048197

Date	Mkt Price $	Mod. Price	Mkt Yield %	Mod. Yield	Rel. Error
3/3/11	106.275	106.714	0.614	-2.804	0.0041304
4/3/11	105.9	106.13	0.749	-2.573	0.0021705
7/3/11	105.631	104.798	1.125	-2.040	0.0078835
8/3/11	105.912	104.042	1.102	-1.729	0.0176566
9/3/11	105.437	106.13	1.059	-2.589	0.0065726
10/3/11	104.946	104.986	1.301	-2.125	0.0003788
11/3/11	105.306	104.941	1.169	-2.109	0.0034633
14/3/11	105.225	105.594	1.2	-2.385	0.0035108
15/3/11	104.618	103.015	1.495	-1.310	0.0153262
16/3/11	103.943	105.37	1.749	-2.299	0.013725
17/3/11	103.95	104.813	1.768	-2.071	0.0083001
21/3/11	104.15	103.528	1.303	-1.539	0.0059756
22/3/11	104.112	108.864	1.379	-3.732	0.0264338
23/3/11	104.456	105.448	1.277	-2.351	0.0094925
24/3/11	104.775	104.77	1.196	-2.070	4.661E-05
25/3/11	104.515	103.486	1.275	-1.528	0.0098412
28/3/11	104.287	104.109	1.427	-1.800	0.0017086
29/3/11	104.35	103.315	1.383	-1.462	0.0099204
30/3/11	104.506	104.174	1.392	-1.833	0.0031813
31/3/11	104.348	103.628	1.429	-1.601	0.0069037
1/4/11	104.275	104.745	1.377	-2.080	0.0045083
4/4/11	104.402	105.449	1.372	-2.386	0.0100328
5/4/11	104.506	103.266	1.347	-1.454	0.0118682
6/4/11	104.743	105.174	1.229	-2.275	0.0041177
7/4/11	104.737	106.614	1.231	-2.883	0.0179217
8/4/11	104.637	104.972	1.316	-2.195	0.003197
11/4/11	104.443	104.877	1.351	-2.163	0.0041599
12/4/11	104.031	103.906	1.497	-1.746	0.0012028
13/4/11	104.162	102.225	1.565	-1.008	0.0185984
14/4/11	103.75	104.743	1.673	-2.113	0.0095679
15/4/11	103.687	102.863	1.679	-1.294	0.0079435
18/4/11	103.5	103.632	1.893	-1.639	0.0012745
19/4/11	103.581	102.137	1.881	-0.976	0.0139432
20/4/11	104.031	100.665	1.612	-0.307	0.0323604
21/4/11	103.687	100.953	1.689	-0.440	0.0263678
25/4/11	103.781	102.565	1.698	-1.177	0.0117214
26/4/11	104.193	100.801	1.458	-0.373	0.0325519
27/4/11	104.451	108.443	1.387	-3.727	0.0382164
28/4/11	104.693	103.032	1.251	-1.392	0.0158645
29/4/11	103.937	105.613	1.594	-2.533	0.0161293
2/5/11	103.745	100.094	1.711	-0.044	0.0351944
3/5/11	103.92	107.094	1.645	-3.184	0.0305447
4/5/11	104.112	106.235	1.656	-2.819	0.0203927
5/5/11	103.895	105.383	1.735	-2.452	0.0143215
6/5/11	103.808	104.538	1.709	-2.082	0.0070282
9/5/11	103.766	105.842	1.764	-2.666	0.020003

Date	Mkt Price $	Mod. Price	Mkt Yield %	Mod. Yield	Rel. Error
10/5/11	103.595	102.472	1.837	-1.157	0.0108395
11/5/11	102.966	100.596	2.022	-0.283	0.0230179
12/5/11	102.95	103.969	2.068	-1.842	0.0098966
13/5/11	102.72	103.926	2.183	-1.825	0.0117374
16/5/11	102.329	100.694	2.351	-0.332	0.0159828
17/5/11	102.454	102.206	2.364	-1.044	0.0024158
18/5/11	102.545	103.342	2.348	-1.571	0.0077712
19/5/11	102.583	100.59	2.399	-0.283	0.0194282
20/5/11	102.391	98.9461	2.435	0.513	0.0336449
23/5/11	102.129	106.835	2.548	-3.154	0.0460764
24/5/11	102.158	101.088	2.566	-0.524	0.0104711
25/5/11	102.187	102.447	2.614	-1.168	0.0025464
26/5/11	102.37	103.179	2.534	-1.511	0.0079066
27/5/11	102.422	104.763	2.511	-2.242	0.0228557
31/5/11	102.6	101.905	2.49	-0.920	0.0067697
1/6/11	102.185	101.002	2.65	-0.488	0.0115752
2/6/11	102.005	103.647	2.781	-1.744	0.0160967
3/6/11	101.77	102.381	2.724	-1.151	0.0060082
6/6/11	101.896	102.719	2.679	-1.316	0.008076
7/6/11	101.851	100.819	2.646	-0.403	0.0101356
8/6/11	101.844	103.649	2.744	-1.759	0.017722
9/6/11	101.837	103.958	2.756	-1.907	0.0208265
10/6/11	101.666	101.668	2.845	-0.818	1.85E-05
13/6/11	101.896	101.042	2.746	-0.516	0.0083773
14/6/11	101.825	99.8752	2.693	0.062	0.0191486
15/6/11	101.5	103.931	2.881	-1.909	0.0239525
16/6/11	101.45	102.867	2.801	-1.405	0.0139667
17/6/11	101.638	102.231	2.751	-1.100	0.0058354
20/6/11	101.831	101.903	2.708	-0.945	0.0007076
21/6/11	101.994	100.762	2.684	-0.382	0.0120774
22/6/11	101.942	99.6344	2.739	0.185	0.0226362
23/6/11	101.912	105.851	2.787	-2.834	0.0386545
24/6/11	101.687	103.699	2.952	-1.822	0.0197902
27/6/11	102.014	102.028	2.699	-1.015	0.0001356
28/6/11	102.174	100.888	2.618	-0.449	0.0125858
29/6/11	102.062	104.061	2.763	-2.009	0.0195846
30/6/11	102.333	102.481	2.623	-1.243	0.0014482

D.5 Daily values of calibrated parameters, date 22 June 2010 - 30 June 2011

TABLE D.13: Daily values of calibrated parameters: Initial intensities, decay rates, constant reversion level and degrees of freedom

Date	$X_0^{(1)}$	$X_0^{(2)}$	$ca^{(1)}$	$ca^{(2)}$	$cb^{(1)}$	$cb^{(2)}$	DoF
22/06/10	9.735E-05	-1.1240	-2.3476	3.3496	1.9320	2.4902	0.0404
23/06/10	9.735E-05	-1.1240	-2.3476	3.3496	1.9320	2.4902	0.0404
24/06/10	0.266	-0.2536	-0.6847	4.0145	1.4306	2.0538	0.0232
25/06/10	0.266	-0.2536	-0.6847	4.0145	1.4306	2.0538	0.0232
28/06/10	0.682	-0.8651	-5.3831	22.1318	1.3521	2.7285	0.0467
29/06/10	0.682	-0.8651	-5.3831	22.1318	1.3521	2.7285	0.0467
30/06/10	0.682	-0.8651	-5.3831	22.1318	1.3521	2.7285	0.0467
1/07/10	0.487	-0.7433	-1.8438	3.4600	1.6702	2.9376	0.0282
2/07/10	0.555	-1.1240	-1.67210	3.3496	2.0220	4.5329	0.0365
6/07/10	0.273	-0.4128	-2.8044	6.6484	1.0898	2.8042	9.94E-12
7/07/10	7.738E-05	-3.0489	-0.0204	1.2424	0.0276	2.6596	0.0100
8/07/10	0.131	-1.5929	-3.6294	6.0653	2.3416	2.0002	0.0499
9/07/10	0.321	-0.0680	-5.2639	3.1271	2.5499	2.0188	0.0030
12/07/10	0.409	-1.0033	-0.5139	1.7040	0.9057	2.3998	0.0344
13/07/10	0.0001	-0.1305	-0.9595	3.2483	4.4748	3.5899	0.0191
14/07/10	6.972E-05	-0.1299	-0.9595	3.2490	4.4745	3.58986	0.0191
15/07/10	7.179E-05	-4.933	-6.6359	4.5301	11.1670	2.8040	0.0276
16/07/10	0.0001	-0.1307	-0.9594	3.248	4.4749	3.5899	0.0191
19/07/10	0.414	-5.7151	-6.4419	6.7748	5.8332	2	0.0217
20/07/10	0.009	-0.8813	-9.3107	2.2962	4.3327	2	0.0348
21/07/10	0.009	-0.8809	-9.3107	2.2962	4.3328	2	0.0348
22/07/10	0.041	-0.6354	-3.7668	0.9419	2.0342	2.00003	0.0091
23/07/10	7.398E-08	-0.9063	-30.2251	0.8245	11.5942	2.9822	0.05
26/07/10	0.175	-0.0000001	-3.4873	3.6904	2.6076	2.9377	0.0373
27/07/10	0.175	-0.0000001	-3.4873	3.6904	2.6076	2.9377	0.0373
28/07/10	9.556E-07	-4.3714	-3.4713	9.6704	7.4741	3.7253	0.0024
29/07/10	9.556E-07	-4.3714	-3.4713	9.6704	7.4741	3.7253	0.0024
30/07/10	0.488	-0.3929	-3.2875	5.7110	2.7355	2.0005	0.0292
2/08/10	2.839E-08	-6.3303	-0.7643	14.5636	1.5550	2.1539	2.96E-09
3/08/10	6.745E-08	-3.2055	-11.0106	14.8234	2.6820	2.0020	0.0186
4/08/10	6.737E-08	-0.1611	-1.6731	0.2474	1.9969	2.6312	2.74E-06
5/08/10	6.737E-08	-0.1611	-1.6731	0.2474	1.9969	2.6312	2.74E-06
6/08/10	0.090	-1.2240	-4.3282	11.0257	2.7768	2.0020	1.93E-08
9/08/10	0.397	-3.0005	-4.6113	38.7958	0.6207	2.0651	0.0263
10/08/10	0.397	-3.0005	-4.6113	38.7958	0.6207	2.0651	0.0263
11/08/10	0.397	-3.0005	-4.6113	38.7958	0.6207	2.0651	0.0263
12/08/10	0.397	-3.0005	-4.6113	38.7958	0.6207	2.0651	0.0263
13/08/10	0.397	-3.0005	-4.6113	38.7958	0.6207	2.0651	0.0263
16/08/10	0.0139	-0.1999	-0.1550	2.2779	0.2802	2.1494	0.0476
17/08/10	9.546E-09	-1.0324	-0.2564	3.5198	0.5447	2.7754	0.0093
18/08/10	0.157	-0.4635	-5.0010	2.3771	7.3369	2.0111	0.00047

TABLE D.14: Daily values of calibrated parameters: Jump sizes, dependence parameter, average jump frequency, diffusion rates copula SD

$\mu^{(1)}$	$\mu^{(2)}$	θ	ρ	ϕ_1	ϕ_2	$\sigma^{(1)}$	$\sigma^{(2)}$
2.5461	1.0000	-0.9911	2.0000	2.3013	0.4726	0.1199	0.2077
2.6122	2.1120	-0.9911	2.0000	2.3013	0.4726	0.1199	0.2077
1.0303	1.4049	0.225214	2.0234	1.9869	10.0689	0.4946	0.3228
1.0303	1.4049	0.225214	2.0234	1.9869	10.0689	0.4946	0.3228
2.9092	2.1783	-0.4045	2.8037	1.1866	0.3883	0.2409	0.2679
2.9092	2.1783	-0.4045	2.8037	1.1866	0.3883	0.2409	0.2679
2.9092	2.1783	-0.4045	2.8037	1.1866	0.3883	0.2409	0.2679
6.6357	2.9766	-0.0408	2.0000	0.5217	6.5279	0.1405	0.2087
2.5337	1.2732	0.955655	2.0000	1.0834	4.7760	0.1196	0.2724
2.8019	2.3383	0.017395	2.0154	0.0003	5.9327	0.1053	0.4120
1.1235	3.1486	-0.92403	2.1144	2.6137	1.4002	0.3880	0.1534
1.0000	2.3502	-0.34669	2.0001	1.8703	2.9851	0.4311	0.4997
3.0536	1.1610	0.84513	2.1641	0.9743	1.2955	0.1876	0.4131
4.1816	1.0618	-0.7078	2	2.2930	1.4902	0.1991	0.4578
3.2596	1.1281	-0.4378	2.0004	6.1086	3.041E-05	0.4724	0.4265
3.2596	1.1281	-0.4378	2.0004	6.1086	3.041E-05	0.4722	0.4265
1.7737	1.9490	0.2228	2.0006	5.7058	2.656E-08	0.2626	0.3264
3.2596	1.1281	-0.4378	2.0005	6.1086	3.041E-05	0.4725	0.4264
4.3072	1.0000	-0.2497	2.1080	7.0996	3.7597	0.3213	0.2453
2.2676	1.0000	0.4904	2.2520	0.5645	3.9394	0.2403	0.3492
2.2676	1.0000	0.4904	2.2520	0.5645	3.9394	0.2403	0.3492
1.0000	1.9548	-0.77163	2	0.5659	0.6204	0.1670	0.4009
1.6539	1.0000	1	2.0009	3.2907	2.24E-06	0.1945	0.1491
1.0885	1.0000	0.0950	2.00002	2.1916	3.6374	0.2926	0.2434
1.0885	1.0000	0.0950	2.00002	2.1916	3.6374	0.2926	0.2434
1.0000	1.0000	-0.9758	2	3.0555	2.2945	0.3109	0.3373
1.0000	1.0000	-0.9758	2	3.0555	2.2945	0.3109	0.3373
2.6154	1.0000	0.58710	2.00001	0.4704	1.4316	0.3298	0.2258
5.8232	1.3842	-0.5313	2	0.6209	4.4984	0.2703	0.5000
3.8487	1.4889	-0.1774	2.0397	1.8363	3.084E-05	0.1625	0.1232
2.4114	1.0000	-0.3873	2.00007	3.2849	4.307	0.0965	0.1363
2.4114	1.0000	-0.3873	2.00007	3.2849	4.307	0.0965	0.1363
1.9267	1.7797	1	2	2.4562	3.9494	0.5000	0.4124
1.7496	2.1635	-0.1731	2	2.8367	3.5544	0.2316	0.5
1.7496	2.1635	-0.1731	2	2.8367	3.5544	0.2316	0.5
1.7496	2.1635	-0.1731	2	2.8367	3.5544	0.2316	0.5
1.7496	2.1635	-0.1731	2	2.8367	3.5544	0.2316	0.5
1.7496	2.1635	-0.1731	2	2.8367	3.5544	0.2316	0.5
1.8414	1.4312	-0.4317	2.1401	1.8357	2.3904	0.4218	0.0476
2.7716	1.2457	1	2.0018	0.2298	4.4771	0.2927	0.2202
2.8560	2.8835	0.5240	2.0549	1.15855	3.1033	0.3982	0.5

Date	$X_0^{(1)}$	$X_0^{(2)}$	$ca^{(1)}$	$ca^{(2)}$	$cb^{(1)}$	$cb^{(2)}$	DoF
19/08/10	0.1574	-0.4635	-5.0010	2.3771	7.3369	2.0111	0.0005
20/08/10	0.8241	-5.0020	-5.1719	48.0240	3.0622	4.5753	0.0252
23/08/10	1.51E-06	-1.9411	-22.9683	4.1558	11.7333	4.3069	0.05
24/08/10	0.2493	-5.0699	-7.5002	28.3618	6.4460	3.4705	0.05
25/08/10	4.91E-05	-1.5159	-17.6032	10.8233	5.5072	2.6808	0.0482
26/08/10	9.17E-08	-0.1172	-8.4297	3.8587	1.9142	3.5091	0.0091
27/08/10	9.38E-07	-2.6928	-8.9855	8.9770	7.6954	3.7819	0.0467
30/08/10	0.2649	-0.6305	-1.4335	2.2413	0.9793	3.2179	0.05
31/08/10	0.0871	-1.5008	-0.7795	7.5866	0.3497	2	0.0244
1/09/10	0.0871	-1.5008	-0.7795	7.5866	0.3497	2	0.0244
2/09/10	3.27E-05	-5.5627	-14.9375	7.1119	12.7490	3.4086	4.75E-06
3/09/10	0.1705	-0.7094	-0.0996	1.0800	0.1915	2.00001	2.05E-09
7/09/10	8.52E-08	-3.1824	-1.5918	13.9519	2.7638	2.3424	0.0047
8/09/10	8.52E-08	-3.1824	-1.5918	13.9519	2.7638	2.3424	0.0047
9/09/10	0.0014	-1.4251	-19.1068	5.0097	5.6229	2.0245	0.0294
10/09/10	0.1717	-1.2824	-6.9647	19.1493	7.5643	2.2433	0.0052
13/09/10	0.8461	-3.5291	-2.2093	3.8858	2.9513	2.1490	0.0285
14/09/10	0.0626	-0.9036	-0.1061	7.5451	0.2080	4.7795	2.07E-07
15/09/10	0.0271	-3.3203	-7.8479	37.1217	8.5617	2.00001	0.0500
16/09/10	0.0271	-3.3203	-7.8479	37.1217	8.5617	2.00001	0.0500
17/09/10	0.1100	-2.4061	-0.1093	24.8176	0.5579	3.0047	0.0500
20/09/10	3.28E-08	-3.6411	-5.8544	0.1265	3.0591	2.8933	0.0152
21/09/10	0.0453	-1.8505	-14.0345	8.8942	5.9173	2.0010	0.0152
22/09/10	0.3077	-5.9716	-9.0447	11.3245	7.7898	3.8295	0.0308
23/09/10	0.5073	-1.0316	-7.5379	6.1799	7.4291	3.8295	0.0288
24/09/10	0.4363	-8.1569	-0.4053	32.0351	1.1427	2.4981	0.0497
27/09/10	0.0597	-7.2548	-2.3011	49.8531	1.5050	3.9885	0.0146
28/09/10	0.0475	-0.6194	-12.5921	9.7207	8.7117	4.3844	0.05
29/09/10	0.2905	-5.3765	-14.5871	16.6569	14.2082	3.8817	0.0360
30/09/10	0.2905	-5.3765	-14.5871	16.6569	14.2082	3.8817	0.0360
1/10/10	0.2905	-5.3765	-14.5871	16.6569	14.2082	3.8817	0.0360
4/10/10	2.85E-07	-0.0281	-3.3848	9.6622	1.8138	4.709	0.0428
5/10/10	2.85E-07	-0.0281	-3.3848	9.6622	1.8138	4.709	0.0428
6/10/10	2.85E-07	-0.0281	-3.3848	9.6622	1.8138	4.709	0.0428
7/10/10	0.2275	-4.4362	-0.2576	60.6703	0.2843	2	0.05
8/10/10	0.2051	-0.0340	-39.3459	3.8163	4.4769	2.3289	0.0393
12/10/10	0.4137	-1.1178	-8.9411	8.9344	2.5828	3.3273	0.0310
13/10/10	5.79E-05	-0.0865	-0.0863	1.8163	0.1045	3.3820	0.0385
14/10/10	5.79E-05	-0.0865	-0.0863	1.8163	0.1045	3.3820	0.0385
15/10/10	1.79E-08	-0.0853	-0.0843	1.8176	0.1022	3.3820	0.0383
18/10/10	0.3521	-1.2232	-0.2896	17.8707	0.4167	4.2149	0.0418
19/10/10	0.5699	-3.6568	-0.2167	117.8035	0.1760	2	0.0317
20/10/10	0.2163	-0.3896	-4.3377	5.4074	2.9531	2.9720	0.0227
21/10/10	0.2845	-2.4622	-0.2848	13.1863	0.8211	4.5782	0.0158
22/10/10	0.0014	-0.1229	-1.3214	1.6691	1.6554	2.0245	0.0075
25/10/10	0.9088	-0.8568	-0.0567	44.6152	0.0390	2.7237	2.43E-05
26/10/10	0.4614	-3.1510	-2.9427	122.6518	4.2144	2.8686	0.0273
27/10/10	0.0001	-1.4767	-0.5713	0.4454	1.6298	2	0.0278

$\mu^{(1)}$	$\mu^{(2)}$	θ	ρ	ϕ_1	ϕ_2	$\sigma^{(1)}$	$\sigma^{(1)}$
2.8560	2.8835	0.5240	2.0549	1.1586	3.1033	0.3982	0.5
3.8109	1.6994	1.0000	2.2107	3.5028	2.2112	0.5	0.3454
1.1335	2.7716	1	2	4.9175	0.6935	0.3248	0.5
3.3372	2.4643	1	2.0203	2.2526	1.173	0.5	0.2605
2.7128	2.1906	1	2	4.6221	0.9539	0.5	0.1125
2.3000	3.2896	0.4871	2.0630	5.25E-08	0.4804	0.2911	0.4380
2.0176	1.5704	-1	2.0189	1.7391	1.3336	0.2549	0.5000
2.4905	1.2537	0.4329	2.0641	0.8129	7.53E-08	0.5	0.2597
2.7150	2.7391	-1	2.0058	7.98E-08	0.0241	0.1880	0.3738
2.7150	2.7391	-1	2.0058	7.98E-08	0.0241	0.1880	0.3738
1.4219	1.4069	1	2.3427	0.2616	0.5043	0.3386	0.2546
2.3456	2.4047	-0.0551	2	3.0258	2.5372	0.3028	0.5000
5.8077	4.9066	-0.0180	2.2636	0.0072	0.3601	0.4426	0.4403
5.8077	4.9066	-0.0180	2.2636	0.0072	0.3601	0.4426	0.4403
1.0050	3.7302	-1	2.6791	2.2859	2.8078	0.1607	0.1184
2.3911	3.352	1	2	0.5759	2.3983	0.3038	0.1539
2.5250	1.00001	0.7205	2.0001	1.2092	0.8110	0.2947	0.1360
1.0000	1.6567	-0.0540	2.0511	2.7685	0.3320	0.3180	0.4647
4.1904	1.7404	-0.4632	2	0.3769	1.0760	0.2760	0.4168
4.1904	1.7404	-0.4632	2	0.3769	1.0760	0.2760	0.4168
2.7047	1.00001	-0.6769	2.4180	1.1506	0.1391	0.4614	0.0890
1.4260	2.2892	0.2808	2.8617	0.1850	1.1817	0.2879	0.2969
2.8744	2.8742	0.0105	2.0608	3.8863	9.67E-08	0.3679	0.5
1.0442	2.6333	0.1871	2	2.0563	4.37E-07	0.3381	0.2215
2.2006	1.0070	-0.2825	2.0003	2.2401	0.4720	0.4734	0.3853
4.5883	2.8819	0.3873	2.0005	5.62E-07	0.4569	0.3513	0.1121
3.8861	3.0144	0.5589	2	0.9883	1.6271	0.4122	0.3853
2.0034	3.0437	0.4847	2.0001	1.8844	7.96E-07	0.3006	0.5
2.8246	2.3429	-0.8213	2.5000	0.4851	0.4415	0.2711	0.5000
2.8246	2.3429	-0.8213	2.5000	0.4851	0.4415	0.2711	0.5000
2.8246	2.3429	-0.8213	2.5000	0.4851	0.4415	0.2711	0.5000
1.1894	1.0339	0.5633	2.0634	0.9989	6.77E-07	0.3091	0.3111
1.1894	1.0339	0.5633	2.0634	0.9989	6.77E-07	0.3091	0.3111
1.1894	1.0339	0.5633	2.0634	0.9989	6.77E-07	0.3091	0.3111
1.4453	3.6266	1.0000	2.0117	1.6305	1.1080	0.5	0.4448
1.0000	1.6951	0.4149	2.0010	0.8364	0.9356	0.3695	0.3111
1.5940	1.8117	-0.2484	2.0470	0.1202	1.2034	0.4714	0.4062
1.5430	3.1813	-0.8889	2.1007	0.9290	0.6800	0.1042	0.0892
1.5430	3.1813	-0.8889	2.1007	0.9290	0.6800	0.1042	0.0892
1.5430	3.1813	-0.8889	2.1007	0.9290	0.6800	0.1042	0.0892
1.0000	3.1294	0.7087	2.1202	4.37E-06	1.2441	0.3430	0.0612
1.1199	1.3231	-1	2.0038	6.58E-08	2.2752	0.4080	0.0505
2.7032	1.3626	-1	2	0.0005	2.1855	0.5	0.4066
1.5158	1.9360	-0.2018	2.1672	1.5529	0.5582	0.2498	0.2021
3.3575	1.9058	-0.0036	2.0003	4.7844	9.12E-08	0.4019	0.1658
1.8423	1.1829	-0.9996	2.0059	4.7472	1.47E-07	0.1882	0.2034
2.9664	1.0000	0.2867	2.0998	7.5898	3.5642	0.2498	0.5
4.3448	1.8784	0.5872	2.5165	2.5902	0.2592	0.4350	0.4429

Date	$X_0^{(1)}$	$X_0^{(2)}$	$ca^{(1)}$	$ca^{(2)}$	$cb^{(1)}$	$cb^{(2)}$	DoF
28/10/10	0.0001	-1.4767	-0.5713	0.4454	1.6298	2	0.0278
29/10/10	0.0014	-0.2046	-0.8769	1.6691	1.7222	2.5196	4.64E-06
1/11/10	0.0008	-1.0945	-2.3031	1.5092	1.5848	3.1558	0.0236
2/11/10	0.0061	-0.0269	-2.4077	3.5171	3.0414	2.14254	0.0288
3/11/10	5.41E-06	-20.2494	-95.8812	119.0661	42.7405	2	0.0041
4/11/10	0.1691	-0.5348	-2.4729	4.3079	6.1854	2.2153	0.0500
5/11/10	0.0400	-0.5442	-4.5541	0.6669	0.9835	2.0001	0.0256
8/11/10	0.1605	-1.8327	-44.1404	19.3013	20.3448	2	0.0500
9/11/10	0.1136	-4.2166	-8.4187	12.5165	6.0417	3.0086	0.0178
10/11/10	0.1724	-0.2764	-3.5448	9.0007	0.9255	2.2351	0.0454
12/11/10	8.54E-06	-2.0148	-3.56282	59.9189	0.6388	2.8382	4.80E-07
15/11/10	0.0914	-1.4200	-5.3686	15.8379	6.2238	2.7206	0.0500
16/11/10	0.0914	-1.4200	-5.3686	15.8379	6.2238	2.7206	0.0500
17/11/10	0.0914	-1.4200	-5.3686	15.8379	6.2238	2.7206	0.0500
18/11/10	0.1705	-0.4134	-1.6320	5.6121	2.8003	3.2140	0.0324
19/11/10	0.0266	-5.7731	-0.13701	36.1224	0.3640	2	0.0144
22/11/10	2.37E-07	-6.0095	-2.1391	30.9573	1.1159	4.3023	0.0221
23/11/10	6.42E-07	-6.6002	-14.9443	24.0659	10.8939	4.2420	0.0425
24/11/10	0.0914	-1.4200	-1.3990	6.0577	1.6219	2.7206	0.0500
26/11/10	0.1622	-2.5585	-0.1114	7.4426	1.1014	3.4522	0.0392
29/11/10	0.0475	-4.9602	-13.113	36.4385	7.6573	2.0981	3.60E-08
30/11/10	0.1789	-11.8809	-11.6242	13.4626	5.5607	2	0.0391
1/12/10	0.1277	-0.7248	-3.6604	13.6385	3.1872	2.7429	3.94E-06
2/12/10	0.0586	-0.3275	-6.0130	8.8351	4.8794	2.3053	0.0214
3/12/10	0.1947	-0.4129	-9.2830	6.0916	2.2565	3.4940	0.0112
6/12/10	0.0098	-4.3107	-13.6164	24.7967	8.2083	2.0000	0.0118
7/12/10	0.2828	-2.8925	-13.0599	21.7289	7.5485	3.3729	0.0439
8/12/10	0.0366	-1.8686	-17.1479	2.0907	6.0152	3.2317	0.0217
9/12/10	0.2007	-0.3640	-14.3323	4.9200	8.4189	2.5189	3.52E-06
10/12/10	0.2007	-0.3640	-14.3323	4.9200	8.4189	2.5189	3.52E-06
13/12/10	0.1458	-1.3535	-10.426	10.1191	7.3807	2.3381	0.0083
14/12/10	0.5885	-3.6082	-3.8449	31.7312	3.3255	2	0.0145
15/12/10	0.3144	-0.4307	-23.5281	16.8458	9.7489	3.4450	0.0025
16/12/10	0.1824	-0.6932	-5.5442	8.7435	2.2681	2	0.0329
17/12/10	0.4519	-1.4200	-7.3064	14.0041	6.2238	2	0.0247
20/12/10	0.3707	-10.4143	-15.0458	16.8044	3.5166	2.6397	2.18E-08
21/12/10	0.2259	-1.9689	-14.3477	8.5814	9.0330	2.8589	0.0223
22/12/10	6.33E-08	-2.2894	-18.5814	37.7187	7.9422	4.7839	0.0168
23/12/10	0.4190	-3.0453	-3.1856	20.6277	4.0509	6.3169	0.0129
27/12/10	0.0008	-1.5835	-3.8217	4.5184	4.9727	2.3254	0.0456
28/12/10	3.08E-07	-0.6272	-14.4119	11.29654	2.5792	4.6460	0.0347
29/12/10	0.7899	-8.7888	-11.9934	23.7208	10.6287	3.1784	0.0028
30/12/10	0.0631	-0.2485	-1.3527	17.4834	0.6130	2.0111	0.0465
31/12/10	0.3524	-0.7538	-0.2734	6.6120	0.7241	3.7367	3.15E-09
3/01/11	0.0922	-2.3359	-21.3341	3.9512	5.5483	4.3907	0.0057
4/01/11	1	-7.1184	-4.6586	22.8797	8.0838	4.5661	0.0011
5/01/11	3.11E-06	-0.8629	-6.7245	0.8284	4.4823	2.3131	0.0268
6/01/11	2.90E-07	-5.4062	-6.4001	14.1772	6.8107	4.2542	0.0057

$\mu^{(1)}$	$\mu^{(2)}$	θ	ρ	ϕ_1	ϕ_2	$\sigma^{(1)}$	$\sigma^{(1)}$
4.3448	1.8784	0.5872	2.5165	2.5902	0.2592	0.4350	0.4429
3.8306	2.1517	1.0000	2.0003	1.1008	0.8047	0.4029	0.2208
4.2176	1.3932	1.0000	2.0124	3.3108	3.1798	0.1673	0.2832
4.1063	1.6136	0.9785	2	5.2543	3.7896	0.3377	0.1309
1.0000	1.5416	0.3775	2.0328	2.1866	2.4454	0.4501	0.3652
2.3688	4.6442	-1	2.00001	6.86E-06	1.2362	0.2183	0.3309
2.0493	1.9263	-0.3271	2.2225	3.0483	2.2867	0.4154	0.3926
3.4414	4.5622	0.0087	2	4.0118	2.9952	0.2960	0.5
1.2467	1.9588	0.2416	2.01252	4.8847	1.5098	0.5	0.4074
1.0000	3.1667	-1	2.1307	2.1087	5.55E-07	0.3272	0.5
1.5197	1.0000	-0.6387	2.2959	1.6179	2.53E-06	0.3110	0.4072
2.5266	2.2776	0.1017	2.2436	0.9598	0.6250	0.3439	0.2003
2.5266	2.2776	0.1017	2.2436	0.9598	0.6250	0.3439	0.2003
2.5266	2.2776	0.1017	2.2436	0.9598	0.6250	0.3439	0.4998
2.4927	2.9794	0.5116	2.00001	0.1639	0.7413	0.3325	0.5
2.7107	1.8105	0.9906	2.00001	2.8205	2.3064	0.5000	0.1685
1.8268	1.1781	-0.3971	2.0412	1.1786	4.83E-05	0.4766	0.5
1.0000	3.1999	-0.3412	2.1031	3.0256	2.8389	0.3529	0.3339
2.5266	2.6431	0.1017	2.4446	0.9598	1.6488	0.3439	0.1386
2.4644	1.0000	0.5604	2.3713	0.7807	0.4141	0.4055	0.2680
1.4250	1.0000	0.1199	2.00003	2.4179	0.4155	0.3844	0.2920
1.4557	1.6855	-1	2.2355	2.0160	5.43E-09	0.3945	0.2555
3.1069	1.2963	0.9298	2.2370	0.3423	3.1455	0.4281	0.1100
2.7146	4.0149	-0.7561	2.0107	1.8946	9.19E-05	0.3131	0.2927
1.9282	5.5297	-0.9040	2.5900	2.7732	0.3199	0.3195	0.5
3.2793	1.8709	-0.5684	2	0.4947	0.0001	0.2695	0.2295
1.6993	3.7273	-1	2.0304	2.4946	0.2757	0.2126	0.3021
2.3027	2.0973	-0.1628	2.0975	1.2319	0.7714	0.3923	0.1914
3.0630	1.4804	1	2.1538	1.6777	1.5497	0.5	0.1259
3.0630	1.4804	1	2.1538	1.6777	1.5497	0.5	0.1259
3.2115	1.0001	-0.9168	2.0366	1.2078	2.1219	0.2788	0.3538
2.6457	2.2212	0.3841	2.00001	1.37459	2.70E-06	0.2909	0.3147
5.2064	4.9461	0.4841	2	3.0409	0.8590	0.4162	0.5
3.1993	1.2096	-1	2.1491	0.8172	1.7795	0.1232	0.3415
1.7593	1.0000	0.5256	2.2436	7.46E-07	2.8855	0.4583	0.4317
6.0481	3.9146	1.0000	2	1.3934	5.41E-07	0.4774	0.2690
2.2224	2.8027	-1	2.0136	0.7266	2.8177	0.2136	0.5
1.0000	4.4157	-0.9260	2.1192	1.78E-08	1.6255	0.2744	0.4209
2.0152	3.7690	0.9286	2.1709	9.95E-07	3.3665	0.3219	0.4725
2.9809	3.0838	-0.4182	2.085	0.8149	0.8840	0.3785	0.2832
1.0000	2.1281	-0.9008	2.0102	3.8188	0.7165	0.1194	0.2864
1.0496	1.0000	0.2984	2.0285	1.4523	5.19E-06	0.2937	0.5000
2.9504	2.5960	-0.3283	2.0064	3.8917	2.2866	0.4322	0.1401
2.1374	1.7184	0.5277	2.2656	0.2136	1.6817	0.1268	0.3169
3.0001	1.3028	-0.3974	2	1.6079	0.1643	0.4027	0.5000
1.6046	1.2741	0.0553	2.0132	0.4589	2.8170	0.5	0.4864
1.3753	1.4737	0.5277	2.00001	7.5728	5.65E-08	0.3943	0.1761
2.2834	2.6024	0.0553	2.0793	1.6931	0.0094	0.3019	0.2968

Date	$X_0^{(1)}$	$X_0^{(2)}$	$ca^{(1)}$	$ca^{(2)}$	$cb^{(1)}$	$cb^{(2)}$	DoF
7/01/11	0.2709	-3.8056	-0.1647	10.9187	0.4225	2.9387	0.0138
10/01/11	0.2389	-0.6538	-3.8405	9.4805	1.4239	2.4578	5.82E-08
11/01/11	0.2389	-0.6538	-3.8405	9.4805	1.4239	2.4578	5.82E-08
12/01/11	0.4240	-3.6036	-9.8295	22.9918	7.6700	2	3.03E-08
13/01/11	0.2088	-0.7309	-2.7731	11.1088	2.9617	2	0.0052
14/01/11	0.2088	-0.7301	-2.7731	11.1089	2.9618	2	0.0052
18/01/11	0.6359	-3.1177	-2.5413	20.2886	1.8264	2	0.0097
19/01/11	0.1130	-0.0218	-2.6487	0.3197	1.6777	2.5911	0.0360
20/01/11	0.4214	-6.5712	-33.5094	16.8780	12.3153	3.1226	0.0152
21/01/11	0.4214	-6.5712	-33.5094	16.8780	12.3153	3.1226	0.0152
24/01/11	2.94E-08	-0.0292	-44.7616	15.7704	3.3092	2.2895	0.0118
25/01/11	0.0466	-0.8609	-5.4039	16.9678	1.4995	5.9544	0.0104
26/01/11	0.2272	-0.2147	-10.1406	1.9226	7.9210	2.6650	1.63E-08
27/01/11	0.2801	-1.4423	-2.5151	10.0920	2.8440	2.2866	0.0001
28/01/11	3.20E-05	-0.2617	-3.7202	3.8096	2.7107	2.1776	0.015
31/01/11	0.2799	-1.4386	-2.5143	10.0946	2.8432	2.2866	3.08E-08
1/02/11	0.0168	-2.9040	-13.9721	12.4729	11.3540	4.0192	0.0024
2/02/11	0.2526	-0.6826	-1.8444	14.1926	1.9809	2.7986	0.0274
3/02/11	0.1497	-0.2953	-2.5933	5.8613	2.2608	4.5639	0.0226
4/02/11	2.51E-06	-1.1298	-0.5475	13.1175	0.3281	2.1732	1.89E-07
7/02/11	0.1324	-0.2617	-1.3004	3.8096	0.9092	2.1776	0.0148
8/02/11	0.1324	-0.2617	-1.3004	3.8096	0.9092	2.1776	0.0148
9/02/11	0.0002	-1.4000	-5.7703	3.5979	6.6512	2.6707	0.0342
10/02/11	0.0002	-1.4000	-5.7703	3.5979	6.6512	2.6707	0.0342
11/02/11	0.0002	-1.4000	-5.7703	3.5979	6.6512	2.6707	0.0342
14/02/11	1.07E-05	-0.2737	-0.7387	4.3974	0.6734	3.7894	0.0138
15/02/11	1.07E-05	-0.2737	-0.7387	4.3974	0.6734	3.7894	0.0138
16/02/11	6.57E-05	-0.2730	-0.0294	5.9064	0.1179	3.1805	0.0147
17/02/11	0.0898	-0.9885	-0.0813	20.2457	0.0907	4.2462	1.48E-07
18/02/11	0.2228	-7.8810	-0.2417	32.9345	0.3013	2.1322	0.0136
22/02/11	0.1497	-0.2953	-2.5933	7.9309	2.2608	2	1.56E-06
23/02/11	0.0484	-0.2485	-0.2267	5.0459	0.1906	2	0.0145
24/02/11	0.0718	-1.8622	-8.6119	4.1583	7.9378	2	0.0028
25/02/11	0.5057	-0.7314	-8.2872	11.1088	2.7370	3.7240	0.05
28/02/11	1.07E-05	-0.2737	-3.4338	4.3974	1.6609	2.0001	0.0311
1/03/11	0.0852	-2.8775	-31.3635	17.5079	15.0268	3.5533	0.0256
2/03/11	4.59E-08	-3.4320	-11.1587	12.9219	12.9943	2.5675	0.0296
3/03/11	1.07E-05	-0.2737	-1.1895	4.3974	0.6734	2.0001	0.0016
4/03/11	5.65E-07	-3.8679	-3.1779	9.5993	1.8284	2.2061	0.0153
7/03/11	0.7696	-0.8976	-4.3412	4.5719	1.5206	3.6653	0.0280
8/03/11	0.4749	-1.6375	-1.1703	11.7998	0.7827	2.8974	0.0046
9/03/11	0.8766	-1.4465	-2.8506	2.9231	2.7810	2.4138	0.0034
10/03/11	0.8766	-1.4465	-2.8506	2.9231	2.7810	2.4138	0.0034
11/03/11	0.0016	-0.2485	-0.7911	5.0459	0.1906	3.7926	3.05E-09
14/03/11	0.2480	-1.1E-07	-1.3161	4.2300	0.4659	2.9504	0.0154
15/03/11	0.3045	-5.9937	-4.6538	21.0515	3.7578	3.0516	0.0316
16/03/11	0.4666	-0.4505	-0.0190	9.6171	0.2005	2.7242	3.01E-07
17/03/11	4.24E-07	-0.6779	-3.6831	9.2274	1.9453	2.2756	0.0009

$\mu^{(1)}$	$\mu^{(2)}$	θ	ρ	ϕ_1	ϕ_2	$\sigma^{(1)}$	$\sigma^{(1)}$
2.3922	2.1593	-0.6212	2.0004	3.2932	2.2099	0.1958	0.4411
1.0000	1.4187	0.4539	2	1.5504	0.0008	0.2345	0.3582
1.0000	1.4187	0.4539	2	1.5504	0.0008	0.2345	0.3582
1.9137	1.5046	1.0000	2.0000	1.4026	3.3622	0.3352	0.3821
3.3043	1.0000	0.9994	2.0000	4.9015	0.7086	0.4795	0.3191
3.3043	1.0000	0.9994	2.0000	4.9015	0.7086	0.4795	0.3191
1.4379	1.0000	0.5094	2	1.1992	1.1445	0.3347	0.3409
1.8645	2.23676	-0.1365	2	1.88E-07	0.2974	0.2857	0.3127
2.8602	1.0000	-1	2.0144	3.0214	0.8390	0.4999	0.3619
2.8602	1.0000	-1	2.0144	3.0214	0.8390	0.4999	0.3619
3.1905	2.5687	-0.9430	2.3259	2.2468	0.8439	0.5	0.1992
3.5448	3.9261	0.4897	2.9357	2.4377	8.99E-08	0.2935	0.3347
1.0000	2.4754	1	2.00004	1.6289	1.2964	0.4513	0.3962
1.8531	3.0512	-0.9999	2.1685	2.0619	1.7113	0.3361	0.5000
2.5913	3.6848	-0.8428	2.0138	1.9112	1.9621	0.2937	0.3492
1.8531	3.0512	-0.9999	2.1681	2.0619	1.7113	0.3359	0.5
2.1496	3.5276	0.3675	2.0048	1.5025	2.374	0.2436	0.3087
3.9432	1.0000	-0.8104	2.0006	0.4079	3.92E-07	0.2737	0.1465
2.9160	1.0000	-0.2509	2.1672	7.21E-08	2.4309	0.2064	0.1886
1.0495	3.5617	0.9384	2.0043	3.8232	3.36E-06	0.1988	0.2874
2.5913	3.6848	-0.8428	2.0138	1.9112	0.4269	0.2937	0.1595
2.5913	3.6848	-0.8428	2.0138	1.9112	0.4269	0.2937	0.1595
2.8056	3.5746	1.0000	2	2.2200	8.09E-06	0.1886	0.5
2.8056	3.5746	1.0000	2	2.2200	8.09E-06	0.1886	0.5
2.8056	3.5746	1.0000	2	2.2200	8.09E-06	0.1886	0.5
1.6624	4.2488	-0.7231	2.0308	0.9150	0.0005	0.23079	0.4449
1.6624	4.2488	-0.7231	2.0308	0.9150	0.0005	0.23079	0.4449
2.4356	1.0042	-0.0584	2.0004	0.3182	0.6117	0.2909	0.2801
2.8347	1.0041	-0.8089	2.00001	3.6448	1.5837	0.2484	0.4569
1.1909	2.4754	0.0268	2.6027	0.0197	0.5876	0.4996	0.3337
2.12592	1.171345	-1	2.0033	0.5173	3.4234	0.2618	0.2380
2.9160	3.4518	-0.3092	2.0095	1.8418	0.0073	0.2064	0.4340
2.4584	2.8745	1.0000	2.1542	4.9805	6.94E-07	0.1701	0.3752
2.03743	2.1483	-1	2.00004	8.98E-07	0.4781	0.2607	0.2670
2.95727	3.9913	-0.7231	2	1.0308	0.7709	0.2032	0.2846
1.0000	2.2316	-0.1700	2.0092	2.7613	3.1235	0.4840	0.4290
1.6679	1.2753	0.4018	2	4.2934	1.7000	0.2449	0.2721
1.6624	3.1785	-0.7231	2.0292	0.9150	0.5700	0.2283	0.5
2.5919	1.8438	-0.1354	2	2.9679	8.37E-07	0.4958	0.0748
1.00001	2.1483	0.9994	2	1.14E-08	0.7086	0.26071	0.5
3.4672	2.6199	-0.3951	2.2657	1.0997	1.4218	0.1680	0.1179
1.00001	3.0373	-1.0000	2.0031	1.3718	2.5253	0.2621	0.3484
1.00001	3.0373	-1.0000	2.0031	1.3718	2.5253	0.2621	0.3484
2.9160	3.4518	0.0171	2.0033	1.0753	4.4019	0.2064	0.4340
1.00001	1.0000	0.9728	2.0179	1.7786	0.8910	0.3752	0.4355
2.9455	2.0019	0.2273	2.0048	1.7319	0.6744	0.3673	0.3616
2.6847	1.9890	-0.7365	2.0193	7.26E-06	0.5116	0.5000	0.5000
1.6490	3.8112	0.5362	2.3118	2.3886	0.7769	0.5000	0.0834

Date	$X_0^{(1)}$	$X_0^{(2)}$	$ca^{(1)}$	$ca^{(2)}$	$cb^{(1)}$	$cb^{(2)}$	DoF
21/03/11	0.1344	-4.1549	-0.0831	25.0322	1.2047	2.8016	0.0220
22/03/11	2.90E-06	-0.9384	-2.7200	6.0266	2.3348	2.5466	0.0045
23/03/11	0.9768	-0.5915	-1.0868	11.4437	0.5324	2.1714	4.21E-09
24/03/11	4.24E-07	-0.6779	-1.7151	19.7531	1.6774	2	0.0010
25/03/11	4.24E-07	-0.6779	-1.7151	19.7531	1.6774	2	0.0010
28/03/11	0.3145	-3.5643	-8.5471	48.1993	6.4393	2.0001	0.0049
29/03/11	0.3145	-3.5643	-8.5471	48.1993	6.4393	2.0001	0.0049
30/03/11	8.59E-08	-0.3457	-0.0790	2.2747	0.1074	2	4.02E-09
31/03/11	0.0319	-0.0000001	-0.3650	4.0371	0.1969	2.9234	2.16E-08
1/04/11	0.2376	-0.5579	-25.4347	11.1547	8.7994	2	4.86E-08
4/04/11	9.63E-08	-0.0208	-7.6878	6.5916	12.1383	2.4018	0.0324
5/04/11	9.63E-08	-0.0208	-7.6878	6.5916	12.1383	2.4018	0.0324
6/04/11	0.1746	-1.1884	-6.4154	12.9980	6.6474	2.2853	0.0319
7/04/11	1	-1.3165	-8.9688	22.7131	16.9850	2.0008	0.0350
8/04/11	0.2123	-1.0947	-6.6013	11.1103	2.2852	3.26	3.53E-09
11/04/11	0.3696	-1.267	-4.8555	0.0113	2.1075	3.0145	1.37E-09
12/04/11	0.0415	-0.0208	-2.5905	2.3247	3.8423	2.4658	0.0227
13/04/11	0.1293	-0.1704	-18.121	38.6082	2.5137	3.7075	0.0081
14/04/11	0.3435	-2.6324	-1.2746	9.7180	1.3621	2.4906	0.0064
15/04/11	0.2377	-2.9643	-0.0272	8.3591	0.1249	2.8618	0.0242
18/04/11	0.2903	-0.4460	-2.3723	15.2947	1.3689	2.3670	0.0333
19/04/11	0.2903	-0.4460	-2.3723	15.2947	1.3689	2.3670	0.0333
20/04/11	0.2903	-0.4460	-2.3723	15.2947	1.3689	2.3670	0.0333
21/04/11	0.4225	-0.5826	-4.4100	20.1075	3.7206	2.3037	0.0339
25/04/11	0.5690	-0.2335	-9.1072	2.3890	7.5356	2	0.0466
26/04/11	2.06E-08	-1.9905	-8.6903	11.2931	5.5287	2.3853	0.0131
27/04/11	0.0415	-0.0208	-4.1754	2.3247	6.1931	2.4658	4.88E-09
28/04/11	0.5644	-2.2001	-4.9223	4.6988	10.2542	2.0081	0.0268
29/04/11	0.2684	-0.5383	-2.6794	5.9361	3.0333	2.7278	0.0279
2/05/11	0.5449	-3.2234	-12.007	33.1080	4.7490	2.4635	0.0497
3/05/11	0.2761	-3.6129	-11.6206	1.7098	7.8303	3.8882	0.0253
4/05/11	0.2761	-3.6129	-11.6206	1.7098	7.8303	3.8882	0.0253
5/05/11	0.2761	-3.6129	-11.6206	1.7098	7.8303	3.8882	0.0253
6/05/11	0.2761	-3.6129	-11.6206	1.7098	7.8303	3.8882	0.0253
9/05/11	0.1621	-5.0099	-8.6967	57.9674	7.8243	2.9072	0.0350
10/05/11	0.2710	-0.7298	-21.4999	47.8204	1.5202	2	0.0214
11/05/11	0.2957	-0.4041	-31.8352	10.4071	4.2639	3.2764	0.0500
12/05/11	0.1982	-0.5552	-2.7042	28.3903	0.9086	2.3450	0.0200
13/05/11	0.3568	-1.5508	-22.0567	32.4401	8.4067	2.1915	4.90E-09
16/05/11	0.1973	-0.6951	-12.9618	27.8488	6.3225	2.5585	0.05
17/05/11	0.4866	-2.6636	-4.2351	67.1381	2.5186	2	0.0405
18/05/11	1.69E-05	-1.8482	-7.5681	4.6880	2.6279	2	0.0256
19/05/11	3.53E-07	-0.6146	-0.3718	4.8596	0.0371	2.0668	0.0405
20/05/11	0.0865	-9.4916	-11.6127	25.4041	4.9425	2.4981	0.0342
23/05/11	1.50E-05	-0.4346	-3.6171	1.7965	7.6732	2	0.0414
24/05/11	1.00E-06	-1.9042	-4.1478	0.0760	3.8197	2.3591	0.0313
25/05/11	0.1727	-0.6886	-0.8611	3.7025	0.6728	3.4434	0.0220
26/05/11	0.0776	-2.3757	-6.9307	4.2653	4.2834	2.3556	0.0239

$\mu^{(1)}$	$\mu^{(2)}$	θ	ρ	ϕ_1	ϕ_2	$\sigma^{(1)}$	$\sigma^{(1)}$
4.5321	2.5100	0.4958	2.0040	2.6164	1.6448	0.5	0.5
3.3056	1.6544	-1.0000	2.0001	2.5234	1.5386	0.3728	0.2441
3.3878	2.5950	0.1908	2.0922	2.9733	1.1006	0.5	0.4818
4.4542	3.8112	-0.8700	2.0214	1.0648	2.2037	0.5	0.4551
4.4542	3.8112	-0.8700	2.0214	1.0648	2.2037	0.5	0.4551
3.6247	1.2363	-0.0588	2.0107	2.91E-08	0.4485	0.2787	0.3881
3.6247	1.2363	-0.0588	2.0107	2.91E-08	0.4485	0.2787	0.3881
4.4232	1.0000	-0.7551	2.0681	0.9908	9.53E-08	0.2685	0.2993
1.5525	3.2116	0.2206	2	3.08E-08	1.8752	0.4628	0.5
3.8931	3.0723	-0.3688	2.1956	1.7238	1.2574	0.2202	0.1958
1.0000	1.8338	-0.3166	2.41	3.0314	8.75E-08	0.2648	0.1560
1.0000	1.8338	-0.3166	2.41	3.0314	8.75E-08	0.2648	0.1560
2.7707	1.0382	-0.3047	2.0000	2.1676	0.5003	0.5000	0.1801
2.4142	2.7013	-0.4902	2	0.3580	3.57E-08	0.2201	0.5
2.7281	1.7395	0.9837	2.0000	0.4672	0.9103	0.3073	0.3368
1.9180	1.6718	0.5220	2	8.27E-08	1.4580	0.2189	0.5
1.8288	1.3950	1.0000	2.1752	3.0314	9.16E-07	0.291071	0.2063
1.7599	1.8168	0.9729	2.0000	6.04E-06	0.0177	0.3121	0.4472
2.6908	4.4402	0.5305	2.0003	1.9060	0.2844	0.4556	0.5
1.5576	1.0439	0.8998	3.0871	1.6158	3.8357	0.5	0.0566
3.3972	5.2958	0.9932	2.0000	5.46E-08	0.9094	0.4275	0.4940
3.3972	5.2958	0.9932	2.0000	5.46E-08	0.9094	0.4275	0.4940
3.3972	5.2958	0.9932	2.0000	5.46E-08	0.9094	0.4275	0.4940
3.8429	4.3060	0.7391	2	1.89E-07	1.0688	0.5000	0.5
3.9529	1.2322	-0.7105	2.2135	2.3655	0.5162	0.3133	0.5
2.5105	2.5381	0.1718	2.0000	2.5434	0.7140	0.5	0.2705
1.8288	4.2433	1.0000	2.0292	3.0314	1.2339	0.1733	0.2063
3.3694	3.3497	-0.0221	2.3285	0.0549	0.5761	0.2195	0.2514
3.4594	2.7583	0.6932	2.1281	0.5904	0.5922	0.3080	0.2206
2.9005	2.7104	0.9611	2	1.775	1.6237	0.4128	0.5
2.5052	1.0001	-0.8524	2	6.1477	4.26E-08	0.4306	0.4030
2.5052	1.0001	-0.8524	2	6.1477	4.26E-08	0.4306	0.4030
2.5052	1.0001	-0.8524	2	6.1477	4.26E-08	0.4306	0.4030
2.5052	1.0001	-0.8524	2	6.1477	4.26E-08	0.4306	0.4030
5.249	2.4516	0.7016	2	0.6051	6.20E-07	0.3167	0.5000
2.8698	4.7004	-0.0901	2.0367	0.6079	1.4203	0.3106	0.1496
2.4222	3.4115	-1	2.2842	5.2475	0.4998	0.2161	0.5
4.7195	3.7592	0.1461	2.0909	2.6589	0.8223	0.1232	0.2161
2.0013	2.3757	-0.1048	2.1109	1.6994	0.8438	0.2054	0.5
3.1992	1.9878	0.9491	2.2135	2.6815	1.3973	0.2064	0.2993
2.3363	3.4639	-0.5022	2.2291	3.1332	1.7704	0.3665	0.4329
1.9696	2.6326	-0.7440	2.0001	0.5569	2.2187	0.2996	0.3931
1.1016	2.0458	-0.8027	2	0.0661	0.5391	0.3958	0.1487
2.2395	1.5954	-0.5609	2.1446	2.92E-07	0.9264	0.4786	0.4819
1.0000	1.0000	-0.9792	3.5046	1.7388	0.7049	0.3277	0.5
4.1872	2.4343	0.6254	2	4.4613	2.3991	0.2518	0.3077
1.0000	3.7393	-0.6818	2.0948	0.7508	0.8155	0.201	0.5
5.1028	1.6195	1	2.2055	2.6768	2.3658	0.2734	0.3651

Date	$X_0^{(1)}$	$X_0^{(2)}$	$ca^{(1)}$	$ca^{(2)}$	$cb^{(1)}$	$cb^{(2)}$	DoF
27/05/11	0.1940	-13.6601	-3.2117	15.3348	5.1681	2	0.05
31/05/11	0.4287	-2.6847	-2.0927	0.6311	1.2508	2.3713	0.0118
1/06/11	0.4287	-2.6847	-2.0927	0.6311	1.2508	2.3713	0.0118
2/06/11	1	-0.7420	-0.0965	0.1192	2.9303	2.6371	0.0249
3/06/11	1	-0.7420	-0.0965	0.1192	2.9303	2.6371	0.0249
6/06/11	1.0000	-0.1991	-1.0412	4.1226	0.4675	3.3619	0.0079
7/06/11	1.0000	-0.1991	-1.0412	4.1226	0.4675	3.3619	0.0079
8/06/11	7.94E-07	-3.4912	-3.4610	13.8931	2.3007	4.0460	0.0010
9/06/11	0.1209	-5.1323	-7.7045	14.5030	4.0763	4.7742	2.46E-07
10/06/11	0.0084	-1.7765	-0.3199	8.5620	0.9658	5.2475	0.0442
13/06/11	0.0572	-1.2879	-2.885	3.7382	1.0675	2	0.0500
14/06/11	0.0572	-1.2879	-2.885	3.7382	1.0675	2	0.0500
15/06/11	0.1756	-3.7876	-16.4364	44.8178	2.2790	2	0.0268
16/06/11	7.54E-08	-0.7903	-15.7724	13.3633	1.9608	3.7116	0.0468
17/06/11	0.1756	-3.7876	-16.4364	44.8178	2.2790	2	0.0268
20/06/11	0.0572	-1.5130	-7.1288	17.7215	2.6378	2	0.0500
21/06/11	0.0572	-1.5130	-7.1288	17.7215	2.6378	2	0.0500
22/06/11	0.0572	-1.5130	-7.1288	17.7215	2.6378	2	0.0500
23/06/11	0.1468	-0.0800	-0.6976	3.3665	4.6058	2.0008	0.0409
24/06/11	0.1468	-0.0800	-0.6976	3.3665	4.6058	2.0008	0.0409
27/06/11	6.97E-06	-1.5130	-7.1288	17.7215	2.6378	2	0.050
28/06/11	6.97E-06	-1.5130	-7.1288	17.7215	2.6378	2	0.050
29/06/11	0.5614	-0.6205	-0.3723	14.5869	0.5845	4.0330	1.16E-07
30/06/11	0.5614	-0.6205	-0.3723	14.5869	0.5845	4.0330	1.16E-07

$\mu^{(1)}$	$\mu^{(2)}$	θ	ρ	ϕ_1	ϕ_2	$\sigma^{(1)}$	$\sigma^{(1)}$
2.0709	1.0000	1.0000	2	5.5748	1.9513	0.5	0.3398
3.7412	2.2955	-0.5180	2	1.2978	7.99E-08	0.2807	0.2707
3.7412	2.2955	-0.5180	2	1.2978	7.99E-08	0.2807	0.2707
1.0000	2.3138	1	2	3.21E-08	0.0398	0.3426	0.2203
1.0000	2.3138	1	2	3.21E-08	0.0398	0.3426	0.2203
1.0000	1.0455	1.0000	2.4540	0.9557	1.9273	0.5000	0.1118
1.0000	1.0455	1.0000	2.4540	0.9557	1.9273	0.5000	0.1118
3.55853	1.694291	1.0000	2.3508	1.5291	2.3016	0.2021	0.4998
1.6139	3.1725	-0.4863	2.0121	3.79E-05	2.9037	0.2601	0.1638
2.3055	1.3879	1.0000	2.0832	1.1311	1.4773	0.2400	0.2926
3.9892	1.0000	0.2725	2.1072	1.9889	0.0117	0.4501	0.2584
3.9892	1.0000	0.2725	2.1072	1.9889	0.0117	0.4501	0.2584
3.8506	1.0000	-0.2885	2	1.76086	1.74765	0.5	0.1055
4.5508	1.0000	-0.7991	3.0832	1.4605	0.2147	0.4356	0.1632
3.8506	1.0000	-0.28851	2	1.7609	1.7477	0.5	0.1055
2.4568	1.0000	0.2725	2.1072	0.0008	1.8087	0.4501	0.5
2.4568	1.0000	0.2725	2.1072	0.0008	1.8087	0.4501	0.5
2.4568	1.0000	0.2725	2.1072	0.0008	1.8087	0.4501	0.5
1.8600	1.3285	-0.5978	2.9407	1.1919	1.5596	0.5000	0.2756
1.8600	1.3285	-0.5978	2.9407	1.1919	1.5596	0.5000	0.2756
1.2205	1.3050	0.2725	2.0639	0.0008	1.8087	0.5000	0.5000
1.2205	1.3050	0.2725	2.0639	0.0008	1.8087	0.5000	0.5000
4.2444	2.2070	0.3949	2.0000	3.1513	0.8155	0.1886	0.3391
4.2444	2.2070	0.3949	2.0000	3.1513	0.8155	0.1886	0.3391

Acronyms

The following list is neither exhaustive nor exclusive, but may be helpful.

cdf	cumulative distribution function
CDS	credit default swap
DoF	degrees of freedom
FGM	Farlie-Gumbel-Mogenstern
GDP	gross domestic product
IE	integral equation
NAB	National Australia Bank
pdf	probability distribution function
PDMP	piecewise-deterministic Markov process
RC	reference credit
RV	random variable
SD	standard deviation
US	United States
VIE	Volterra Integral Equation
w.r.t.	with respect to

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