Gray Tensor Product and Kontsevich's Swiss-Cheese Conjecture

By

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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Abstract

We study connections between two seemingly very distant constructions: Gray-product of higher categories and famous Kontsevich Swiss-Cheese conjecture. Gray-product of 2-categories is known for almost 50 years and it is an extremely important construction in 2-category theory. It was proved by Crans and later by Bourke and Gurski that a naive analogue of Gray-product in higher dimensions does not exist. Nevertheless, there is a conjecture that there exists a weaker version of this product in all dimensions such that it descends to a closed structure on homotopy level.

Swiss-Cheese conjecture was proposed by Fields medalist M.Kontsevich in 1998 to handle a problem of the existence of higher order Hochschild complexes. It is geometrical in nature and is very important in deformation quantisation theory.

In the thesis we outline a surprising relationship between these two important conjectures, which was not observed before. Namely, the existence of a homotopically closed Gray-product of $\mathcal{V}$-enriched categories implies the Swiss-Cheese conjecture in $\mathcal{V}$. We provide a full proof of this statement for $\mathcal{V} = \mathbf{Set}$, $\mathbf{Ab}$ and $\mathbf{Cat}$ using the idea of categorification.

Contents

1
Introduction

The idea of symmetry is one of the main pillars of classical and modern mathematics. Since Galois this idea was formalised in the concept of a group action on some other object. It was understood later that one can obtain a fruitful generalisation of this concept if one replaces groups by monoids. The concept of an A-module over a k-algebra A revolutionised many fields of mathematics in the first half of the 20th century. With the emergence of Category Theory in the second half of 20th century, this concept of action received a full treatment in the context of an action of a monoid in a monoidal category.

Another extremely useful and important construction of classical algebra closely related to the action of a monoid is the centre construction. This construction is not functorial and this creates a problem of how to interpret it within Category Theory. There are different approaches to this. One approach is to consider a monoid M as a one object category ΣM. Then its centre is the (commutative) monoid of endomorphisms of the identity functor $Id : \Sigma M \to \Sigma M$. It is obvious already from this definition that a proper understanding of centre requires a process called categorification. We interpret a monoid as a category, an object which lives in the next dimensional (in the sense of higher category theory) universe. It was understood by many people in higher category theory (Joyal, Day, Street, Baez, Dolan, Crans and others) that one can define centres of higher dimensional k-monoidal categories using a similar approach by categorification.

One shortcoming of this definition of centre (and higher centres) is that it is still non-fuctorial and does not provide a definition by universal property, which is highly desirable if one wish to understand a construction from a categorical perspective.

Inspired by Deligne's conjecture on Hochschild cochains such a universal property approach was suggested by Kontsevich in his seminal 1998 paper [19].

For this he uses a beautiful object, called the Swiss-Cheese operad SC_d, introduced by Voronov in [31]. This is an operad which combines together two classical operads: the operad of little $(d-1)$-disks and little d-disks. The operad of little d-disks (invented by Boardman-Vogt-May in the 1960s) came from algebraic topology. Algebras of this operad (called E_d-algebras) describe structure which algebraically characterises (up to group completion) the class of d-fold loop spaces in homotopy theory.

The algebras of Voronov's operad SC_d consists of a pair of an E_d-algebra B and an E_{d-1}-algebra A together with some structure maps which connect these two algebra structures. Kontsevich proposed to see these maps as an action of B on A. For $d = 1$ it is obvious, for example, that such a structure amounts to a monoid up to higher homotopy structure on B (that is an E_1-algebra structure) and an action of B (again up to higher homotopies) on a topological space A.

One can then fix a E_{d-1}-algebra A and consider the homotopy category of E_d-algebras B acting on A in the above sense. Kontsevich conjectured that this category has a terminal object $C(A)$. This means that up to homotopy, any action of B on A amounts to a uniquely determined map of E_d-algebras $B \to C(A)$. The object $C(A)$ can be considered as a d-centre of the E_{d-1}-algebra A. This is the famous Kontsevich Swiss-Cheese conjecture.

This conjecture can be formulated for E_d-algebras in any nice enough symmetric monoidal model category $(\mathcal{V}, \otimes, I)$. It was recently proved in such a general setting in [30]. The proof is not conceptually transparent and technically hard.

The purpose of this thesis is to show that there is a possibility to combine Kontsevich's understanding of the higher centre problem with the higher categorical approach described at the beginning of the introduction. The idea of this combination was sketched out in 2009 by Batanin in a talk in Adelaide University. In this approach we connect the Swiss-Cheese conjecture with another famous problem in higher category theory and homotopy theory, that of the existence of a Gray-product of higher categories. This potentially can lead to a very clear and simple proof of the Swiss-Cheese conjecture but also can be useful in the converse direction.

The Gray-product of 2-categories was defined by Joyal (there is an earlier nonsymmetric version due to Gray [12]) and used by Gordon-Power-Street to prove their famous coherence theorem for tricategories [10]. Also Steve Lack proved that the Gray-product, in contrast to the cartesian product, satisfies the push-out product axiom and so descends to the closed symmetric monoidal structure on the homotopy category of 2-categories (with Lack's model structure). Unfortunately, there is a no-go theorem due to Crans and Bourke-Gurski which says that it is impossible to define a closed symmetric monoidal structure on Gray-categories satisfying the natural conditions which we expect from the next level Gray-product construction. Nevertheless, there is a hope that some weaker version of the Gray-product conjecture may still be true. More generally, there may exist a symmetric lax-monoidal structure on the model category of $\mathcal{V}$-categories (provided it exists) for any reasonable monoidal category, which descends to a closed symmetric monoidal structure on the homotopy level. Some evidence for such a possibility can be found in the paper of Batanin-Cisinski-Weber [3] and recent paper of Shoikhet [29].

The main result of our thesis is to show, that in the case where we know that the Gray-product exists the $d = 1$ Swiss-Cheese conjecture in $\mathcal{V}$-**Cat** implies the $d = 2$ Swiss-Cheese conjecture in $\mathcal{V}$. The arguments are extremely simple in this case. Since E_1-algebras are homotopy equivalent to monoids we use monoids with respect to the Gray-product as models of E_1-algebras in $\mathcal{V}$-**Cat**. Then the $d = 1$ Swiss-Cheese conjecture is simply a statement that an action of a strict monoidal (with respect to Gray-product) $\mathcal{V}$-category B on a $\mathcal{V}$-category A amounts to a strict monoidal $\mathcal{V}$-functor $B \to \mathrm{End}(A)$, where $\mathrm{End}(A)$ is the internal endomorphism object. Suppose now that B and A both are one object $\mathcal{V}$-categories. Then B can be identified with a E_2-algebra in $\mathcal{V}$ (by the Eckman-Hilton argument) and A to a monoid in $\mathcal{V}$. The action of B on A is then an action of the Swiss-Cheese operad on the pair (B, A) (this is Batanin's symmetrization theorem for Swiss-Cheese 2-operads [2]). The monoidal functor $B \to \mathrm{End}(A)$ amounts to a E_2-algebra map $B \to Z(A)$ where $Z(A)$ is the

Hochschild complex of A. This is the $d = 2$ Swiss-Cheese conjecture.

In this thesis we did not have an intention to work in full possible generality. Instead, we wanted to write detailed proofs of this result in some special cases where the Gray-product is well understood. We will then use this proof for generalisation in our future work. For example, we would like to formulate precise conditions on the tensor product on $\mathcal{V}$-**Cat** under which the above approach works. We are also going to generalise this statement for higher degree d. We conjecture that there exists a Gray-product of E_{d-1}-algebras in $\mathcal{V}$-**Cat** such that the Swiss-Cheese conjecture in the degree d implies the Swiss-Cheese conjecture in the degree $d + 1$ in $\mathcal{V}$ after restriction to the one object case.

The thesis is constructed as follows. In Chapter 2, we will provide basic prerequisite tools with a reminder on elements of monoidal categories, where we will also present a statement about monoid actions in a symmetric closed monoidal category, which will be used in situations where the Gray-product is well known, namely for **Cat**, **Ab-Cat** and **2Cat**. This will be the $d = 1$ statement of the Swiss-Cheese conjecture in **Cat**, **Ab-Cat** and **2Cat**. In Chapter 3, we present the main result in the special case of $\mathcal{V} = \mathbf{Set}$ and go on to a brief digression into enriched category to present the main result for $\mathcal{V} = \mathbf{Ab}$. In Chapter 4, we will begin with cubical functors and introduce the Gray tensor product of 2-categories, giving **2Cat** the structure of a symmetric closed monoidal category, which will then transit into Chapter 5 in the presentation of the main result for the special case of $\mathcal{V} = \mathbf{Cat}$. Lastly in Chapter 6, we will provide the precise formulation of the Swiss-Cheese conjecture we make connections with.

2
Background

While many of the mathematics is done in **Set** with sets and functions, every set is more or less a category with no arrows between objects except for identity arrows. And every function between sets can also more or less be regarded as a functor between such categories. And there is a bigger universe **Cat** which embrace the world of **Set**. Mathematics we do in the language of Set Theory can be reconstructed with Category-theoretic analogues by replacing sets to categories, functions to functors, equation between functions to natural isomorphisms between functors. This process is called **categorification** [1].

In this section, we introduce the necessary background in elements of Monoidal Category [23] which is the categorification of the usual monoids. The category **Set** is an example of a monoidal category and a lot of the constructions in **Set** can be abstracted as constructions in an arbitrary monoidal category. We introduce concepts such as monoid objects, monoid morphisms and monoid actions in an arbitrary monoidal category. Furthermore, we will discuss *Coherence Theorems* and *Closed Monoidal Categories*. The definitions are due to [23].

2.1 Monoidal Categories

Monoidal Category is a categorification of usual the concept of Monoid where rather than a set and a function, a monoidal category $\mathcal{M}$ is a category $\mathcal{M}$ with a bifunctor $\otimes : \mathcal{M}\times\mathcal{M} \to \mathcal{M}$ satisfying coherent associativity and unitality.

Definition 2.1 (Monoidal Category)**.** A *Monoidal Category* is a category $\mathcal{M}$ equipped with a functor

$$\mathcal{M} \times \mathcal{M} \xrightarrow{\otimes} \mathcal{M}$$

and an object $I \in \mathcal{M}$ with

1. components of natural isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ in each arguments $A, B, C \in \mathcal{M}$, called *associators*

2. components of natural isomorphisms $r_A : A \otimes I \cong A$ and $l_A : I \otimes A \cong A$ in each objects $A \in \mathcal{M}$ called *right unitors* and *left unitors* respectively

such that the following diagrams commute:

- Pentagon identity

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes id_D"] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "id_A\otimes\alpha_{B,C,D}"']
\end{tikzcd}$$

- Triangle identity

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "r_A\otimes id_B"'] & & A \otimes (I \otimes B) \arrow[dl, "id_A\otimes l_B"] \\
& A \otimes B &
\end{tikzcd}$$

Example 2.1. 1. Any category with all finite products with the unit being the terminal object (*Cartesian Monoidal Category*): ***Set***, ***Cat***, ***Grp***, ***Top***, ***CRng***, $\boldsymbol{Vect}_k$.

2. Dually, any category with all finite coproducts with the unit object being the initial object.

3. For a commutative ring R, category ***R-Mod*** of modules with tensor product $\otimes_R$ and with R, being the module over itself, as the unit.

Definition 2.2 (Strict Monoidal Category)**.** A monoidal category is called *strict* if the associators and unitors are identity.

Example 2.2. For one example, let C be any category. The functor category $\mathrm{End}(C) = [C, C]$ which we call *endofunctor category of* C has endofunctors $F : C \to C$ as objects and natural transformations $\alpha : F \Rightarrow G : C \to C$ as morphisms. We can compose objects $F : C \to C$ and $G : C \to C$ as $G \circ F : C \to C$ which we have the functor:

$$\circ : \mathrm{End}(C) \times \mathrm{End}(C) \to \mathrm{End}(C)$$

and the unit object $\mathrm{id}_C : C \to C$. The endofunctor category $\mathrm{End}(C)$ together with the above functor is a strict monoidal category.

As (**Set**, $\times$, 1) is an example of a monoidal category, the usual constructions like monoids and monoid morphisms can be reconstructed in an arbitrary monoidal category. Recall that the usual definition of monoid is a set N equipped with functions

$$N \times N \xrightarrow{m} N \xleftarrow{e} 1$$

where 1 here is a singleton set, such that the following diagram commutes, expressing the associativity and unitality:

1.
$$\begin{tikzcd}
(N \times N) \times N \arrow[rr, "\cong"] \arrow[d, "m\times N"] & & N \times (N \times N) \arrow[d, "N\times m"] \\
N \times N \arrow[dr, "m"] & & N \times N \arrow[dl, "m"'] \\
& N &
\end{tikzcd}$$

2.
$$\begin{tikzcd}
N \times 1 \arrow[r, "\cong"] \arrow[d, "N\times e"] & N \arrow[dd, "N"] & 1 \times N \arrow[l, "\cong"'] \arrow[d, "e\times N"] \\
N \times N \arrow[dr, "m"] & & N \times N \arrow[dl, "m"'] \\
& N &
\end{tikzcd}$$

Now, instead of $(\mathbf{Set}, \times, 1)$, by considering an arbitrary monoidal category $(C, \otimes, I)$ we introduce the notion of a monoid object M in the monoidal category C.

Definition 2.3 (Monoid object in a monoidal category)**.** A *monoid* $\mathcal{M}$ in a monoidal category $(C, \otimes, I, \alpha, l, r)$ is an object $M \in C$ equipped with morphisms

$$M \otimes M \xrightarrow{m} M \xleftarrow{e} I$$

satisfying the following conditions:

1. m satisfies associativity, that is, the following diagram commutes

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[rr, "\cong", "\alpha"'] \arrow[d, "m\otimes M"] & & M \otimes (M \otimes M) \arrow[d, "M\otimes m"] \\
M \otimes M \arrow[dr, "m"] & & M \otimes M \arrow[dl, "m"'] \\
& M &
\end{tikzcd}$$

2. e satisfies unitality, that is, the following diagram commutes

$$\begin{tikzcd}
M \otimes I \arrow[r, "\cong", "\lambda"'] \arrow[d, "M\otimes e"] & M \arrow[dd, "M"] & I \otimes M \arrow[l, "\cong"', "\eta"] \arrow[d, "e\otimes M"] \\
M \otimes M \arrow[dr, "m"] & & M \otimes M \arrow[dl, "m"'] \\
& M &
\end{tikzcd}$$

Example 2.3.

1. We have seen that a Monoid in $(\mathbf{Set}, \times, 1)$ is a usual monoid
2. Monoid in $(\mathbf{Cat}, \times, 1)$ is a strict monoidal category

3. Monoid in the monoidal category ***Mon*** of monoids with direct product of monoids is a commutative monoid
4. Monoid in the monoidal category ***R-Mod*** of modules over commutative ring R with tensor product $\otimes_R$ is an R-Algebra.

In the similar way, we have the following definition of *monoid morphism*:

Definition 2.4 (Morphism of monoids)**.** Let (M, m, e) and (M', m', e') be two monoids in a monoidal category $(C, \otimes, I, a, l, r)$. A *monoid morphism* $f : (M, m, e) \to (M', m', e')$ is a morphism $f : M \to M'$ of C such that the following diagrams commute:

1.
$$\begin{tikzcd} M \otimes M \arrow[r, "f \otimes f"] \arrow[d, "m"] & M' \otimes M' \arrow[d, "m'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

2.
$$\begin{tikzcd} I \arrow[r, "e"] \arrow[d, "e'"] & M \arrow[dl, "f"] \\ M' & \end{tikzcd}$$

Remark. Given a monoidal category $(C, \otimes, I)$, we can form a category $\mathbf{Mon}_C$ of monoids in C and monoid morphisms.

While the notion of Monoidal Category is a categorification of usual monoid, we have two categorified notions of commutative monoid: braided mononoidal category and symmetric monoidal category.

Definition 2.5 (Braided Monoidal Category)**.** A *Braided monoidal category* $\mathcal{M}$ is a monoidal category equipped with additional natural isomorphism in $\mathcal{M}$ called *braiding*:

$$B_{A,B} : A \otimes B \to B \otimes A$$

such that we have two commutative diagrams called Hexagon identities:

$$\begin{tikzcd} & A \otimes (B \otimes C) \arrow[r, "B_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha"] & \\ (A \otimes B) \otimes C \arrow[ur, "\alpha"] \arrow[dr, "B_{A,B} \otimes C"'] & & & B \otimes (C \otimes A) \\ & (B \otimes A) \otimes C \arrow[r, "\alpha"'] & B \otimes (A \otimes C) \arrow[ur, "B \otimes B_{A,C}"'] & \end{tikzcd}$$

$$\begin{tikzcd} & (A \otimes B) \otimes C \arrow[r, "B_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}"] & \\ A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}"] \arrow[dr, "A \otimes B_{B,C}"'] & & & (C \otimes A) \otimes B \\ & A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}"'] & (A \otimes C) \otimes B \arrow[ur, "B_{A,C} \otimes B"'] & \end{tikzcd}$$

Remark.

1. By inverting all arrows in the first Hexagon identity, it is easy to see that the first Hexagon identity implies the second one for $B^{-1}_{B,A} : A \otimes B \to B \otimes A$. Likewise, the second Hexagon identity implies the first one for $B^{-1}_{B,A}$. So B^{-1} is also a braiding, generally a distinct one from B.

2. The braiding is compatible with unitors [17]:

$$\begin{tikzcd} A \otimes I \arrow[rr, "B_{A,I}"] \arrow[dr, "r_A"'] & & I \otimes A \arrow[dl, "l_A"] \\ & A & \end{tikzcd} \quad \begin{tikzcd} I \otimes A \arrow[rr, "B_{I,A}"] \arrow[dr, "l_A"'] & & A \otimes I \arrow[dl, "r_A"] \\ & A & \end{tikzcd}$$

   comutes.

3. Braided monoidal category is same thing as a tricategory with one 0-cell and one 1-cell.

Definition 2.6 (Symmertric Monoidal Category)**.** Symmetric monoidal category $\mathcal{M}$ is a monoidal category equipped with additional natural isomorphism in $\mathcal{M}$ called *braiding*:

$$B_{A,B} : A \otimes B \to B \otimes A$$

such that $B_{B,A} \circ B_{A,B} = id_{A \otimes B}$ and the first Hexagon identity (from the definition of Braided Monoidal Category) holds:

$$\begin{tikzcd} & A \otimes (B \otimes C) \arrow[r, "B_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha"] & \\ (A \otimes B) \otimes C \arrow[ur, "\alpha"] \arrow[dr, "B_{A,B}\otimes id_C"'] & & & B \otimes (C \otimes A) \\ & (B \otimes A) \otimes C \arrow[r, "\alpha"'] & B \otimes (A \otimes C) \arrow[ur, "id_B \otimes B_{A,C}"'] & \end{tikzcd}$$

It is precisely a Braided Monoidal Category with additional condition that the two braidings B and B^{-1} are equal $B_{A,B} = B^{-1}_{B,A}$, where every diagrams made by associators, left and right unitors, braidings, their inverses and identity morphism commute [24]. Cartesian monoidal categories are examples of symmetric monoidal categoy.

2.2 Monoidal functor, monoidal natural transformation and monoidal equivalence

Algebraic structures are studied with maps preserving its structure. We know monoid homomorphisms are the maps between monoids that preserves the monoid multiplication and identity. We now introduced *monoidal functor* between monoidal categories which preserves the monoidal structure.

Definition 2.7 (Monoidal Functor)**.** Let $(\mathcal{M}, \otimes, I, \alpha, r, l)$ and $(\mathcal{M}', \otimes', I', \alpha', r', l')$ be two monoidal categories. A *(strong) Monoidal Functor* (F, ϕ, φ) from $\mathcal{M}$ to $\mathcal{M}'$ is a functor $F : \mathcal{M} \to \mathcal{M}'$ equipped with

1. a natural isomorphism ϕ between the following functors $\phi : F(-)\otimes' F(-) \cong F(-\otimes-) : \mathcal{M}^2 \to \mathcal{M}'$ with components $\phi_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$ for all $A, B \in \mathcal{M}$, and
2. an isomorphism $\psi : I' \to F(I)$.

such that the following diagrams commute:

-
$$\begin{tikzcd}
(F(A) \otimes' F(B)) \otimes' F(C) \arrow[r, "\alpha'"] \arrow[d, "\phi\otimes' 1"] & F(A) \otimes' (F(B) \otimes' F(C)) \arrow[d, "1\otimes'\phi"] \\
F(A \otimes B) \otimes' F(C) \arrow[d, "\phi"] & F(A) \otimes' F(B \otimes C) \arrow[d, "\phi"] \\
F((A \otimes B) \otimes C) \arrow[r, "F(\alpha)"] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

-
$$\begin{tikzcd}
F(A) \otimes' I' \arrow[r, "1\otimes'\psi"] \arrow[d, "r'"] & F(A) \otimes' F(I) \arrow[d, "\phi"] & I' \otimes' F(B) \arrow[r, "\psi\otimes' 1"] \arrow[d, "l'"] & F(I) \otimes' F(B) \arrow[d, "\phi"] \\
F(A) & F(A \otimes I) \arrow[l, "F(r)"'] & F(B) & F(I \otimes B) \arrow[l, "F(l)"']
\end{tikzcd}$$

Remark. A monoidal functor is called *strict* if $\phi_{A,B}$ for all $A, B \in \mathcal{M}$ and ψ are identity maps.

The axioms ensure that for more than 3 objects involved, every extension of the first diagram commutes for any 2 different arrangements of brackets v and w [23].

$$\begin{tikzcd}
v(F(A_1), ..., F(A_n)) \arrow[r, "\alpha' s"] \arrow[d, "\phi' s"] & w(F(A_1), ..., F(A_n)) \arrow[d, "\phi' s"] \\
F(v(A_1, ..., A_n)) \arrow[r, "F(\alpha' s)"] & F(w(A_1, ...A_n))
\end{tikzcd}$$

Suppose the monoidal categories $\mathcal{M}$ and $\mathcal{M}'$ are braided (possibly symmetric) with braidings c and c' respectively. A monoidal functor $(F, \phi, \psi) : \mathcal{M} \to \mathcal{M}'$ is called a *braided monoidal functor* if in addition the following diagram

$$\begin{tikzcd}
F(A) \otimes' F(B) \arrow[r, "c'"] \arrow[d, "\phi"] & F(B) \otimes' F(A) \arrow[d, "\phi"] \\
F(A \otimes B) \arrow[r, "F(c)"] & F(B \otimes A)
\end{tikzcd}$$

commutes, respecting the structure of the braidings. With regards to symmetric monoidal categories, *symmetric monoidal functors* are braided monoidal functors with no extra conditions.

While monoidal functor preserves the monoidal structure, *monoidal natural transformations* are natural transformations between monoidal functors that respects the structures of monoidal functors.

Definition 2.8 (Monoidal Natural Transformation)**.** Let $(\mathcal{M}, \otimes, I, \alpha, r, l)$ and $(\mathcal{M}', \otimes', I', \alpha', r', l')$ be two monoidal categories. And suppose that (F, ϕ, ψ) and (G, γ, η) are monoidal functors from $\mathcal{M}$ to $\mathcal{M}'$. A *monoidal natural transformation* is a natural transformation $\beta : F \Rightarrow G$ such that the following diagrams commutes:

$$\begin{tikzcd} F(A) \otimes' F(B) \arrow[r, "\beta_A \otimes' \beta_B"] \arrow[d, "\phi"] & G(A) \otimes' G(B) \arrow[d, "\gamma"] \\ F(A \otimes B) \arrow[r, "\beta_{A \otimes B}"] & G(A \otimes B) \end{tikzcd}$$

and

$$\begin{tikzcd} I' \arrow[d, "\psi"] \arrow[dr, "\eta"] & \\ F(I) \arrow[r, "\beta_I"] & G(I) \end{tikzcd}$$

Remark. Braided and symmetric monoidal natural transformations are plain monoidal natural transformations requiring no further additional conditions.

Monoidal equivalence and Mac Lane's Strictness Theorem

With the definitions introduced above, monoidal categories together with monoidal functors and monoidal natural transformations form a 2-category **MonCat**. Similarly, we have 2-categories **BrMonCat** and **SymmMonCat** with braided (respectively, symmetric) monoidal categories, braided (respectively, symmetric) monoidal functors and braided (respectively, symmetric) monoidal natural transformations.

We all know that equivalence of two categories C and $\mathcal{D}$ is given by functors $K : C \to \mathcal{D}$ and $H : \mathcal{D} \to C$ together with natural isomorphisms $\theta : \mathrm{id}_C \Rightarrow HK$ and $\vartheta : KH \Rightarrow \mathrm{id}_D$. Analogously, two monoidal categories $\mathcal{M}$ and $\mathcal{N}$ are *monoidally equivalent* if there are monoidal functors $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{M}$ together with monoidal natural isomorphisms $\eta : \mathrm{id}_{\mathcal{M}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{N}}$ (that is, if $\mathcal{M}$ and $\mathcal{N}$ are *internally equivalent* objects in the 2-category **MonCat**). Equivalently, a monoidal equivalence between two monoidal categories $\mathcal{M}$ and $\mathcal{N}$ is given by a monoidal functor $F : \mathcal{M} \to \mathcal{N}$ which is an equivalence of $\mathcal{M}$ and $\mathcal{N}$ as ordinary categories ([8], Definition 2.4.1, Remark 2.4.10). *Monoidal isomorphism* refers to when η and ϵ are identities.

In the similar way, braided monoidal equivalence and symmetric monoidal equivalence refers to when two objects in the 2-category **BrMonCat** and respectively, two objects in **SymmMonCat**, are internally equivalent.

It is easier to work with strict monoidal categories than general monoidal categories. Thanks to Mac Lane's Strictness Theorem, we can practically regard monoidal categories to be strict.

Theorem 2.1 (Mac Lane's Strictness Theorem)**.** *Any monoidal category C is monoidally equivalent to some strict monoidal category C'.*

Proof. We refer to Theorem 2.8.5 [8] for the verification of the proof. Alternatively, from the fact that every bicategory is biequivalent (Chapter 4) to a 2-category [22], one could verify this by considering the one-object case this biequivalence where a pseudofunctor between one-object bicategories can be identified with a monoidal functor. □

The Strictness Theorem means that studies of monoidal categories can conveniently be conducted with equivalent strict ones where findings about them in the strict ones also hold in equivalent general ones.

The *Coherence Theorem of monoidal categories* [23] then follows as a corollary of the Strictness Theorem ([8], Theorem 2.9.2) which states that: suppose we have $P_1, P_2 \in \mathcal{M}$ obtained by sensible arrangement of brackets on $A_1 \otimes \cdots \otimes A_n$ with random insertions of I's, such as

$$P_1 = (A_1 \otimes I) \otimes ((A_2 \otimes A_3) \otimes ...) \otimes (A_n \otimes I)$$

and

$$P_2 = (A_1 \otimes A_2) \otimes (A_3 \otimes ...(I \otimes A_4)).$$

Then any isomorphisms of P_1 and P_2 obtained by compositions of instances of associators, right unitors, left unitors and their inverses and (by the bifunctor $\otimes$) identity morphism are equal.

2.3 Action of monoid and Closed monoidal categories

In this section, we introduce what it means for a monoidal category to be closed. We will also define what action of a monoid is and what it is equivalent to in symmetric closed monoidal categories. This result will be used throughout the various symmetric closed monoidal categories appearing in the thesis.

Definition 2.9 (Monoid action)**.** Let $(C, \otimes, I, a, r, l)$ be a monoidal category. Let (M, m, e) be a monoid in $(C, \otimes, I)$. An *action of Monoid object M on an object $C \in \mathrm{ob}\, C$* in a monoidal category $(C, \otimes, I)$ is given by a morphism $\mu : M \otimes C \to C$ in C such that the following diagrams commutes:

1.
$$\begin{tikzcd}
 & M \otimes (M \otimes C) \arrow[r, "1_M \otimes \mu"] & M \otimes C \arrow[d, "\mu"] \\
(M \otimes M) \otimes C \arrow[ur, "a"] \arrow[d, "m \otimes 1_C"] & & \\
M \otimes C \arrow[rr, "\mu"] & & C
\end{tikzcd}$$

2.
$$\begin{tikzcd}
I \otimes C \arrow[rr, "e \otimes 1_C"] \arrow[dr, "l"'] & & M \otimes C \arrow[dl, "\mu"] \\
 & C &
\end{tikzcd}$$

Remark. Every monoid object (M, m, e) of a monoidal category $(C, \otimes, I)$ acts on itself with the monoid multiplication map $m : M \otimes M \to M$ as the action map. When our monoidal category is $(\mathbf{Set}, \times, 1)$, we retrieve the usual set-theoretic definition of monoid action.

Suppose we have a usual set-theoretic monoid M. It is elementary to see that *to give an action of a monoid M on a set X is equivalent to give a monoid homomorphism $M \to$ **Set**(X, X)*. This uses the following property of $(\mathbf{Set}, \times, 1)$ that there is a bijection of sets

$$\mathbf{Set}(X \times Y, Z) \cong \mathbf{Set}(X, \mathbf{Set}(Y, Z)).$$

Definition 2.10 (Closed monoidal category)**.** A monoidal category $(C, \otimes, I, a, r, l)$ is (right) *closed* if each functor $(- \otimes X) : C \to C$ has a right adjoint $[X, -] : C \to C$.

This means that there are bijections

$$\theta_{A,B} : C(A \otimes X, B) \cong C(A, [X, B])$$

natural in A and B. Given $f \in C(A \otimes X, B)$, we write $\overline{f} \in C(A, [X, B])$ for the adjunct $\theta_{A,B}(f)$. We also write, given $g \in C(A, [X, B])$, $\overline{g} \in C(A \otimes X, B)$ for the adjunct $\theta^{-1}_{A,B}(g)$.

We call the object $[X, B] \in C$ the *internal hom* from X to B and call the counit the *evaluation map* $\mathrm{ev}_{X,B} : [X, B] \otimes X \to B$. It can be easily seen that they satisfy the property such that for any morphism $f : A \otimes X \to Y$, we have $\mathrm{ev}_{X,Y} \circ (\overline{f} \otimes 1_X) = f$.

(Right) closed monoidal category can also be alternatively defined as the following, which has been reviewed in [27].

Definition 2.11. A monoidal category $(C, \otimes, I, a, r, l)$ is (right) *closed* if it is equipped with

- an object $[X, B] \in C$
- a morphism $\mathrm{ev}_{X,B} : [X, B] \otimes X \to B$ called *evaluation map*

for each $X, B \in C$ such that for every morphism $f : A \otimes X \to B$ there exists a unique morphism $\overline{f} : A \to [X, B]$ such that $f = \mathrm{ev}_{X,B} \circ (\overline{f} \otimes 1_X)$

A monoidal category is *biclosed* if, not only it is closed, but in addition it is also equipped with a *left closed structure*: if every $(X \otimes -) : C \to C$ has a right adjoint $[[X, -]] : C \to C$.

In case of when a monoidal category is symmetric with braiding γ, it is left closed if and only if it is right closed, with $[[A, B]] = [A, B]$ and the functors $(- \otimes A) \cong (A \otimes -) : C \to C$ isomorphic and the *left evaluation map* is $\mathrm{ev}'_{X,B} = \mathrm{ev}_{X,B} \circ \gamma_{X,[X,B]} : X \otimes [X, B] \to B$

In the next propositions, we show that the equivalence between a monoid action $M \otimes C \to C$ and a monoid homomorphism $M \to [C, C]$, like that in $(\mathbf{Set}, \times, 1)$, holds true in general symmetric closed monoidal category. We start by showing that for any symmetric closed monoidal category $(C, \otimes, I)$, the internal hom $[A, A]$ for any $A \in \mathrm{ob}\, C$ is a monoid object of $(C, \otimes, I)$.

Proposition 2.2. *When C is a symmetric closed monoidal category, we have that for every $A \in C$, the object $[A, A] \in C$ is a monoid object with the composition morphism $M : [A, A] \otimes [A, A] \to [A, A]$ and the identity element $i : I \to [A, A]$ defined respectively as the adjuncts under $C(A \otimes X, B) \cong C(A, [X, B])$ of the composite*

$$([A, A] \otimes [A, A]) \otimes A \xrightarrow{a} [A, A] \otimes ([A, A] \otimes A) \xrightarrow{1\otimes \mathrm{ev}} [A, A] \otimes A \xrightarrow{\mathrm{ev}} A$$

and the left unitor $l_A : I \otimes A \to A$.

Proof. This is actually a part of a bigger fact as noted in [18] that when C is symmetric closed monoidal category, C is equipped with a canonical structure of a C-enriched category by the following data:

- objects are those of $\mathrm{ob}\, C$;
- for all $A, B \in \mathrm{ob}\, C$, $[A, B] \in C$ as the hom-object;

- for all $A, B, C \in \operatorname{ob} \mathcal{C}$, the composition map

$$M : [B, C] \otimes [A, B] \to [A, C]$$

corresponding to the adjunct of the composite

$$([B, C] \otimes [A, B]) \otimes A \xrightarrow{a} [B, C] \otimes ([A, B] \otimes A) \xrightarrow{1 \otimes \mathrm{ev}} [B, C] \otimes B \xrightarrow{\mathrm{ev}} C$$

- for all $A \in \operatorname{ob} \mathcal{C}$, the unit $i : I \to [A, A]$ corresponding to the adjunct of the left unitor $l_A : I \otimes A \to A$.

□

Proposition 2.3. *In a symmetric closed monoidal category $\mathcal{C}$, to give an action $\mu : M \otimes C \to C$ of a monoid object $(M, m, e) \in \mathcal{C}$ on an object $C \in \mathcal{C}$ is equivalent to give a monoid morphism $\overline{\mu} : M \to [C, C]$*

Proof. Suppose we have an action $\mu : M \otimes C \to C$ satisfying 1. and 2. of Definition 2.9.

Let the adjunct of μ be the morphism $\overline{\mu} : M \to [C, C]$. The $\mathcal{C}((M \otimes M) \otimes C, C) \cong \mathcal{C}(M \otimes M, [C, C])$-adjunct of the left side

$$(M \otimes M) \otimes C \xrightarrow{m \otimes 1} M \otimes C \xrightarrow{\mu} C$$

of 1. is given by

$$M \otimes M \xrightarrow{m} M \xrightarrow{\overline{\mu}} [C, C].$$

And the right side

$$(M \otimes M) \otimes C \xrightarrow{a} M \otimes (M \otimes C) \xrightarrow{1 \otimes \mu} M \otimes C \xrightarrow{\mu} C$$

of 1. is equal to

$$(M \otimes M) \otimes C \xrightarrow{(\overline{\mu} \otimes \overline{\mu}) \otimes 1} ([C, C] \otimes [C, C]) \otimes C \xrightarrow{a} [C, C] \otimes ([C, C] \otimes C) \xrightarrow{\mathrm{ev} \circ (1 \otimes \mathrm{ev})} C$$

which can be easily seen since

$$\begin{aligned} \mathrm{ev} \circ (1 \otimes \mathrm{ev}) \circ a \circ ((\overline{\mu} \otimes \overline{\mu}) \otimes 1) &= \mathrm{ev} \circ (1 \otimes \mathrm{ev}) \circ (\overline{\mu} \otimes (\overline{\mu} \otimes 1)) \circ a \\ &\qquad \text{(from the naturality of } a\text{)} \\ &= \mathrm{ev} \circ (\overline{\mu} \otimes \mu) \circ a \ \ (\text{by } \mathrm{ev} \circ (\overline{\mu} \otimes 1) = \mu) \\ &= \mu \circ (1 \otimes \mu) \circ a \ \ (\text{by } \mathrm{ev} \circ (\overline{\mu} \otimes 1) = \mu). \end{aligned}$$

The adjunct of this is then

$$M \otimes M \xrightarrow{\overline{\mu} \otimes \overline{\mu}} [C, C] \otimes [C, C] \xrightarrow{M} [C, C]$$

Hence the following diagram commutes

$$\begin{tikzcd} M \otimes M \arrow[r, "\overline{\mu} \otimes \overline{\mu}"] \arrow[d, "m"] & {[C, C] \otimes [C, C]} \arrow[d, "M"] \\ M \arrow[r, "\overline{\mu}"] & {[C, C].} \end{tikzcd}$$

Likewise, consider now the diagram 2. in Definition 2.9. The following diagram:

$$\begin{tikzcd} I \arrow[r, "e"] \arrow[d, "i"] & M \arrow[dl, "\overline{\mu}"] \\ {[C, C]} & \end{tikzcd}$$

is the adjunct of the diagram 2. in the Definition 2.9 under $\mathcal{C}(I \otimes C, C) \cong \mathcal{C}(I, [C, C])$, completing the proof that the axioms of monoid actions corresponds to the axioms of monoid morphisms in symmetric closed monoidal categories. □

3
Result in $(\mathbf{Cat}, \times, I)$

3.1 Action of Monoidal Category

Categorification of the set-theoretic notion of a monoid action is the action of a monoidal category. Introduced in [4] and presented in [15], an *action of a monoidal category* $(\mathcal{M}, \otimes, I, a, r, l)$ on a category $\mathcal{C}$ is a functor $\cdot : \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ with components of two natural isomorphisms

$$\begin{aligned} \alpha_{M,N,X} &: (M \otimes N) \cdot X \cong M \cdot (N \cdot X) \\ \lambda_X &: I \cdot X \cong X \end{aligned}$$

such that the following diagram commutes:

$$\begin{tikzcd}
((M \otimes N) \otimes K) \cdot X \arrow[r, "a_{M,N,K} \cdot id_X"] \arrow[d, "\alpha_{(M \otimes N),K,X}"'] & (M \otimes (N \otimes K)) \cdot X \arrow[dd, "\alpha_{M,(N \otimes K),X}"] \\
(M \otimes N) \cdot (K \cdot X) \arrow[d, "\alpha_{M,N,(K \cdot X)}"'] & \\
M \cdot (N \cdot (K \cdot X)) & M \cdot ((N \otimes K) \cdot X)) \arrow[l, "id_M \cdot \alpha_{N,K,X}"]
\end{tikzcd}$$

$$\begin{tikzcd}
(M \otimes I) \cdot X \arrow[d, "\alpha_{M,I,X}"'] \arrow[dr, "r_M \cdot id_X"] & \\
M \cdot (I \cdot X) \arrow[r, "id_M \cdot \lambda_X"'] & M \cdot X
\end{tikzcd}
\qquad
\begin{tikzcd}
(I \otimes M) \cdot X \arrow[d, "\alpha_{I,M,X}"'] \arrow[dr, "l_M \cdot id_X"] & \\
I \cdot (M \cdot X) \arrow[r, "\lambda_{M \cdot X}"'] & M \cdot X
\end{tikzcd}$$

Strict monoidal action is when the natural isomorphisms α and λ are identities. Under the closed property of $(\mathbf{Cat}, \times, I)$, to give an action of a monoidal category is equivalently to give a monoidal functor $(F, \phi, \varphi) : \mathcal{M} \to \mathbf{Cat}(C, C)$ where $\mathbf{Cat}(C, C)$ is the endofunctor category with the strict monoidal structure as shown in Example 2.2. It is elementary to check that to give components of natural isomorphisms $\alpha_{M,N,X} : (M \otimes N) \cdot X \cong M \cdot (N \cdot X)$ natural in M, N and X is to give components of natural isomorphisms $\phi_{M,N} : F_M \circ F_N \cong F_{M \otimes N}$ natural in M, N and to give $\lambda_X : I \circ X \cong X$ natural in X is to give an isomorphism $\varphi : \mathrm{id}_C \cong F_I$ and to give the coherence axiom for the action is equivalently to give the coherence axiom for the monoidal functor.

The strict case of this equivalence is precisely the statement in Proposition 2.3 in monoidal category $(\mathbf{Cat}, \times, I)$: *to give a strict action of a strict monoidal category $\mathcal{M}$ on a category C is equivalent to give a strict monoidal functor $\mathcal{M} \to \mathbf{Cat}(C, C)$.*

In this chapter, we will illustrate what this statement reduces to for one-object cases and also how enrichment over the monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$ of abelian groups gives a familiar known statement about algebra over a ring.

3.2 One object case

Lemma 3.1. *A monoid object in the monoidal category of monoids is precisely a commutative monoid.*

Proof. Consider the monoidal category ($\mathbf{Mon}$, $\times$, I) monoids where I is the one object monoid and $\times$ is the direct product of monoids. A monoid object in $\mathbf{Mon}$ is a monoid $(M, \circ, 1)$ equipped with a monoid homomorphism $\star : M \times M \to M$ (in other words, a binary function $\star : M \times M \to M$ such that $(m \circ l) \star (n \circ k) = (m \star n) \circ (l \star k)$ and $1 \star 1 = 1$) such that for all $m, n, l \in M$ and the identity element $1 \in M$, we have $(m \star n) \star l = m \star (n \star l)$ and $m \star 1 = 1 \star m = m$.

So we have a set M equipped with 2 binary functions $\circ : M \times M \to M$ and $\star : M \times M \to M$ such that:

1. $\circ$ is associative and unital with identity element 1
2. $\star$ is associative and unital with the same identity element 1
3. interchange law $(m \circ l) \star (n \circ k) = (m \star n) \circ (l \star k)$ holds.

It then follows from Eckmann-Hilton argument [7] that the operations $\circ, \star$ are the same and that M is a commutative monoid:

$$a \star b = (a \circ 1) \star (1 \circ b) = (a \star 1) \circ (1 \star b) = a \circ b$$

$$a \circ b = (1 \circ a) \circ (b \circ 1) = (1 \circ b) \circ (a \circ 1) = b \circ a.$$

Conversely, any commutative monoid $(M, \circ, 1)$, we can make M a monoid object in the category $\mathbf{Mon}$ since it is equipped with the monoid homomorphism $\circ : M \times M \to M$ (the binary function $\circ : M \times M \to M$ such that $(m \circ l) \circ (n \circ k) = (m \circ n) \circ (l \circ k)$ and $1 \circ 1 = 1$) which is associative and unital (and it is the unique way to equip M with associative and unital monoid homomorphism $M \times M \to M$ as seen previously). $\square$

Lemma 3.2. *A strict monoidal category with one object is precisely a commutative monoid.*

Proof. Let $(\mathcal{M}, \otimes, *)$ be a strict monoidal category with one object $*$. Then $\mathcal{M}(*, *) = k$ is a monoid under composition $\circ$. Also it is equipped with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ with the identity natural isomorphisms between the below functors

$$(- \otimes -) \otimes - = - \otimes (- \otimes -) : \mathcal{M}^3 \to \mathcal{M}$$
$$(- \otimes *) = \mathrm{id}_{\mathcal{M}} = (* \otimes -) : \mathcal{M} \to \mathcal{M}$$

making these naturality squares commute:

$$\begin{tikzcd} * \arrow[r, equal] \arrow[d, "(m\otimes n)\otimes j"'] & * \arrow[d, "m\otimes(n\otimes j)"] \\ * \arrow[r, equal] & * \end{tikzcd} \quad \begin{tikzcd} * \arrow[r, equal] \arrow[d, "m\otimes 1"'] & * \arrow[d, "m"] \\ * \arrow[r, equal] & * \end{tikzcd} \quad \begin{tikzcd} * \arrow[r, equal] \arrow[d, "1\otimes m"'] & * \arrow[d, "m"] \\ * \arrow[r, equal] & * \end{tikzcd}$$

i.e. it is equipped with a function $\otimes : k \times k \to k$ such that $(m \otimes n) \circ (l \otimes j) = (m \circ l) \otimes (n \circ j)$ (from functoriality of $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$) and such that $(m \otimes n) \otimes j = m \otimes (n \otimes j)$ and $m \otimes 1 = m = 1 \otimes m$ (from the naturality square above).

It is precisely a monoid object in the monoidal category of monoids and so k is a commutative monoid. $\square$

Lemma 3.3. *Let k be a commutative monoid and A be a monoid. To give a structure on A as a monoid object in the monoidal category of k-sets and k-functions [1] is equivalently to give a strict action of the strict monoidal category with one object $\mathcal{M}(*, *) = k$ on the category with one object $C(*, *) = A$.*

Proof. Regard k as a one-object strict monoidal category $\mathcal{M}(*, *) = k$. And regard A as a one-object category $C(*, *) = A$. A strict action of $(\mathcal{M}, \circ, *)$ on C is a functor $\mathcal{M} \times C \to C$ together with two identity natural isomorphisms $(* \circ *) \cdot * = * \cdot (* \cdot *)$ and $* \cdot * = *$ making the following naturality square commute:

$$\begin{tikzcd} (* \circ *) \cdot * \arrow[r, equal] \arrow[d, "(m\circ n)\cdot x"'] & * \cdot (* \cdot *) \arrow[d, "m\cdot(n\cdot x)"] \\ (* \circ *) \cdot * \arrow[r, equal] & * \cdot (* \cdot *) \end{tikzcd} \quad \begin{tikzcd} * \cdot * \arrow[r, equal] \arrow[d, "1_k\cdot x"'] & * \arrow[d, "x"] \\ * \cdot * \arrow[r, equal] & * \end{tikzcd}$$

To give such functor $\mathcal{M} \times C \to C$ with such natural isomorphisms is equivalently to give a function $k \times A \to A$ such that

$$(mn) \cdot x = m \cdot (n \cdot x) \tag{3.1}$$
$$1_k \cdot x = x \tag{3.2}$$

[1] We call a set X equipped with an action $\cdot$ of a monoid M as an *M-set*. Let X and Y be two M-sets. An *M-function* f is a function $f : X \to Y$ such that $f(m \cdot x) = m \cdot f(x)$. Let Z also be an M-set. A function $h : X \times Y \to Z$ is *M-bilinear* if for all $x \in X$ and $y \in Y$, $h(x, -) : Y \to Z$ and $h(-, y) : X \to Z$ are M-functions.

We can form a category **M-Set** of M-sets and M-functions. And let A and B be two M-sets. Let $A \otimes B$ be the quotient set of $A \times B$ by $(ka, b) \sim (a, kb)$ and $A \otimes B$ can be made into an M-set with monoid action $m[(a, b)] := [(ma, b)] = [(a, mb)]$.

This then forms a monoidal category (**M-Set**, $\otimes$, $\{*\}$).

(from naturality) and

$$(m' \cdot x') \circ_A (m \cdot x) = m'm \cdot (x' \circ_A x) \tag{3.3}$$

(from functoriality).

(3.1) and (3.2) says that the monoid A is equipped with a structure of k-set and (3.3) says that its monoid multiplication $\circ_A : A \times A \to A$ is also a bilinear k-function which is equivalently a (associative and unital) k-function $A \otimes A \to A$ where $A \otimes A$ is the monoidal product in the category $(\mathbf{k\text{-}Set}, \otimes, \{*\})$ [2].

Hence, a strict action of a strict monoidal category with one object $\mathcal{M}(*, *) = k$ on a category with one object $C(*, *) = A$ is equivalently a structure, on A, of a monoid object in the category $(\mathbf{k\text{-}Set}, \otimes, \{*\})$. □

The next theorem assembles what the equivalence between a monoid action $\mathcal{M} \times C \to C$ and a monoid morphism $\mathcal{M} \to \mathbf{Cat}(C, C)$ in $(\mathbf{Cat}, \times, I)$ reduces to for one-object cases of $\mathcal{M}$ and C.

Theorem 3.4. *Let k be a commutative monoid and A be a monoid. Then a structure, on A, of a monoid object in the monoidal category* $(\mathbf{k\text{-}Set}, \otimes, \{*\})$ *is equivalent to a map of commutative monoids:* $k \to Z(A)$ *where* $Z(A)$ *is the centre of the monoid A.*

Proof. Regard k as a one object strict monoidal category $(\mathcal{M}, \circ, *)$ with $\mathcal{M}(*, *) = k$ and regard A as a one object category C with $C(*, *) = A$. We know from Lemma 3.3 that to give a structure, on A, of a monoid object in $(\mathbf{k\text{-}Set}, \otimes, \{*\})$ is equivalently to give a strict action of the strict monoidal category $\mathcal{M}$ on C.

By the Proposition 2.3 in $(\mathbf{Cat}, \times, I)$, this is equivalent to a strict monoidal functor $\mathcal{M} \to \mathbf{Cat}(C, C)$.

Now consider $\mathbf{Cat}(C, C)$ is when C is an one-object category. It is an endofunctor category with:

- **objects**: monoid homomorphisms $f : A \to A$
- **morphisms**: $\alpha : f \Rightarrow g$ is an element $\alpha \in A$ such that the naturality square

$$\begin{tikzcd} * \arrow[r, "\alpha"] \arrow[d, "f(y)"] & * \arrow[d, "g(y)"] \\ * \arrow[r, "\alpha"] & * \end{tikzcd}$$

  commutes for all $y \in A$.

Let's see what a strict monoidal functor $F : \mathcal{M} \to \mathbf{Cat}(C, C)$ reduces to. It is a functor $F : \mathcal{M} \to \mathbf{Cat}(C, C)$ which maps with

1. components of a identity natural isomorphism $F(*) \circ F(*) = F(*)$
2. an identity $1_C = F(*)$

Such functor is equivalent to give a monoid homomorphism from k to the set of natural transformations from 1_A to 1_A.

i.e. a monoid homomorphism $k \to \mathbf{Cat}(C, C)(1_C, 1_C)$ which is equivalent to a monoid homomorphism $k \to \mathrm{Z}(A) = \{x \in A \mid xy = yx \ \forall y \in A\}$. □

[2] There is a bijection between the set of M-functions $A \otimes B \to C$ and the set of bilinear M-functions $A \times B \to C$. Consider a bilinear M-function $h : A \times B \to A \otimes B$ which maps $h(ma, nb) = mn[(a, b)]$. Then any bilinear M-function $f : A \times B \to C$ factors uniquely through h. i.e. for every function $f : A \times B \to C$ such that $f(ma, nb) = mnf(a, b)$, we can define an M-function $\overline{f} : A \otimes B \to C$ by $\overline{f}(m[(a, b)]) = mf(a, b)$. Then $f = \overline{f} \circ h$ and $\overline{f}$ is the unique M-function which satisfies $f = \overline{f} \circ h$. Moreover, the M-set $A \otimes B$ with this property is unique (can be easily shown).

3.3 Enriched case

It is well known that the definition of algebra over a ring can be given in two alternate ways. Let R be a commutative unital ring. An *R-Algebra* is a ring R' together with a ring homomorphism $\varphi : R \to Z(R') \subseteq R'$. ($Z(R') = \{z \in R' : zr = rz\ \forall R\}$ is the centre of R'). Alternatively, an *R-Algebra* is a ring R' which is also an R-module such that the multiplication map $R' \times R' \to R'$ is R-bilinear.

If we enrich the previous result over the monoidal category $(\mathbf{Ab}, \otimes, \mathbb{Z})$, we get the familiar result stating the equivalence of these two alternate definitions of R-algebra. We first begin with a brief digression into the closed monoidal structure of $\mathcal{V}$-**Cat**, the 2-category of $\mathcal{V}$-enriched categories as presented in [18], when the right condition is given for the monoidal category $\mathcal{V}$ we're enriching over. We follow closely with the materials and notations in the text [18].

The idea of a $\mathcal{V}$-category for a monoidal category $(\mathcal{V}, \otimes, I, a, r, l)$ is a generalisation of the notion of a category, the **Set**-category. An ordinary category $\mathcal{A}$ has hom-*sets* $\mathcal{A}(X, Y)$ and composition-*functions* $\mathcal{A}(Y, Z) \times \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)$. In a $\mathcal{V}$-enriched category C, these are replaced by hom-*objects* $C(X, Y)$ and composition-*morphisms* $C(Y, Z) \otimes C(X, Y) \to C(X, Z)$ of the given monoidal category $(\mathcal{V}, \otimes, I)$.

Definitions of (small) $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations are classical and it is well known that these form a 2-category $\mathcal{V}$-**Cat**.

When the base monoidal category $(\mathcal{V}, \otimes, I)$ is symmetric, the 2-category $\mathcal{V}$-**Cat** admits a natural 2-functor $\otimes : \mathcal{V}\text{-}\mathbf{Cat} \times \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ which gives for each pair of small $\mathcal{V}$-categories C and $\mathcal{D}$, the *product $\mathcal{V}$-category* $C \otimes \mathcal{D}$ with $\mathrm{ob}(C \otimes \mathcal{D}) = \mathrm{ob}\, C \times \mathrm{ob}\, \mathcal{D}$ and $C \otimes \mathcal{D}((C, D), (C', D')) = C(C, C') \otimes \mathcal{D}(D, D')$. The definitions of *product $\mathcal{V}$-functor* $F \otimes G : C \otimes \mathcal{D} \to C' \otimes \mathcal{D}'$ and *product $\mathcal{V}$-natural transformation* $\alpha \otimes \beta : F \otimes G \to F' \otimes G'$ which constitute the 2-functor are easy to see.

This 2-functor gives $\mathcal{V}$-**Cat** a structure of symmetric monoidal 2-category in the sense that we have components of coherent 2-natural isomorphisms: $(C \otimes \mathcal{D}) \otimes \mathcal{E} \cong C \otimes (\mathcal{D} \otimes \mathcal{E})$, $\mathcal{I} \otimes C \cong C \cong C \otimes \mathcal{I}$, $C \otimes \mathcal{D} \cong \mathcal{D} \otimes C$. The unit $\mathcal{I}$ for this monoidal product is the $\mathcal{V}$-category with one object $*$ and with $\mathcal{I}(*, *) = I$. Since these are isomorphisms in the 2-category $\mathcal{V}$-**Cat**, it is evident that we can consider $\mathcal{V}$-**Cat** as a mere symmetric monoidal category by considering it without $\mathcal{V}$-natural transformations, the 2-cells.

Ends over $\mathcal{V}$-valued $\mathcal{V}$-functor for symmetric closed monoidal $\mathcal{V}$

It was mentioned that when $\mathcal{V}$ is a symmetric closed monoidal category,

$$\mathcal{V}(A, [B, C]) \cong \mathcal{V}(A \otimes B, C) \cong \mathcal{V}(B, [A, C]) \tag{3.4}$$

then $\mathcal{V}$ is equipped with a structure of a $\mathcal{V}$-category with internal homs $[B, C] \in \mathcal{V}$ as its hom-objects for each pair $B, C \in \mathcal{V}$. Consider a $\mathcal{V}$-valued $\mathcal{V}$-functor $H : C^{op} \otimes C \to \mathcal{V}$. Note that this induces a partial functor $H(C, -) : C \to \mathcal{V}$ which is defined as

$$C \xrightarrow{\cong} \mathcal{I} \otimes C \xrightarrow{C \otimes 1} C^{op} \otimes C \xrightarrow{H} \mathcal{V}. \tag{3.5}$$

And likewise, the partial functor $H(-, C) : C^{op} \to \mathcal{V}$ is defined similarly. The *enriched-end (2.1 [18]) of H* is an object of $\mathcal{V}$ which we write as $\int_{C \in C} H(C, C)$ with universal family of morphisms $\alpha_C : \int_{C \in C} H(C, C) \to H(C, C)$ making each of the below diagrams expressing

$\mathcal{V}$-naturality condition commute

$$\begin{array}{ccc} \int_{C\in\mathcal{C}} H(C,C) & \xrightarrow{\alpha_C} & H(C,C) \\ \downarrow{\scriptstyle \alpha_{C'}} & & \downarrow{\scriptstyle \phi_{C,C'}} \\ H(C',C') & \xrightarrow{\varphi_{C,C'}} & [\mathcal{C}(C,C'),H(C,C')] \end{array} \tag{3.6}$$

such that for any other family of morphisms $\beta_C : K \to H(C,C)$ making each diagrams (3.6) commute, there exists a unique $f : \int_{C\in\mathcal{C}} H(C,C) \to K$ such that $\beta_C = \alpha_C f$.

Here, $\phi_{C,C'}$ is the adjunct (3.4) of $H(C,-)_{C,C'} : \mathcal{C}(C,C') \to [H(C,C),H(C,C')]$ from (3.5) and likewise, $\varphi_{C,C'}$ is the same adjunct of $H(-,C)_{C,C'}$.

When $\mathcal{V}$ is complete, enriched ends exist and can be formulated as an equalizer

$$\int_{C\in\mathcal{C}} H(C,C) \xrightarrow{\alpha} \prod_{C\in\mathcal{C}} H(C,C) \overset{\phi}{\underset{\varphi}{\rightrightarrows}} \prod_{C,C'\in\mathcal{C}} [\mathcal{C}(C,C'),H(C,C')]$$

where ϕ is induced by $\phi_{C,C'} : H(C,C) \to [\mathcal{C}(C,C'),H(C,C')]$ and φ is induced by $\varphi_{C,C'}$: $H(C',C') \to [\mathcal{C}(C,C'),H(C,C')]$.

Enriched functor category and closed structure

We now add the condition that the symmetric monoidal category $(\mathcal{V},\otimes,I)$ is monoidally closed and complete. In this case, the category $\mathcal{V}$-$\mathbf{Cat}(\mathcal{C},\mathcal{D})$ of $\mathcal{V}$-functors $\mathcal{C}\to\mathcal{D}$ and $\mathcal{V}$-natural transformations is given the structure of a $\mathcal{V}$-category; it is the *underlying category* (1.3 [18]) $\mathcal{V}$-$\mathbf{Cat}(\mathcal{I},[\mathcal{C},\mathcal{D}])$ of the *functor* $\mathcal{V}$*-category* $[\mathcal{C},\mathcal{D}]$. The *functor* $\mathcal{V}$*-category* $[\mathcal{C},\mathcal{D}]$ has the $\mathcal{V}$-functors $\mathcal{C}\to\mathcal{D}$ as objects with hom-objects $[\mathcal{C},\mathcal{D}](F,G)\in\mathcal{V}$ defined (2.2 [18]) as enriched ends $\int_{C\in\mathcal{C}} \mathcal{D}(FC,GC)$

$$\int_{C\in\mathcal{C}} \mathcal{D}(FC,GC) \xrightarrow{\alpha} \prod_{C\in\mathcal{C}} \mathcal{D}(FC,GC) \overset{\phi}{\underset{\varphi}{\rightrightarrows}} \prod_{C,C'\in\mathcal{C}} [\mathcal{C}(C,C'),\mathcal{D}(FC,GC')].$$

Moreover, there are isomorphisms (2.3 [18])

$$\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \cong \mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A},[\mathcal{B},\mathcal{C}]) \tag{3.7}$$

of ordinary categories 2-natural in each variable from which we can once again regard $\mathcal{V}$-$\mathbf{Cat}$ as a mere symmetric closed monoidal category by considering it without $\mathcal{V}$-natural transformations and considering the isomorphisms (3.7) as mere bijections of object-sets.

Thus, when our base category $\mathcal{V}$ is a complete symmetric closed monoidal category, we can now talk about monoid actions $\mathcal{M}\otimes\mathcal{C}\to\mathcal{C}$ in the symmetric closed monoidal category $(\mathcal{V}\text{-}\mathbf{Cat},\otimes,\mathcal{I})$ being equivalent (by Proposition 2.3) to the monoid morphism $\mathcal{M}\to[\mathcal{C},\mathcal{C}]$. We draw our attention to the case when $(\mathcal{V},\otimes,I) = (\mathbf{Ab},\otimes,I)$ and consider the one object case of this equivalence.

Lemma 3.5. *Monoid object in* $(\boldsymbol{Ab}, \otimes, \mathbb{Z})$ *is a ring.*

Proof. A monoid object in $(\mathbf{Ab}, \otimes, \mathbb{Z})$ is an abelian group M together with a group homomorphism of abelian groups (equivalently a module homomorphism of $\mathbb{Z}$-modules) $\circ : M \otimes M \to M$ such that it is associative and unital with respect to some element of $1 \in M$. This is equivalent to a $\mathbb{Z}$-bilinear map $\circ : M \times M \to M$ such that it is associative and unital. To spell it out, it is a map such that

$$\begin{gathered} a \circ (b \circ c) = (a \circ b) \circ c \\ a \circ 1 = a = 1 \circ a \\ (a + b) \circ c = a \circ c + b \circ c \\ c \circ (a + b) = c \circ a + c \circ b \end{gathered}$$

which is a ring. □

Proposition 3.6. *A monoid object* $\mathcal{M}$ *in* $(\boldsymbol{Ab\text{-}Cat}, \otimes, \mathcal{I})$ *(strict monoidal* ***Ab****-category* $\mathcal{M}$*) with one object is a commutative ring.*

Proof. An **Ab**-category $\mathcal{M}$ with one object is equivalently a monoid object $k = \mathcal{M}(*, *)$ in $(\mathbf{Ab}, \otimes, \mathbb{Z})$ which is again equivalently a ring k with ring multiplication denoted as $\circ$. Also k is equipped with a group homomorphism of abelian groups $\star : k \otimes k \to k$ such that it is associative, unital and $(g' \star h') \circ (g \star h) = (g' \circ g) \star (h' \circ h)$. By Eckmann-Hilton argument, $\circ = \star$ and the ring multiplication is commutative. Hence a strict monoidal **Ab**-category $\mathcal{M}$ with one object is a commutative ring. □

Theorem 3.7 (Categorical proof of the equivalence between two definitions of an R-algebra)**.** *Let* k *be a commutative ring and* A *be a ring. Then a structure of a* k*-algebra on* A *is equivalent to a ring homomorphism between commutative rings* $k \to \mathrm{Z}(A)$

Proof. Suppose $\mathcal{M}$ and C are **Ab**-enriched categories with one object. Then $k = \mathcal{M}(*, *)$ and $A = C(*, *)$ are monoid objects in the monoidal category of $(\mathbf{Ab}, \otimes)$ which means they are rings.

Also k is a commutative ring since $\mathcal{M}$ is a strict monoidal **Ab**-category.

The axioms for a strict **Ab**-action of $\mathcal{M}$ on C tells us that we have a group homomorphism $\cdot : k \otimes A \to A$ such that for all $g, h \in k$, $x \in A$ and e the unit for the ring k, we have $(gh) \cdot x = g \cdot (h \cdot x)$ and $e \cdot x = x$ (i.e. a ring homomorphism $k \to \mathrm{End}(A, A)$). This means that A is a k-module.

Also the functoriality says that the composition map of A, $\circ : A \otimes A \to A$ (which is an associative and unital group homomorphism of abelian groups) is a k-module homomorphism that is associative and right/left unital: $(gh) \cdot (x \circ y) = (g \cdot x) \circ (h \cdot y)$. This says that A is a monoid object in the monoidal category $(\mathbf{k\text{-}Mod}, \otimes, \{*\})$ which is to say that A is a k-algebra.

A strict **Ab**-action is also equivalent to a strict monoidal **Ab**-functor $\mathcal{M} \to \mathbf{Ab\text{-}Cat}(C, C)$ which, when $\mathcal{M}$ and C are one-object **Ab**-enriched categories, is equivalent to a ring homomorphism $k \to \mathrm{Z}(A)$. □

4

Symmetric closed monoidal structure on **2Cat** with Gray Tensor Product

The term 2-category refers to strict bicategory, having associators and unitors being the identities. And by 2-functor, we mean a strict pseudofunctor between 2-categories.

Note that any bicategory C is biequivalent to some 2-category C' where *biequivalence* means a pseudofunctor $F : C \to C'$ which is surjective up to equivalence on objects and which for all objects $A, B \in C$, $F_{A,B} : C(A, B) \to C'(FA, FB)$ is an equivalence of categories $C(A, B)$ and $C'(A, B)$. Alternatively, it means that there exists pseudofunctors $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$ such that $\mathrm{id}_{\mathcal{D}}$ and FG are internally equivalent in the bicategory $[\mathcal{D}, \mathcal{D}]$ of pseudofunctors, *pseudonatural transformations* and *modifications* and likewise $GF \simeq \mathrm{id}_C$ in $[C, C]$. Due to size issues, we will not go over elements of bicategories. We will refer to [22] for brief overview of basic elements of bicategories.

In this chapter, we will introduce a symmetric closed monoidal structure on the category **2Cat** of 2-categories and 2-functors with what's called the *Gray tensor product*. The internal-hom of this monoidal closed structure is the 2-category $\mathbf{Psd}(\mathcal{A}, \mathcal{B})$ of 2-functors $\mathcal{A} \to \mathcal{B}$, pseudonatural transformations and modifications. In literature, it is introduced in [12] and [11] and has also been organised in [10] and [13]. The materials presented in this chapter is in expository nature of these texts.

4.1 Preliminaries on Cubical functors

In this section, we will introduce cubical functors which in [12], it is called quasi-functors. We will follow closely with [13] and [10].

Definition 4.1 (Cubical functor)**.** Let $C_1, \ldots, C_n, \mathcal{D}$ be 2-categories. A pseudofunctor $F : C_1 \times \cdots \times C_n \to \mathcal{D}$ is *cubical* if its component 2-isomorphism

$$\phi : F(g_1, \ldots, g_n) \circ F(f_1, \ldots, f_n) \to F(g_1 \circ f_1, \ldots, g_n \circ f_n)$$

is the identity whenever for all $j < i$, either f_i or g_j is an identity 1-cell.

Remark. A cubical functor for $n = 1$ is a strict 2-functor.

The next two results are important results which we will use later in unpacking the definition of *Gray-monoid* and *Gray-monoid action*.

Proposition 4.1 (4.2 [10]). *For $n = 2$, a cubical functor $F : C_1 \times C_2 \to \mathcal{D}$ is equivalently to give:*

1. *a strict 2-functor $F_A : C_2 \to \mathcal{D}$ for each object $A \in C_1$ and a strict 2-functor $F_B : C_1 \to \mathcal{D}$ for each object $B \in C_2$ such that $F_B(A) = F_A(B) := F(A, B)$*
2. *for all f_1 and f_2, a component 2-cell of a natural isomorphism*

$$\gamma_{f_1,f_2} : F_{B'}(f_1) \circ F_A(f_2) \to F_{A'}(f_2) \circ F_B(f_1)$$

natural in f_1 and f_2 such that $\gamma_{f_1,1_B} = 1_{F_B(f_1)}$ and $\gamma_{1_A,f_2} = 1_{F_A(f_2)}$

such that for all

$$\begin{tikzcd}
(A,B) \arrow[r, bend left, "{(f_1,f_2)}"] \arrow[r, bend right, "{(g_1,g_2)}"'] \arrow[r, phantom, "\Downarrow(\alpha_1,\alpha_2)"] & (A',B') \arrow[r, "{(h_1,h_2)}"] & (A'',B'')
\end{tikzcd}$$

the following axiom holds:

$$\begin{tikzcd}
F(A,B) \arrow[r, "F_A(f_2)"] \arrow[d, "F_B(f_1)"'] \arrow[dr, phantom, "\Downarrow\gamma_{f_1,f_2}"] & F(A,B') \arrow[r, "F_A(h_2)"] \arrow[d, "F_{B'}(f_1)"] \arrow[dr, phantom, "\Downarrow\gamma_{f_1,h_2}"] & F(A,B'') \arrow[d, "F_{B''}(f_1)"] \\
F(A',B) \arrow[r, "F_{A'}(f_2)"] \arrow[d, "F_B(h_1)"'] \arrow[dr, phantom, "\Downarrow\gamma_{h_1,f_2}"] & F(A',B') \arrow[r, "F_{A'}(h_2)"] \arrow[d, "F_{B'}(h_1)"] \arrow[dr, phantom, "\Downarrow\gamma_{h_1,h_2}"] & F(A',B'') \arrow[d, "F_{B''}(h_1)"] \\
F(A'',B) \arrow[r, "F_{A''}(f_2)"'] & F(A'',B') \arrow[r, "F_{A''}(h_2)"'] & F(A'',B'')
\end{tikzcd}
=
\begin{tikzcd}
F(A,B) \arrow[r, "F_A(h_2f_2)"] \arrow[d, "F_B(h_1f_1)"'] \arrow[dr, phantom, "\Downarrow\gamma_{h_1f_1,h_2f_2}"] & F(A,B'') \arrow[d, "F_{B''}(h_1f_1)"] \\
F(A'',B) \arrow[r, "F_{A''}(h_2f_2)"'] & F(A'',B'')
\end{tikzcd}$$

Remark. For the full proof, we refer to Proposition 3.2 [13]. It is easy to see that a cubical functor $F : C_1 \times C_2 \to \mathcal{D}$ determines a 2-functor F_A by $F_A(f_2) = F(1, f_2)$, $F_A(\alpha_2) = F(1, \alpha_2)$ (likewise determines F_B in the similar way) and determines γ_{f_1,f_2} by the vertical composite of component 2-cells of natural isomorphisms coming from the constraints in pseudofunctor:

$$F(f_1, 1_{B'}) \circ F(1_A, f_2) \xrightarrow[\cong]{\phi} F(f_1, f_2) \xlongequal{\phi^{-1}} F(1_{A'}, f_2) \circ F(f_1, 1_B)$$

hence natural in f_1 and f_2.

Conversely, give the above data, we form a cubical functor by

- $F(A, B) := F_B(A) = F_A(B)$
- $F(f_1, f_2) := F(1, f_2) \circ F(f_1, 1)$
- $F(\alpha_1, \alpha_2) := F(1, \alpha_2) * F(\alpha_1, 1)$

Proposition 4.2 (4.3 [10])**.** *A cubical functor for $n = 3$, $F : \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \to \mathcal{D}$ is equivalently to give:*

1. *cubical functors of two variables for each objects $A \in \mathcal{C}_1$, $B \in \mathcal{C}_2$ and $C \in \mathcal{C}_3$*
   (a) $F_A : \mathcal{C}_2 \times \mathcal{C}_3 \to \mathcal{D}$
   (b) $F_B : \mathcal{C}_1 \times \mathcal{C}_3 \to \mathcal{D}$
   (c) $F_C : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$
2. *such that for all pairs $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$, we have $F_A(B, -) = F_B(A, -)$ and likewise for pairs A, C and B, C*
3. *and such that for all $(f_1, f_2, f_3) : (A, B, C) \to (A', B', C')$ the following holds:*

$$
\begin{tikzcd}
 & F(A,B,C') \arrow[dr, "F_A(f_2,1)"] & \\
F(A,B,C) \arrow[ur, "F_A(1,f_3)"] \arrow[dr, "F_A(f_2,1)"'] \arrow[dd, "F_C(f_1,1)"'] \arrow[rr, phantom, "\Downarrow \gamma^{F_A}_{f_2,f_3}"] & & F(A,B',C') \arrow[dd, "F_{B'}(f_1,1)"] \\
 & F(A,B',C) \arrow[ur, "F_{B'}(1,f_3)"'] \arrow[dd, "F_{B'}(f_1,1)"] \arrow[dr, phantom, "\Downarrow \gamma^{F_{B'}}_{f_1,f_2}"] & \\
F(A',B,C) \arrow[dr, "F_C(1,f_2)"'] \arrow[ur, phantom, "\Downarrow \gamma^{F_C}_{f_1,f_2}"] & & F(A',B',C') \\
 & F(A',B',C) \arrow[ur, "F_{B'}(1,f_3)"'] &
\end{tikzcd}
$$

$$=$$

$$
\begin{tikzcd}
 & F(A,B,C') \arrow[dr, "F_A(f_2,1)"] \arrow[dd, "F_B(f_1,1)"] & \\
F(A,B,C) \arrow[ur, "F_A(1,f_3)"] \arrow[dd, "F_C(f_1,1)"'] \arrow[dr, phantom, "\Downarrow \gamma^{F_B}_{f_1,f_3}"] & & F(A,B',C') \arrow[dd, "F_B(f_1,1)"] \arrow[dl, phantom, "\Downarrow \gamma^{F_{C'}}_{f_1,f_2}"] \\
 & F(A',B,C') \arrow[dr, "F_{A'}(f_2,1)"] & \\
F(A',B,C) \arrow[ur, "F_B(1,f_3)"'] \arrow[dr, "F_C(1,f_2)"'] \arrow[rr, phantom, "\Downarrow \gamma^{F_{A'}}_{f_2,f_3}"] & & F(A',B',C') \\
 & F(A',B',C) \arrow[ur, "F_B(1,f_3)"'] &
\end{tikzcd}
$$

Remark (Proposition 5.2.4 [13]). 2-categories and cubical functors form a multicategory **Cub**.

4.2 Gray Tensor Product

In the cartesian product $\mathcal{C} \times \mathcal{D}$ of 2-categories $\mathcal{C}$ and $\mathcal{D}$, we have the following commutative square:

$$
\begin{tikzcd}
(A,B) \arrow[r, "(1,g)"] \arrow[d, "(f,1)"'] & (A,B') \arrow[d, "(f,1)"] \\
(A',B) \arrow[r, "(1,g)"'] & (A',B')
\end{tikzcd}
$$

which in the Gray tensor product $\mathcal{C}\otimes_G\mathcal{D}$ the commutativity is weakened up to 2-isomorphism:

$$\begin{tikzcd}
(A,B) \arrow[r, "{(1,g)}"] \arrow[d, "{(f,1)}"'] \arrow[dr, phantom, "\gamma_{f,g} \swarrow"] & (A,B') \arrow[d, "{(f,1)}"] \\
(A',B) \arrow[r, "{(1,g)}"'] & (A',B')
\end{tikzcd}$$

This weakening differs from the original version of Gray in [12] and [11] where, instead, the square above commutes only up to 2-morphism $\gamma_{f,g}$ being not necessarily an isomorphism. Nevertheless, we will follow the outline introduced in [12] but with the only refinement made on $\gamma_{f,g}$ to be the isomorphism. This is the same definition given in Gordon, Power, Street's [10] which is also same as Gurski's [13].

Analogously with the tensor product $A \otimes_R B$ of R-modules, the Gray tensor product $\mathcal{C} \otimes_G \mathcal{D}$ for 2-categories $\mathcal{C}$ and $\mathcal{D}$ is the 2-category such that there is a cubical functor $c : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_G \mathcal{D}$ inducing a bijection between cubical functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ and 2-functors $\mathcal{C} \otimes_G \mathcal{D} \to \mathcal{E}$ by factorisation through c. We first introduce the explicit construction of $\mathcal{C} \otimes_G \mathcal{D}$ and later revisit this universal property.

Definition 4.2 (Theorem I.4.9, [12])**.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. The *Gray tensor product* of $\mathcal{C}$ and $\mathcal{D}$ is a 2-category denoted as $\mathcal{C} \otimes_G \mathcal{D}$ such that

- Objects are pairs (A, B) where $A \in \operatorname{ob} \mathcal{C}$ and $B \in \operatorname{ob} \mathcal{D}$

- 1-cells are equivalence classes of composable words generated by two kind of 1-cells:

$$(f, 1) : (A, B) \to (A', B)$$
$$(1, g) : (A, B) \to (A, B')$$

  with 1-cells $f : A \to A'$ and $g : B \to B'$ of $\mathcal{C}$ and $\mathcal{D}$ respectively.

  i.e. A composable word is a string $w = (f_n, g_n)...(f_2, g_2)(f_1, g_1)$ where for each i, either f_i or g_i is an identity, such that it is well-formed in the manner that the compositions $f_n...f_2 f_1$ and $g_n...g_2 g_1$ exist in $\mathcal{C}$ and $\mathcal{D}$ respectively. Composition of words w and v are induced by juxtaposition wv of words.

  Two words are equivalent if they are made so by the following equivalence relations compatible with composition such that

  - $(f', 1)(f, 1) \sim (f' f, 1)$
  - $(1, g')(1, g) \sim (1, g' g)$
  - $wv \sim w'v$ and $uw \sim uw'$ whenever $w \sim w'$.

- 2-cells are generated by three kind of 2-cells:

$$(\alpha, 1) : (f, 1) \Rightarrow (f'1)$$
$$(1, \beta) : (1, g) \Rightarrow (1, g')$$
$$\gamma_{f,g} : (f, 1)(1, g) \Rightarrow (1, g)(f, 1)$$

where $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ are 2-cells of C and $\mathcal{D}$ respectively and $\gamma_{f,g}$'s are isomorphisms for all non-identity 1-cells $f \in C$ and $g \in \mathcal{D}$. When either f or g is an identity 1-cell, then $\gamma_{f,g}$ is the identity.

2-cells are then equivalence classes of horizontal and vertical composites of these generating 2-cells. First, consider a horizontally composable word generated by these 2-cells. i.e. a string $\mu = \lambda_n * \cdots * \lambda_2 * \lambda_1$ where each λ_i is either $\lambda_{f,g}$ for some f and g, or (α_i, β_i) with either α_i or β_i being the identity 2-cell, such that the string is well-formed in the manner that whenever

$$\lambda_{i+1} * \lambda_i = \begin{cases} (\alpha_{i+1}, \beta_{i+1}) * (\alpha_i, \beta_i) & \text{then } \alpha_{i+1} * \alpha_i \text{ and } \beta_{i+1} * \beta_i \\ (\alpha_{i+1}, \beta_{i+1}) * \gamma_{f,g} & \text{then } \alpha_{i+1} * f \text{ and } \beta_{i+1} * g \\ \gamma_{f,g} * (\alpha_i, \beta_i) & \text{then } f * \alpha_i \text{ and } g * \beta_i \\ \gamma_{f',g'} * \gamma_{f,g} & \text{then } f' * f \text{ and } g' * g \end{cases}$$

are defined 2-cells in C and $\mathcal{D}$ respectively. Horizontal composition of words μ and σ are induced by $*$ as $\mu * \sigma$.

Two horizontally composable words are equivalent if they are made so by the following equivalence relations compatible with $*$ such that

- $(\alpha', 1) * (\alpha, 1) \sim (\alpha' * \alpha, 1)$
- $(1, \beta') * (1, \beta) \sim (1, \beta' * \beta)$
- $\mu * \sigma \sim \mu'\sigma$ and $\phi * \mu \sim \phi * \mu'$ whenever $\mu \sim \mu'$.

Let $[\mu]$ denote the above equivalence class of a horizontal word μ. 2-cells are then equivalence classes of vertically composable words, with vertical composition induced by juxtaposition $[\mu_n]...[\mu_2][\mu_1]$ of words. Two words are equivalent if they are made so by the following equivalence relations:

- $\big(\gamma_{f',g} * (f, 1)\big)\big((f', 1) * \gamma_{f,g}\big) \sim \gamma_{f'f,g}$
- $\big((1, g') * \gamma_{f,g}\big)\big(\gamma_{f,g'} * (1, g)\big) \sim \gamma_{f,g'g}$
- $\big((1, g') * (f', 1) * \gamma_{f,g}\big)\big(\gamma_{f',g'} * (f, 1) * (1, g)\big) \sim \big(\gamma_{f',g'} * (1, g) * (f, 1)\big)\big((f', 1) * (1, g') * \gamma_{f,g}\big)$
- If $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ are 2-cells of C and $\mathcal{D}$ respectively, then $\big((1, \beta) * (\alpha, 1)\big)\gamma_{f,g} \sim \gamma_{f',g'}\big((\alpha, 1) * (1, \beta)\big)$
- $(\alpha', 1)(\alpha, 1) \sim (\alpha'\alpha, 1)$
- $(1, \beta')(1, \beta) \sim (1, \beta'\beta)$
- $[\alpha][\beta] \sim [\alpha'][\beta']$ and $[\eta][\alpha] \sim [\eta][\alpha']$ whenever $[\alpha] \sim [\alpha']$.

Now, given 2-cells, vertical composition is given by concatenation of strings. And for the horizontal composition, let $\Gamma = [\mu_n] \cdots [\mu_2][\mu_1]$ and $\Lambda = [\eta_m] \cdots [\eta_2][\eta_1]$ such that the 0-cell source of Γ is equal to the 0-cell target of Λ. If $m \neq n$, say if $m < n$, then we form $[\mu'_n]...[\mu'_2][\mu'_1]$ by inserting $n - m$ identity 2-cells randomly into $[\mu_m]...[\mu_2][\mu_1]$ and define $\Gamma * \Lambda$ as the equivalence class of $[\mu'_n * \eta_n] \cdots [\mu'_2 * \eta_2][\mu'_1 * \eta_1]$.

Remark. This Gray tensor product provides $(\mathbf{2Cat}, \otimes_G, \mathcal{I})$ a structure of a symmetric monoidal category [11] with $\mathcal{I}$ being the 2-category with one object, one 1-cell and one 2-cell.

Consider a cubical functor $c : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes_G \mathcal{B}$ determined by 2-functors $c_A : \mathcal{B} \to \mathcal{A} \otimes_G \mathcal{B}$ and $c_B : \mathcal{A} \to \mathcal{A} \otimes_G \mathcal{B}$ with $c_A(B) = (A, B)$, $c_A(f_2) = (1_A, f_2)$ and $c_A(\alpha_2) = (1_{1_A}, \alpha_2)$ (and c_B defined similarly) and by

$$\gamma^c_{f_1,f_2} : c_{B'}(f_1) \circ c_A(f_2) \to c_{A'}(f_2) \circ c_B(f_1)$$

of Proposition 4.1 being equal to

$$\gamma_{f_1,f_2} : (f_1, 1_{B'})(1_A, f_2) \to (1_{A'}, f_2)(f_1, 1_B)$$

of Definition 4.2.

Lemma 4.3. *It is easy to see that* $c : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes_G \mathcal{B}$ *is natural in* $\mathcal{A}$ *and* $\mathcal{B}$*:*

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "c"] \arrow[d, "F \times G"'] & \mathcal{A} \otimes_G \mathcal{B} \arrow[d, "F \otimes_G G"] \\
\mathcal{A}' \times \mathcal{B}' \arrow[r, "c'"'] & \mathcal{A}' \otimes_G \mathcal{B}'
\end{tikzcd}$$

commutes for 2*-functors* F *and* G*.*

Theorem 4.4 (Theorem 3.7 [13], Theorem I.4.9 [12])**.** *The gray tensor product* $\mathcal{A} \otimes_G \mathcal{B}$ *of* 2*-categories* $\mathcal{A}$ *and* $\mathcal{B}$ *is the target* 2*-category of the cubical functor* $c : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \otimes_G \mathcal{B}$ *such that for all cubical functor* $F \in \boldsymbol{Cub}(\mathcal{A} \times \mathcal{B}, C)$ *the* 2*-functor* $\overline{F} \in \boldsymbol{2Cat}(\mathcal{A} \otimes_G \mathcal{B}, C)$ *mapping the objects and the generators of* $\mathcal{A} \otimes_G \mathcal{B}$ *by*

- $\overline{F}(A, B) = F(A, B)$
- $\overline{F}(f_1, 1) = F_B(f_1)$
- $\overline{F}(1, f_2) = F_A(f_2)$
- $\overline{F}(\alpha_1, 1) = F_B(\alpha_1)$
- $\overline{F}(1, \alpha_2) = F_A(\alpha_2)$
- $\overline{F}(\gamma_{f_1,f_2}) = \gamma^F_{f_1,f_2}$

is the unique 2*-functor making the diagram*

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "F"] \arrow[d, "c"'] & C \\
\mathcal{A} \otimes_G \mathcal{B} \arrow[ur, "\overline{F}"'] &
\end{tikzcd}$$

commute hence giving a bijection $\boldsymbol{Cub}(\mathcal{A} \times \mathcal{B}, C) \cong \boldsymbol{2Cat}(\mathcal{A} \otimes_G \mathcal{B}, C)$ *naturally in all argument* $\mathcal{A}, \mathcal{B}$ *and* C*.*

Proof. We refer to Theorem 3.7 of [13] for the well-definedness of $\overline{F}$ and the bijection part of the proof. The bijection $- \circ c : \mathbf{2Cat}(\mathcal{A} \otimes_G \mathcal{B}, C) \cong \mathbf{Cub}(\mathcal{A} \times \mathcal{B}, C)$ is the component of natural isomorphism between functors

$$\mathbf{2Cat}^{\mathrm{op}} \times \mathbf{2Cat}^{\mathrm{op}} \times \mathbf{2Cat} \xrightarrow{\otimes_G^{\mathrm{op}} \times \mathbf{2Cat}} \mathbf{2Cat}^{\mathrm{op}} \times \mathbf{2Cat} \xrightarrow{\text{hom}} \mathbf{Set}$$

and

$$\mathbf{2Cat}^{\mathrm{op}} \times \mathbf{2Cat}^{\mathrm{op}} \times \mathbf{2Cat} \xrightarrow{\times^{\mathrm{op}} \times \mathbf{2Cat}} \mathbf{2Cat}^{\mathrm{op}} \times \mathbf{2Cat} \xrightarrow{\text{Cub}} \mathbf{Set}$$

where the naturality condition

$$\begin{tikzcd}
\mathbf{2Cat}(\mathcal{A} \otimes_G \mathcal{B}, C) \arrow[r, "- \circ c"] \arrow[d, "K \circ - \circ (G \otimes_G H)"'] & \mathbf{Cub}(\mathcal{A} \times \mathcal{B}, C) \arrow[d, "K \circ - \circ (G \times H)"] \\
\mathbf{2Cat}(\mathcal{A} \otimes_G \mathcal{B}, C) \arrow[r, "- \circ c'"'] & \mathbf{Cub}(\mathcal{A} \times \mathcal{B}, C)
\end{tikzcd}$$

follows easily from Lemma 4.3. □

4.3 Monoidal closed structure

In this section, we illustrate the monoidal closed structure of the symmetric monoidal category $(\mathbf{2Cat}, \otimes_G, \mathcal{I})$. Given 2-categories $\mathcal{A}$ and $\mathcal{B}$, 2-functors $\mathcal{A} \to \mathcal{B}$, pseudonatural transformations and modifications form a 2-category which we will denote as $\mathbf{Psd}(\mathcal{A}, \mathcal{B})$. From now on, notations involving pseudonatural transformations and modifications will be adopted from [22]. Peudonatural transformation is called strong transformation in [22].

Consider the evaluation map

$$e : \mathbf{Psd}(\mathcal{A}, \mathcal{B}) \times \mathcal{A} \to \mathcal{B}$$

which

$e(F, A) = F(A)$ for a 2-functor $F : \mathcal{A} \to \mathcal{B}$ and an object $A \in \mathcal{A}$

$e(\sigma, f) = F'(f) \circ \sigma_A$ for a pseudonatural transformation $\sigma : F \to F'$ and a 1-cell $f \in \mathcal{A}$

$e(\Gamma, \alpha) = F'(\alpha) * \Gamma_A$ for a modification $\Gamma : \sigma \to \sigma'$ and a 2-cell $\alpha : f \to f'$.

It can be easily seen that this is a cubical functor since, for 1-cells (pseudonatural transformations) $\sigma : F \to F'$ and $\gamma : F' \to F''$ of $\mathbf{Psd}(\mathcal{A}, \mathcal{B})$ and for 1-cells $f : A \to A'$ and $g : A' \to A''$ of $\mathcal{A}$, the constraint 2-isomorphism

$$\phi : e(\gamma, g) \circ e(\sigma, f) \to e(\gamma\sigma, gf)$$

of $e : \mathbf{Psd}(\mathcal{A}, \mathcal{B}) \times \mathcal{A} \to \mathcal{B}$ is given by

$$1_{F''(g)} * \gamma_f^{-1} * 1_{\sigma_A} = \phi : F''(g)\gamma_{A'}F'(f)\sigma_A \to F''(g)F''(f)\gamma_A\sigma_A$$

which is precisely the identity whenever either γ or f is the identity hence satisfying the cubicality.

This evaluation map above induces a bijection (Theorem I.4.2 [12] and Proposition 5.3.3 [13])

$$\mathbf{2Cat}(\mathcal{A}, \mathbf{Psd}(\mathcal{B}, C)) \cong \mathbf{Cub}(\mathcal{A} \times \mathcal{B}, C)$$

by sending $F \in \mathbf{2Cat}(\mathcal{A}, \mathbf{Psd}(\mathcal{B}, C))$ to the composite

$$\mathcal{A} \times \mathcal{B} \xrightarrow{F\times 1_{\mathcal{B}}} \mathbf{Psd}(\mathcal{A}, \mathcal{B}) \times \mathcal{A} \xrightarrow{e} \mathcal{B}$$

which is indeed a cubical functor by the fact that **Cub** is a multicategory. The bijection $e \circ (- \times 1) : \mathbf{2Cat}(\mathcal{A}, \mathbf{Psd}(\mathcal{B}, C)) \cong \mathbf{Cub}(\mathcal{A} \times \mathcal{B}, C)$ is the component bijection of a natural isomorphism making the following diagram commute:

$$\begin{tikzcd}
\mathbf{2Cat}(\mathcal{A}, \mathbf{Psd}(\mathcal{B}, C)) \arrow[r, "e \circ (- \times 1)"] \arrow[d, "\mathbf{Psd}(\mathcal{B}, K) \circ - \circ G"'] & \mathbf{Cub}(\mathcal{A} \times \mathcal{B}, C) \arrow[d, "K \circ - \circ (G \times 1)"] \\
\mathbf{2Cat}(\mathcal{A}', \mathbf{Psd}(\mathcal{B}, C')) \arrow[r, "e \circ (- \times 1)"'] & \mathbf{Cub}(\mathcal{A}' \times \mathcal{B}, C')
\end{tikzcd}$$

This, together with Theorem 4.4, gives us the following result:

Theorem 4.5. *The monoidal category* $(\mathbf{2Cat}, \otimes_G, \mathcal{I})$ *is closed with the internal-hom* $\mathbf{Psd}(\mathcal{B}, C)$

$$\mathbf{2Cat}(\mathcal{A} \otimes_G \mathcal{B}, C) \cong \mathbf{2Cat}(\mathcal{A}, \mathbf{Psd}(\mathcal{B}, C))$$

As the category **2Cat** is a symmetrical closed monoidal category also with respect to the cartesian product, from now on, we will denote separately as **Gray** for the symmetrical closed monoidal category with respect to the Gray tensor product $\otimes_G$.

5

Result in $(\mathbf{2Cat}, \otimes_G, \mathcal{I})$

5.1 Gray-monoid and monoidal 2-functor

Next we have an extension of the notion of monoidal structure to bicategories, called *monoidal bicategories*. A concise definition of a monoidal bicategory is that it is a tricategory with one object which, to briefly spell out, is a bicategory equipped with the composition pseudofunctor giving the bicategory a monoidal structure up to pseudonatural equivalence. For a full detailed definition, we refer to Appendix C of [28]. It is a weak notion of monoidal structure on bicategory and thus it is difficult to work with; but it can be delt with semi-strict form of a monoidal structure on 2-categories called Gray-monoids.

Just as the coherence of bicategories gave the fact, as a corollary, that any monoidal category is monoidally equivalent to a strict monoidal category, the coherence of tricategories shown in [10] gives the fact that any monoidal bicategory is monoidally biequivalent to a Gray-monoid.

Tricategory is a weak 3-category which is a one step higher generalisation from bicategories. Likewise, *trihomomorphism*, *tritransformation* and *trimodification* extends the notions of pseudofunctor, pseudonatural transformation and modification for bicategories. For two tricategories $\mathcal{T}$ and $\mathcal{R}$, there is a tricategory $\mathbf{Tricat}(\mathcal{T}, \mathcal{R})$ consisting of trihomomorphisms from $\mathcal{T}$ to $\mathcal{R}$ as objects, tritransformations as 1-cells, trimodifications as 2-cells and *perturbations* as 3-cells. We refer to [10] for full definitions of these terms. Within a tricategory $\mathcal{T}$, two objects A and B of $\mathcal{T}$ are *internally biequivalent* if there exist 1-cells $f : A \to B$ and $g : B \to A$ such that fg and fg are internally equivalent to 1_B and 1_A respectively in each of their hom-bicategories. This generalises the definition of internal equivalence within a bicategory.

Triequivalence is a notion of equivalence between tricategories which extends the notion of biequivalence between bicategories. Two tricategories $\mathcal{T}$ and $\mathcal{R}$ are *triequivalent* if there exists a trihomomorphisms $K : \mathcal{T} \to \mathcal{R}$ and $H : \mathcal{R} \to \mathcal{T}$ such that $1_{\mathcal{T}}$ and HK are internally biequivalent in the tricategory $\mathbf{Tricat}(\mathcal{T}, \mathcal{T})$ and $1_{\mathcal{R}}$ and KH are internally biequivalent in the tricategory $\mathbf{Tricat}(\mathcal{R}, \mathcal{R})$. Equivalently, two tricategories $\mathcal{T}$ and $\mathcal{R}$ are triequivalent if and only if there is a trihomomorphism $K : \mathcal{T} \to \mathcal{R}$ which is surjective up to internal biequivalence on objects (*triessentially surjective*) and each $K_{A,B} : \mathcal{T}(A, B) \to \mathcal{R}(KA, KB)$

are biequivalence of hom-bicategories (*locally a biequivalence*).

Tricategories aren't triequivalent to a fully strict 3-category, but it has been shown in [10] *Coherence for tricategories* by Gordon, Power and Street that every tricategory is triequivalent to a **Gray**-category, a type of semi-strict 3-category. A concise definition of a monoidal bicategory is that it is a tricategory with one object. And a Gray-monoid which we will define soon is also precisely a one-object **Gray**-category. And the one-object case of this Coherence theorem for tricategories is precisely the statement that any monoidal bicategory is *monoidally biequivalent* (one-object case of triequivalence) to a Gray-monoid. This section, we introduce Gray-monoids which simplifies the computational complications with monoidal bicategories.

Definition 5.1 (Gray monoid)**.** *Gray monoid* $(\mathcal{M}, \otimes, i)$ is a monoid object in the monoidal category $(\mathbf{Gray}, \otimes_G, \mathcal{I}, \alpha, \lambda, \eta)$. It is a 2-category $\mathcal{M}$ equipped with 2-functors

$$\mathcal{M} \otimes_G \mathcal{M} \xrightarrow{\otimes} \mathcal{M} \xleftarrow{i} \mathcal{I}$$

such that the following diagrams commute:

1. Associativity axiom:

$$\begin{tikzcd}
(\mathcal{M} \otimes_G \mathcal{M}) \otimes_G \mathcal{M} \arrow[rr, "\cong", "\alpha"'] \arrow[d, "\otimes \otimes_G \mathcal{M}"'] & & \mathcal{M} \otimes_G (\mathcal{M} \otimes_G \mathcal{M}) \arrow[d, "\mathcal{M} \otimes_G \otimes"] \\
\mathcal{M} \otimes_G \mathcal{M} \arrow[dr, "\otimes"'] & & \mathcal{M} \otimes_G \mathcal{M} \arrow[dl, "\otimes"] \\
& \mathcal{M} &
\end{tikzcd}$$

2. Unit axiom

$$\begin{tikzcd}
\mathcal{M} \otimes_G \mathcal{I} \arrow[r, "\cong", "\lambda"'] \arrow[d, "\mathcal{M} \otimes_G i"'] & \mathcal{M} \arrow[dd, "\mathcal{M}"] & \mathcal{I} \otimes_G \mathcal{M} \arrow[l, "\cong"', "\eta"] \arrow[d, "i \otimes_G \mathcal{M}"] \\
\mathcal{M} \otimes_G \mathcal{M} \arrow[dr, "\otimes"'] & & \mathcal{M} \otimes_G \mathcal{M} \arrow[dl, "\otimes"] \\
& \mathcal{M} &
\end{tikzcd}$$

Remark. This is a one-object **Gray**-category.

Definition 5.2 (Strict monoidal 2-functor)**.** *Strict monoidal 2-functor* $F : \mathcal{M} \to \mathcal{M}'$ is a monoid morphism between Gray-monoids $(\mathcal{M}, \otimes, i)$ and $(\mathcal{M}', \otimes', i')$. It is a 2-functor $F : M \to M'$ of C making the following diagrams commute:

1.

$$\begin{tikzcd}
\mathcal{M} \otimes_G \mathcal{M} \arrow[r, "F \otimes_G F"] \arrow[d, "\otimes"'] & \mathcal{M}' \otimes_G \mathcal{M}' \arrow[d, "\otimes'"] \\
\mathcal{M} \arrow[r, "F"'] & \mathcal{M}'
\end{tikzcd}$$

2.

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "i"] \arrow[d, "i'"'] & \mathcal{M} \arrow[dl, "F"] \\
\mathcal{M}' &
\end{tikzcd}$$

5.2 Centre of monoidal categories

Definition 5.3 ([16])**.** Let $(\mathcal{C}, \otimes, I, a, r, l)$ be a monoidal category. The *centre* $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is the category with

1. Objects as pairs (Z, μ) where $Z \in \mathcal{C}$ and $\mu : (Z \otimes -) \Rightarrow (- \otimes Z)$ is a natural isomorphism such that the following diagrams commute:

$$\begin{tikzcd} Z \otimes I \arrow[rr, "\mu_I"] \arrow[dr, "r_Z"'] & & I \otimes Z \arrow[dl, "l_Z"] \\ & Z & \end{tikzcd}$$

$$\begin{tikzcd} & Z \otimes (X \otimes Y) \arrow[r, "\mu_{X \otimes Y}"] & (X \otimes Y) \otimes Z \arrow[dr, "a_{X,Y,Z}"] & \\ (Z \otimes X) \otimes Y \arrow[ur, "a_{Z,X,Y}"] \arrow[dr, "\mu_X \otimes 1_Y"'] & & & X \otimes (Y \otimes Z) \\ & (X \otimes Z) \otimes Y \arrow[r, "a_{X,Z,Y}"'] & X \otimes (Z \otimes Y) \arrow[ur, "1_X \otimes \mu_Y"'] & \end{tikzcd} \tag{5.1}$$

2. A morphism $f : (Z, \mu) \to (Z', \mu')$ is an arrow $f : Z \to Z'$ in $\mathcal{C}$ such that the following diagram commutes for all $X \in \mathcal{C}$:

$$\begin{tikzcd} & X \otimes Z \arrow[dr, "1 \otimes f"] & \\ Z \otimes X \arrow[ur, "\mu_X"] \arrow[dr, "f \otimes 1"'] & & X \otimes Z' \\ & Z' \otimes X \arrow[ur, "\mu'_X"'] & \end{tikzcd}$$

Remark. The centre $\mathcal{Z}(\mathcal{C})$ is a monoidal category with functor

$$\mathcal{Z}(\mathcal{C}) \times \mathcal{Z}(\mathcal{C}) \xrightarrow{\otimes} \mathcal{Z}(\mathcal{C})$$

$$\begin{pmatrix} ((Z, \mu), (W, \eta)) \\ \Big\downarrow{\scriptstyle (f,g)} \\ ((Z', \mu'), (W, \eta')) \end{pmatrix} \longmapsto \begin{pmatrix} (Z \otimes W, \mu \otimes \eta) \\ \Big\downarrow{\scriptstyle f \otimes g} \\ (Z' \otimes W', \mu' \otimes \eta') \end{pmatrix}$$

where $(\mu \otimes \eta)_X : (Z \otimes W) \otimes X \longrightarrow X \otimes (Z \otimes W)$ is defined by the below commutative diagram:

$$\begin{tikzcd}
(Z\otimes W)\otimes X \arrow[r, "(\mu\otimes\eta)_X"] \arrow[d, "a_{Z,W,X}"'] & X\otimes(Z\otimes W) \\
Z\otimes(W\otimes X) \arrow[d, "1_Z\otimes\eta_X"'] & (X\otimes Z)\otimes W \arrow[u, "a_{X,Z,W}"'] \\
Z\otimes(X\otimes W) \arrow[r, "a^{-1}_{Z,X,W}"'] & (Z\otimes X)\otimes W \arrow[u, "\mu_X\otimes 1_W"']
\end{tikzcd} \tag{5.2}$$

and the unit object $(I, r^{-1}l) \in \mathcal{Z}(C)$.

The associators and unitors of $(C, \otimes, I, a, r, l)$ are morphisms in $\mathcal{Z}(C)$ and so that $\big(\mathcal{Z}(C), \otimes, (I, r^{-1}l), a, r, l\big)$ is a monoidal category.

Proposition 5.1. *The centre $\mathcal{Z}(C)$ is a braided monoidal category with braiding given by:*

$$B_{(Z,\mu),(W,\eta)} := \mu_W : (Z\otimes W, \mu\otimes\eta) \longrightarrow (W\otimes Z, \eta\otimes\mu)$$

Proof. To show that $\mu_W : (Z\otimes W, \mu\otimes\eta) \longrightarrow (W\otimes Z, \eta\otimes\mu)$ is an arrow in $\mathcal{Z}(C)$ we have to show that $(1\otimes\mu_W)\circ(\mu\otimes\eta)_X = (\eta\otimes\mu)_X\circ(\mu_W\otimes 1)$.

$$\begin{aligned}
(1\otimes\mu_W)\circ(\mu\otimes\eta)_X &= (1\otimes\mu_W)(\mu_X\otimes 1)(1\otimes\eta_X) \ \text{ (by equation (5.2))}\\
&= \mu_{X\otimes W}(1\otimes\eta_X) \ \text{ (by equation (5.1))}\\
&= (\eta_X\otimes 1)\mu_{W\otimes X} \ \text{ (from naturality of } \mu)\\
&= (\eta_X\otimes 1)(1\otimes\mu_X)(\mu_W\otimes 1) \ \text{ (by equation (5.1))}\\
&= (\eta\otimes\mu)_X\circ(\mu_W\otimes 1) \ \text{ (by equation (5.2))}
\end{aligned}$$

To show that this gives braiding for the monoidal category $\mathcal{Z}(C)$, we have to show that $B_{(Z,\mu),(W\otimes V,\eta\otimes\theta)} = (1_{(W,\eta)}\otimes B_{(Z,\mu),(V,\theta)})\circ(B_{(Z,\mu),(W,\eta)}\otimes 1_{(V,\theta)})$ and $B_{(Z\otimes W,\mu\otimes\eta),(V,\theta)} = (B_{(Z,\mu),(V,\theta)}\otimes 1_{(W,\eta)})\circ(1_{(Z,\mu)}\otimes B_{(W,\eta),(V,\theta)})$. These can be easily seen as:

$$\begin{aligned}
B_{(Z,\mu),(W\otimes V,\eta\otimes\theta)} &= \mu_{W\otimes V}\\
&= (1_W\otimes\mu_V)(\mu_W\otimes 1_V) \ \text{ (by equation(5.1))}\\
&= (1_{(W,\eta)}\otimes B_{(Z,\mu),(V,\theta)})\circ(B_{(Z,\mu),(W,\eta)}\otimes 1_{(V,\theta)})
\end{aligned}$$

$$\begin{aligned}
B_{(Z\otimes W,\mu\otimes\eta),(V,\theta)} &= (\mu\otimes\eta)_V\\
&= (\mu_V\otimes 1_W)(1_Z\otimes\eta_V) \ \text{ (by equation(5.2))}\\
&= (B_{(Z,\mu),(V,\theta)}\otimes 1_{(W,\eta)})\circ(1_{(Z,\mu)}\otimes B_{(W,\eta),(V,\theta)}).
\end{aligned}$$

□

5.3 One-object case of the action of a Gray monoid on a 2-category

Consider the monoidal category $(\mathbf{Gray}, \otimes_G, \mathcal{I})$. Following the Definition 2.9 of monoid action, consider giving an action of a Gray-monoid $\mathcal{M}$ on a 2-category C. The Proposition 2.3 in the symmetric closed monoidal category $(\mathbf{Gray}, \otimes_G, \mathcal{I})$ says that: *to give an action $\mathcal{M} \otimes_G C \to C$ is equivalently to give a strict monoidal 2-functor $\mathcal{M} \to \boldsymbol{Psd}(C, C)$* where the internal-hom $\mathbf{Psd}(C, C)$ is a Gray-monoid as we've seen in Proposition 2.2.

In this section we will show what this statement reduces to when $\mathcal{M}$ and C are one-object 2-categories.

Proposition 5.2. *Gray-monoid $(\mathcal{M}, \otimes, i)$ with one object is a braided strict monoidal category.*

Proof. We begin with unpacking the Definition 5.1 of a Gray-monoid. As we have seen in Theorem 4.4 that $\mathbf{Cub}(\mathcal{A} \times \mathcal{B}, C) \cong \mathbf{2Cat}(\mathcal{A} \otimes_G \mathcal{B}, C)$, to give the 2-functor $\otimes : \mathcal{M} \otimes_G \mathcal{M} \to \mathcal{M}$ is equivalently to give a cubical functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, which by the Proposition 4.1, it is equivalently to give the following data:

1. For each object $A \in \mathcal{M}$, a 2-functor

$$A \otimes - : \mathcal{M} \to \mathcal{M}$$

   giving $A \otimes B \in \operatorname{ob} \mathcal{M}$, $A \otimes f : A \otimes B \to A \otimes B'$ and $A \otimes \alpha : 1_A \otimes f \to 1_A \otimes f'$ for each object B, 1-cell $f : B \to B'$ and 2-cell $\alpha : f \to f'$ of $\mathcal{M}$

2. For each object $A \in \mathcal{M}$, a 2-functor

$$- \otimes A : \mathcal{M} \to \mathcal{M}$$

   in the likewise manner

3. for all f and g, a component 2-cell of a natural isomorphism

$$\begin{tikzcd} A \otimes B \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] \arrow[dr, phantom, "\gamma_{f,g}\ \Swarrow"] & A \otimes B' \arrow[d, "f\otimes 1"] \\ A' \otimes B \arrow[r, "1\otimes g"'] & A' \otimes B' \end{tikzcd}$$

   natural in f and g such that $\gamma_{f,g}$ is the identity when either f or g is the identity

such that for all

$$\begin{tikzcd} (A, B) \arrow[r, bend left, "(f_1, f_2)"] \arrow[r, bend right, "(g_1, g_2)"'] \arrow[r, phantom, "\Downarrow(\alpha_1, \alpha_2)"] & (A', B') \arrow[r, "(h_1, h_2)"] & (A'', B'') \end{tikzcd}$$

the following axiom holds:

$$
\begin{tikzcd}
A \otimes B \arrow[r, "A \otimes f_2"] \arrow[d, "B \otimes f_1"'] \arrow[dr, phantom, "\Downarrow\gamma_{f_1,f_2}"] & A \otimes B' \arrow[r, "A \otimes h_2"] \arrow[d, "B' \otimes f_1"'] \arrow[dr, phantom, "\Downarrow\gamma_{f_1,h_2}"] & A \otimes B'' \arrow[d, "B'' \otimes f_1"] \\
A' \otimes B \arrow[r, "A' \otimes f_2"] \arrow[d, "B \otimes h_1"'] \arrow[dr, phantom, "\Downarrow\gamma_{h_1,f_2}"] & A' \otimes B' \arrow[r, "A' \otimes h_2"] \arrow[d, "B' \otimes h_1"'] \arrow[dr, phantom, "\Downarrow\gamma_{h_1,h_2}"] & A' \otimes B'' \arrow[d, "B'' \otimes h_1"] \\
A'' \otimes B \arrow[r, "A'' \otimes f_2"'] & A'' \otimes B' \arrow[r, "A'' \otimes h_2"'] & A'' \otimes B''
\end{tikzcd}
=
\begin{tikzcd}
A \otimes B \arrow[r, "A \otimes h_2 f_2"] \arrow[d, "B \otimes h_1 f_1"'] \arrow[dr, phantom, "\Downarrow\gamma_{h_1 f_1, h_2 f_2}"] & A \otimes B'' \arrow[d, "B'' \otimes h_1 f_1"] \\
A'' \otimes B \arrow[r, "A'' \otimes h_2 f_2"'] & A'' \otimes B''
\end{tikzcd}
\tag{5.3}
$$

The given cubical functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ satisfies the associativity axiom

$$
\begin{tikzcd}
(\mathcal{M} \times \mathcal{M}) \times \mathcal{M} \arrow[rr, "\cong", "\alpha"'] \arrow[d, "\otimes\times\mathcal{M}"'] & & \mathcal{M} \times (\mathcal{M} \times \mathcal{M}) \arrow[d, "\mathcal{M}\times\otimes"] \\
\mathcal{M} \times \mathcal{M} \arrow[dr, "\otimes"'] & & \mathcal{M} \times \mathcal{M} \arrow[dl, "\otimes"] \\
& \mathcal{M} &
\end{tikzcd}
$$

The left and the right hand side of the diagram $\otimes \circ (\otimes \times \mathcal{M})$ and $\otimes \circ (\mathcal{M} \times \otimes)$ are cubical functors of 3 variables (as **Cub** is a multicategory) which determines three cubical functors of 2 variables by the Proposition 4.2. The above associativity axiom means that these are equal, which again by the Proposition 4.1 and Proposition 4.2 reduces to the following:

4. For each object $A, B \in \operatorname{ob} \mathcal{M}$, the 2-functors $A \otimes (B \otimes -) : \mathcal{M} \to \mathcal{M}$ and $(A \otimes B) \otimes - : \mathcal{M} \to \mathcal{M}$ are equal.

   And likewise, $A \otimes (- \otimes B) = (A \otimes -) \otimes B$ and $- \otimes (A \otimes B) = (- \otimes A) \otimes B$.

5. $\gamma_{A\otimes g,h} = A \otimes \gamma_{g,h}$ and $\gamma_{f\otimes B,h} = \gamma_{f,B\otimes h}$ and $\gamma_{f,g\otimes C} = \gamma_{f,g} \otimes C$.

And lastly, the cubical functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ satisfies the unit axiom which gives the following data:

6. a unit object $I \in \mathcal{M}$

7. the 2-functors $- \otimes I : \mathcal{M} \to \mathcal{M}$ and $I \otimes - : \mathcal{M} \to \mathcal{M}$ and $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ are equal.

The datas from 1. to 7. gives the alternate unpacked definition of a Gray-monoid which, in several references like [21], has been called as *semi-strict monoidal 2-category*.

One-object Gray-monoid $\mathcal{M}$ reduces to a braided strict monoidal category $K = \mathcal{M}(I, I)$ with monoidal product $\circ : K \times K \to K$ being the composition functor for the 2-category

$$\circ : \mathcal{M}(I, I) \times \mathcal{M}(I, I) \to \mathcal{M}(I, I)$$

and with the component of natural isomorphism $\gamma_{f,g} : f \circ g \to g \circ f$ being the braiding as the equation in axiom (5.3) is precisely the axioms of braided monoidal category outlined in Definition 2.5. The rest of the datas vanish since the only object of $\mathcal{M}$ is the unit object I. $\square$

Consider a braided strict monoidal category $(\mathcal{K}, \otimes, I_{\mathcal{K}})$ and a strict monoidal category $(\mathcal{A}, \otimes', I_{\mathcal{A}})$. By *a structure of a $\mathcal{K}$-algebra on $\mathcal{A}$*, we mean a monoidal functor $\odot : \mathcal{K} \times \mathcal{A} \to \mathcal{A}$ such that

1. $(M \otimes M') \odot X = M \odot (M' \odot X)$ and $I_{\mathcal{K}} \odot X = X$

and by a *a strict structure of a $\mathcal{K}$-algebra on $\mathcal{A}$*, we require an additional condition that

2. for $M, M' \in \mathrm{ob}\,\mathcal{K}$ and $X, X' \in \mathrm{ob}\,\mathcal{A}$, the constraint natural isomorphism

$$\phi : (M' \odot X') \otimes' (M \odot X) \to (M' \otimes M) \odot (X' \otimes' X)$$

for the monoidal functor $\odot$ is the identity if either $M' = I_{\mathcal{K}}$ or $X = I_{\mathcal{A}}$.

Theorem 5.3. *Let $\mathcal{K}$ be a braided strict monoidal category and $\mathcal{A}$ be a strict monoidal category. Then a strict structure of a $\mathcal{K}$-algebra on $\mathcal{A}$ is equivalent to a braided strict monoidal functor $\mathcal{K} \to \mathcal{Z}(\mathcal{A})$.*

Proof. Regard $(\mathcal{K}, \otimes, I_{\mathcal{K}})$ as a one-object Gray-monoid $(\mathcal{M}, \otimes, i)$ with $\mathcal{K} = \mathcal{M}(I, I)$ and $I_{\mathcal{K}} = \mathrm{id}_I$. And regard $(\mathcal{A}, \otimes', I_{\mathcal{A}})$ as a one-object 2-category C with $\mathcal{A} = C(*, *)$ with $I_{\mathcal{A}} = \mathrm{id}_*$. Then a strict structure of a $\mathcal{K}$-algebra on $\mathcal{A}$ is equivalent to a monoid action $\odot : \mathcal{M} \otimes_G C \to C$ of the Gray-monoid $\mathcal{M}$ on the 2-category C which by Proposition 2.3 in $(\mathbf{2Cat}, \otimes_G, \mathcal{I})$ is equivalent to a strict monoidal 2-functor $\mathcal{M} \to \mathbf{Psd}(C, C)$.

Let's see what this strict monoidal 2-functor reduces to when $\mathcal{M}$ and C have one object. We first begin with unpacking the Definition 5.2 of a general strict monoidal 2-functor $F : \mathcal{M} \to \mathcal{M}'$. Consider the first diagram of the Definition 5.2. Under the bijection $\mathbf{2Cat}(\mathcal{M} \otimes_G \mathcal{M}, \mathcal{M}') \cong \mathbf{Cub}(\mathcal{M} \times \mathcal{M}, \mathcal{M}')$, the 2-functor $F \circ \otimes : \mathcal{M} \otimes_G \mathcal{M} \to \mathcal{M}'$ uniquely determines a cubical functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}'$ which, by Proposition 4.1, is uniquely determined by

a. the 2-functor $F(A \otimes -) : \mathcal{M} \to \mathcal{M}'$ for each object $A \in \mathrm{ob}\,\mathcal{M}$

b. the 2-functor $F(- \otimes A) : \mathcal{M} \to \mathcal{M}'$ for each object $A \in \mathrm{ob}\,\mathcal{M}$

c. the component 2-cell of the natural isomorphism $F(\gamma_{f,g}) : F(f \otimes B) \circ F(A \otimes g) \to F(A' \otimes g) \circ F(f \otimes B)$.

Likewise, the 2-functor $\otimes \circ (F \otimes_G F) : \mathcal{M} \otimes_G \mathcal{M} \to \mathcal{M}'$ is uniquely determined by

d. the 2-functor $F(A) \otimes F(-) : \mathcal{M} \to \mathcal{M}'$ for each object $A \in \mathrm{ob}\,\mathcal{M}$

e. the 2-functor $F(-) \otimes F(A) : \mathcal{M} \to \mathcal{M}'$ for each object $A \in \mathrm{ob}\,\mathcal{M}$

f. the component 2-cell of the natural isomorphism $\gamma_{F(f),F(g)} : (F(f) \otimes F(B)) \circ (F(A) \otimes F(g)) \to (F(A') \otimes F(g)) \circ (F(f) \otimes F(B))$.

We require that these datas are the same:

1. $F(A \otimes -) = F(A) \otimes F(-)$

2. $F(- \otimes A) = F(-) \otimes F(A)$

3. $F(\gamma_{f,g}) = \gamma_{F(f),F(g)}$.

The second diagram in the Definition 5.2 gives the condition that

4. $F(I_{\mathcal{M}}) = I_{\mathcal{M}'}$

These datas from 1. to 4. gives the alternate unpacked definition of a strict monoidal 2-category as used in [20].

It now becomes clear that a strict monoidal 2-functor between one-object Gray-monoids is precisely a braided and strict monoidal functor since the additional axiom for the *braided* monoidal functor is precisely the condition $F(\gamma_{f,g}) = \gamma_{F(f),F(g)}$.

It is routine to check that $\mathbf{Psd}(C, C)(\mathrm{id}_C, \mathrm{id}_C)$ is precisely the centre $\mathcal{Z}(\mathcal{A})$ and we conclude that a strict monoidal 2-functor $\mathcal{M} \to \mathbf{Psd}(C, C)$ is precisely a braided strict monoidal functor $\mathcal{K} \to \mathcal{Z}(\mathcal{A})$. (Omission of 2. corresponds to the equivalence to a braided monoidal functor $\mathcal{K} \to \mathcal{Z}(\mathcal{A})$ not necessarily strict.) $\square$

6

Kontsevich's Swiss-Cheese Conjecture

6.1 Operads and Algebras

In this section, we outline elements of Operad and Algebras. Operad is an algebraic structure parametrising n-ary operations with a notion of composition governing the associative and unital conditions.

Definition 6.1. Let $(\mathcal{V}, \otimes, I)$ be a braided monoidal category. A *nonsymmetric 1-coloured operad* A consists of:

1. for each $n \in \mathbb{N}$, an object $A(n) \in \mathrm{ob}\,\mathcal{V}$
2. a *unit* morphism $i : I \to A(1)$
3. a *composition* morphism

$$m : A(n) \otimes A(k_1) \otimes \cdots \otimes A(k_n) \to A(k_1 + \cdots + k_n)$$

such that the composition map is associative and unital in the sense that

$$\begin{tikzcd}
A(n) \otimes (A(k_1) \otimes \cdots \otimes A(k_n)) \otimes \bigotimes_{j=1}^{n} \bigotimes_{h=1}^{k_j} A(i_{k_j,h}) \arrow[r, "m\otimes \mathrm{id}"] \arrow[d, "\cong"] & A(\sum_{u=1}^{n} k_u) \otimes \bigotimes_{j=1}^{n} \bigotimes_{h=1}^{k_j} A(i_{k_j,h}) \arrow[dd, "m"] \\
A(n) \otimes \bigotimes_{j=1}^{n} A(k_j) \otimes \bigotimes_{h=1}^{k_j} A(i_{k_j,h}) \arrow[d, "\mathrm{id}\otimes \bigotimes_{j=1}^{n} m"] & \\
A(n) \otimes \bigotimes_{j=1}^{n} A(\sum_{h=1}^{k_j} i_{k_j,h}) \arrow[r, "m"] & A(\sum_{j=1}^{n} \sum_{h=1}^{k_j} i_{k_j,h})
\end{tikzcd}$$

$$\begin{tikzcd} A(n)\otimes I^{\otimes n} \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\otimes i^{\otimes n}"'] & A(n) \\ A(n)\otimes A(1)^{\otimes n} \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} I\otimes A(n) \arrow[r, "\cong"] \arrow[d, "i\otimes \mathrm{id}"'] & A(n) \\ A(1)\otimes A(n) \arrow[ur, "m"'] & \end{tikzcd}$$

Definition 6.2. An operad A is called *reduced* if $A(0)$ is the unit I of $\mathcal{V}$

Definition 6.3 (Symmetric Operad)**.** If $(\mathcal{V}, \otimes, I)$ is symmetric, we can also define symmetric operad in $\mathcal{V}$. A *symmetric operad* A in $\mathcal{V}$ is an operad A in $\mathcal{V}$ together with a symmetric group action given by a monoid morphism $S_n \to \mathcal{V}(A(n), A(n))$ for each $n \in \mathbb{N}$, giving for each $\sigma \in S_n$, a morphism $\sigma : A(n) \to A(n)$ such that the composition map m equivariant with σ's in the sense that the following diagrams commute

$$\begin{tikzcd} A(n)\otimes (A(k_1)\otimes\cdots\otimes A(k_n)) \arrow[r, "\sigma\otimes\sigma*"] \arrow[d, "m"] & A(n)\otimes (A(k_{\sigma(1)})\otimes\cdots\otimes A(k_{\sigma(n)})) \arrow[d, "m"] \\ A(k_1+\cdots+k_n) \arrow[r, "\sigma(k_1,\dots,k_n)"] & A(k_{\sigma(1)}+\cdots+k_{\sigma(n)}) \end{tikzcd}$$

$$\begin{tikzcd} A(n)\otimes A(k_1)\otimes\cdots A(k_n) \arrow[r, "\mathrm{id}\otimes\tau_1\otimes\cdots\otimes\tau_n"] \arrow[d, "m"] & A(n)\otimes A(k_1)\otimes\cdots\otimes A(k_n) \arrow[d, "m"] \\ A(k_1+\cdots+k_n) \arrow[r, "\tau_1\oplus\cdots\oplus\tau_n"] & A(k_1+\cdots+k_n) \end{tikzcd}$$

where $\sigma* : A(k_1)\otimes\cdots\otimes A(k_n) \to A(k_{\sigma(1)})\otimes\cdots\otimes A(k_{\sigma(n)})$ is the isomorphism induced by the braidings of $\mathcal{V}$ and $\sigma(k_1, \dots, k_n) \in S_{k_1+\cdots+k_n}$ is to be thought of as the permutation of n blocks of letters by $\sigma \in S_n$ and where $\tau_1 \oplus \cdots \oplus \tau_n \in S_{\sum_{j=1}^n k_j}$ is the natural inclusion $\prod_{j=1}^n S_{k_j} \hookrightarrow S_{\sum_{j=1}^n k_j}$ of $(\tau_1, \dots, \tau_n) \in \prod_{j=1}^n S_{k_j}$.

We now introduce *coloured operads* which we would be a generalisation of the above notion of operads. It captures the nature of n-ary operations from multiple types of arguments.

Definition 6.4 (Coloured Operad)**.** Let $(\mathcal{V}, \otimes, I)$ be a symmetric monoidal category. A *coloured operad* A in $\mathcal{V}$ consists of:

1. a set C of colours
2. for each $c_1, \dots, c_n, c \in C$, an object $A(c_1, \dots, c_n; c) \in \mathrm{ob}\,\mathcal{V}$
3. for each $c \in C$, a *unit* morphism $i_c : I \to A(c; c)$
4. a *composition* morphism

   $$m : A(c_1, \dots c_n; c) \otimes A(d_{1,1}, \dots d_{1,k_1}; c_1) \otimes \cdots A(d_{n,1}, \dots d_{n,k_n}; c_n) \to A(d_{1,1}, \dots d_{n,k_n}; c)$$

5. for all $n \in \mathbb{N}$ and $\sigma \in S_n$ (where S_n is the symmetric group on n-variables), we have a morphism $\sigma : A(c_1, \dots, c_n; c) \to A(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c)$

such that the composition morphism is associative, unital and equivariance with S_n-action in the obvious sense.

Remark. From now on, we will simply use the term *operad* to mean *symmetric 1-coloured operad*. *Nonsymmetric* and/or *coloured* operads will explicitly be called as such unless it has a specific name attached to it from which we would know that it's nonsymmetric and/or coloured. The same goes to operad morphisms and algebras where the nonsymmetric variant is the omition of the symmetric group action from the definition.

Remark. Nonsymmetric coloured operad is the same notion to the notion of a multicategory enriched in $(\mathcal{V}, \otimes, I)$.

Definition 6.5 (Operad morphism)**.** An *operad morphism* $F : A \to B$ for operads A and B consists of a morphism $F_n : A(n) \to B(n)$ in $\mathcal{V}$ for each $n \in \mathbb{N}$, such that they are compatible with the composition and unit morphism and with the S_n-action expressed by following commutative diagrams

$$\begin{tikzcd}
A(n) \otimes A(k_1) \otimes \cdots \otimes A(k_n) \arrow[r, "m"] \arrow[d, "F_n \otimes F_{k_1} \otimes \cdots \otimes F_{k_n}"] & A(k_1 + \cdots + k_n) \arrow[d, "F_{k_1+\cdots+k_n}"] \\
B(n) \otimes B(k_1) \otimes \cdots \otimes B(k_n) \arrow[r, "m"] & B(k_1 + \cdots + k_n)
\end{tikzcd}$$

$$\begin{tikzcd}
I \arrow[r, "i"] \arrow[dr, "i"'] & A(1) \arrow[d, "F_1"] \\
& B(1)
\end{tikzcd}
\qquad
\begin{tikzcd}
A(n) \arrow[r, "\sigma"] \arrow[d, "F_n"] & A(n) \arrow[d, "F_n"] \\
B(n) \arrow[r, "\sigma"] & B(n)
\end{tikzcd}$$

Definition 6.6. An operad morphism $F : A \to B$ between reduced operads A and B is called *reduced* if $F_0 : A(0) \to B(0)$ is the identity.

Definition 6.7 (Algebra over an operad)**.** Consider a operad A over a symmetric monoidal category $(\mathcal{V}, \otimes, I)$. An *algebra X over the operad A* consists of a morphism $\mu_n : A(n) \otimes X^{\otimes n} \to X$ for each $n \in \mathbb{N}$ such that they are associative, unital and equivariant with the symmetric group action $\sigma \in S_n$ in the following sense:

$$\begin{tikzcd}
A(n) \otimes A(k_1) \otimes \cdots \otimes A(k_n) \otimes X^{\otimes(k_1+\cdots+k_n)} \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\cong"'] & A(k_1 + \cdots + k_n) \otimes X^{\otimes(k_1+\cdots+k_n)} \arrow[dd, "\mu_{k_1+\cdots+k_n}"] \\
A(n) \otimes A(k_1) \otimes X^{\otimes k_1} \otimes \cdots \otimes A(k_n) \otimes X^{\otimes k_n} \arrow[d, "\mathrm{id}\otimes m^{\otimes n}"'] & \\
A(n) \otimes X^{\otimes n} \arrow[r, "\mu_n"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes X \arrow[r, "\cong"] \arrow[d, "i\otimes\mathrm{id}"'] & X \\
A(1) \otimes X \arrow[ur, "\mu_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A(n) \otimes X^{\otimes n} \arrow[r, "\sigma\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\sigma"'] & A(n) \otimes X^{\otimes n} \arrow[d, "\mu_n"] \\
A(n) \otimes X^{\otimes n} \arrow[r, "\mu_n"'] & X
\end{tikzcd}$$

Definition 6.8 (Algebra over coloured operad)**.** Consider a coloured operad A with colours C. An *algebra X over the coloured operad A* consists of:

1. C-indexed collection $X_c \in \operatorname{ob} \mathcal{V}$ for all $c \in C$
2. a collection of morphisms $A(c_1, ..., c_n; c) \otimes (X_{c_1} \otimes \cdots \otimes X_{c_n}) \to X_c$

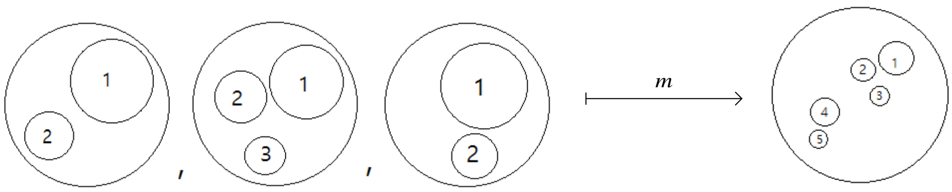
such that they are associative, unital and S_n-equivariant in the obvious sense.

Example 6.1. Commutative operad Com is a 1-coloured operad with $\mathrm{Com}(n) = I$. Category of algebras of Com in $(\mathcal{V}, \otimes, I)$ is same as the category of commutative monoid objects in $(\mathcal{V}, \otimes, I)$.

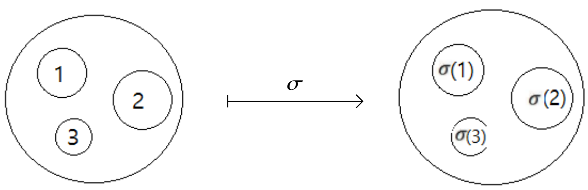
Example 6.2. Associative operad Ass in a cocomplete symmetric monoidal category $(\mathcal{V}, \otimes, I)$ is a 1-coloured operad with $\mathrm{Ass}(n) = \bigsqcup_{S_n} I$ with symmetric group action given by group multiplication. Category of algebras of Ass in $(\mathcal{V}, \otimes, I)$ is same as the category of monoid objects in $(\mathcal{V}, \otimes, I)$.

Example 6.3. For each object X of a symmetric monoidal category $(\mathcal{V}, \otimes, I)$, we can associate a **Set**-operad called the *endomorphism operad* $\mathrm{End}(X)$ with $\mathrm{End}(X)(n) = \mathcal{V}(X^{\otimes^n}, X)$. When $(\mathcal{V}, \otimes, I)$ is monoidally closed, we can define a $\mathcal{V}$-enriched version of endomorphism operad $\underline{\mathrm{End}}(X)$ with $\underline{\mathrm{End}}(X)(n) = [X^{\otimes n}, X]$ being the internal-hom from $X^{\otimes n}$ to X. An algebra X over the operad A is then an operad morphism $A \to \underline{\mathrm{End}}(X)$.

Example 6.4 (Little d-disk operad D_d [26], [6])**.** Little d-disk operad D_d is a reduced topological operad given by configuration spaces of non-overlapping ordered little d-disks inside the unit d-disk with the multiplication map $m : D_d(n) \times (D_d(k_1) \times \cdots \times D_d(k_n)) \to D_d(k_1 + \cdots + k_n)$ being the iterated embeddings into the little disks in the first unit disk as in the following way (for $d = 2$)



and the symmetric group action $\sigma : D_d(n) \to D_d(n)$ for acts by permuting the ordering of little disks as in the following way (for $n = 2$)



The little d-disk operad D_d was defined in order to recognize when a given connected and pointed topological space (X, x) is weak homotopy equivalent to a d-fold loop space $\mathrm{Maps}((\mathbb{S}^d, *), (Y, y))$, namely the *recognition theorem* (Theorem 2.7 [25]) stating that if a connected space is an algebra over D_d, it is weak homotopy equivalent to a d-fold loop space.

Little n-disks operad exists in any symmetric monoidal model category $(\mathcal{V}, \otimes, I)$. For example, from a topological operad, we can form an operad over **Set** through path component functor π_0 : **Top** $\to$ **Set** and an operad over **Cat** through fundamental groupoids functor π_1 : **Top** $\to$ **Cat** where for a topological space X, the category $\pi_1(X)$ with objects being the elements of X, and morphisms between $x \in X$ and $y \in X$ is the homotopy class of paths $p : [0,1] \to X$ with $p(0) = x$ and $p(1) = y$.

Definition 6.9 (Weak equivalence of operads)**.** Let $(\mathcal{V}, \otimes, I)$ be a symmetric monoidal model category. For operads A, B in $\mathcal{V}$, a weak equivalence $F : A \to B$ is an operad morphism such that $F_n : A(n) \to B(n)$ is a weak equivalence in $\mathcal{V}$ for all $n \in \mathbb{N}$.

Remark (Theorem 3.1 [5]). If $(\mathcal{V}, \otimes, I)$ is a "good" enough monoidal model category in the sense of [5], then the category of reduced operads and reduced operad morphism is equipped with a cofibrantly generated model structure with weak equivalences are defined as above with classes of fibrations and cofibrations defined similarly.

Definition 6.10. In a symmetric monoidal model category $(\mathcal{V}, \otimes, I)$, an operad A is called an *E_d-operad* if it is weakly equivalent to the little d-disk operad D_d, meaning that they can be connected by zig-zag of weak equivalences of operads

$$D_d \xrightarrow{\sim} A_1 \xleftarrow{\sim} A_2 \xrightarrow{\sim} A_3 \xleftarrow{\sim} A_4 \xrightarrow{\sim} \cdots \xleftarrow{\sim} A_n \xrightarrow{\sim} A$$

Example 6.5. As $\pi_0(D_1(n)) \cong S_n$, E_1-algebras in **Set** are monoids. As $\pi_o(D_2(n)) \cong I$, the E_2-algebras in **Set** are commutative monoids.

Example 6.6. E_1-algebras in **Ab** are rings. E_2-algebras in **Ab** are commutative rings.

Example 6.7. There exist E_1-operads in **Cat** such that their category of algebras are equivalent to the category of monoidal categories and strict monoidal categories correspondingly. There exist E_2-operads in **Cat** such that their category of algebras are equivalent to the category of braided strict monoidal categories (5.2.18 [9]).

6.2 Swiss-Cheese conjecture

In this section, we formulate the Swiss-Cheese conjecture made by Kontsevich. Swiss-Cheese operad was defined by Voronov in [31] and used by Kontsevich to formulate his conjecture in [19].

Definition 6.11 (Swiss-Cheese type operad [2])**.** A Swiss-Cheese type operad A is a 2-coloured operad with colours $\{O, C\}$, standing for open and closed colours, such that for $C_1 = \cdots = C_n = C$ and $O_1 = \cdots = O_m = O$ we have $A(C_1, ..., C_n, O_1, ..., O_m; C) = \emptyset$ if $m \geq 1$ and appropriate action of symmetric group $S_n \times S_m$.

It consists of two colours and objects $A(n, m) \in \mathrm{ob}\,\mathcal{V}$ with action of symmetric group $S_n \times S_m$ and $B(n) \in \mathrm{ob}\,\mathcal{V}$ with action of symmetric group S_n together with appropriate composition morphisms

$$c : A(n, m) \otimes \left(\bigotimes_{i=1}^{m} A(k_i, j_i) \right) \otimes \left(\bigotimes_{u=1}^{n} B(l_u) \right) \to A\left(\sum_{i=1}^{m} k_i + \sum_{u=1}^{n} l_u, \sum_{i=1}^{m} j_m \right)$$

$$c_B : B(n) \otimes (B(h_1) \otimes \cdots B(h_n) \to B(h_1 + \cdots h_n))$$

giving rise to an operad B (of closed coloured part) and an operad S (of open coloured part) with $S(m) = A(0, m)$.

Definition 6.12 (Swiss-Cheese operad [31])**.** Swiss-Cheese operad SC_d is a topological Swiss-Cheese type operad with $\mathrm{SC}_d(n, m)$ being the configuration space of n non-overlapping ordered little d-disks and m separately ordered little upper semi-d-disks inside the unit upper semi-d-disk such that the little upper semi-d-disks are sitting on the diameter of the unit unit upper semi-d-disk

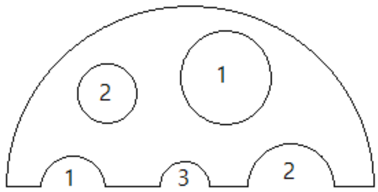


Figure 6.1: An example of $\mathrm{SC}_2(2, 3)$

and with objects $B(n) = D_d(n)$ and the composition morphism c is given by the ordered embeddings of $\mathrm{SC}_d(k_i, j_i)$'s into the little upper semi-d-disks and $D_d(l_u)$'s into the little d-disks and c_B is the same one given by that of the operad D_d.

Remark. The closed coloured part of SC operad is E_d-operad and the open coloured part of SC operad is E_{d-1}-operad.

Definition 6.13 (Action μ of E_d-algebra on E_{d-1}-algebra)**.** Let $X_C \in \mathrm{ob}\,\mathcal{V}$ be an E_d-algebra and $X_O \in \mathrm{ob}\,\mathcal{V}$ be an E_{d-1}-algebra. We say that X_C acts on X_O if there is a structure of SC_d-algebra on $\{X_C, X_O\}$ such that the collection of morphisms

$$\mu_{n,m} : \mathrm{SC}(C_1, \ldots, C_n, O_1, \ldots, O_m; z) \otimes X_C^{\otimes n} \otimes X_O^{\otimes m} \to X_z$$

restricts to the given structure of E_d-algebra on X_C when $m = 0$ and $z = C$ and to the given structure of E_{d-1}-algebra on X_O when $n = 0$ and $z = O$.

Swiss-Cheese Conjecture

For an E_{d-1}-algebra A, consider a category $\mathbf{Act}_d(A)$ of pairs (B, μ^B) where B is an E_d-algebra acting μ^B on A and morphisms from (B, μ^B) to $(B', \mu^{B'})$ are the morphisms $f : B \to B'$ in $\mathcal{V}$ such that $(B, A) \xrightarrow{(f,1)} (B', A)$ is a map of SC_d-algebras. Let our $\mathcal{V}$ be a monoidal model category. $f \in \mathbf{Act}_d(A)$ is a weak equivalence if f is a weak equivalence in $\mathcal{V}$, from which we can talk about the homotopy category $\mathrm{Ho}(\mathbf{Act}_d(A))$ obtained by localisation $\mathbf{Act}_d(A)[\mathcal{W}^{-1}]$ with respect to the class $\mathcal{W}$ of weak equivalences. The conjecture is that $\mathrm{Ho}(\mathbf{Act}_d(A))$ has a terminal object which for $d = 2$, is the classical Hoschild complex (derived centre) of A.

In the case $\mathcal{V} = \mathbf{Set}$, let A be an E_1-algebra which is a monoid. The Hochschild of A is the centre $Z(A)$ of A. For a commutative monoid k which is an E_2-algebra, the structure of SC_2-action of k on A is precisely an associative and unital monoid homomorphism $k \times A \to A$ (the structure on A as a monoid object in **k-Set**). The Swiss-Cheese conjecture amounts to the claim that to give a such structure on A with k is equivalently to give a map of commutative monoids $k \to Z(A)$ which has been shown in Theorem 3.4 as the one-object case of the equivalence between a monoid action $\mathcal{M} \times \mathcal{C} \to \mathcal{C}$ and a monoid morphism $\mathcal{M} \to \mathbf{Cat}(\mathcal{C}, \mathcal{C})$ in the world of $(\mathbf{Cat}, \times, I)$. Notice also that the equivalence between strict action of a monoid action $\mathcal{M} \times C \to C$ and a monoid morphism $\mathcal{M} \to \mathbf{Cat}(C, C)$ in $(\mathbf{Cat}, \times, I)$ is the Swiss-Cheese conjecture for $d = 1$ in **Cat** which is, again, a direct result of having the symmetric monoidally closed structure on **Cat** (Proposition 2.3).

Similarly in $\mathcal{V} = \mathbf{Cat}$, given an E_1-algebra $\mathcal{A}$, up to equivalence, a strict monoidal category and let $\mathcal{K}$ be an E_2-algebra, a braided strict monoidal category up to equivalence.

Remark. To see it we can use a description of fundamental grouppoid $\pi_1(\mathrm{SC}_2)$ of Swiss-Cheese operad SC_2 in terms of coloured braids [14] which extends a description of fundamental grouppoid of operad D_2 given by Fresse [9]. The homotopy category of action $\mathrm{Ho}(\mathbf{Act}_d(\mathcal{A}))$ of this operad $\pi_1(\mathrm{SC}_2)$ is equivalent to the homotopy category of action of braided monoidal categories on a monoidal category $\mathcal{A}$. We can then strictify everything up to homotopy.

The $d = 2$ Swiss-Cheese conjecture says that to give a structure of an action of $\mathcal{K}$ on $\mathcal{A}$ is to give a braided monoidal functor $\mathcal{K} \to \mathcal{Z}(\mathcal{A})$. Generic model theoretic argument allows us to replace any structure of an action of $\mathcal{K}$ on $\mathcal{A}$ to a strict cofibrant algbera. With the homotopy category of action $\mathrm{Ho}(\mathbf{Act}_d(\mathcal{A}'))$ by cofibrant replacement $\mathcal{A}'$ being equivalent to $\mathrm{Ho}(\mathbf{Act}_d(\mathcal{A}))$, the strict algebra case of this has been shown in Theorem 5.3 as a one-object case of the Swiss-Cheese conjecture for $d = 1$ in **2Cat**. This statement for $d = 1$ in **2Cat** is that the choice of a monoid action $\mathcal{M} \otimes_G C \to C$ is a monoid morphism $\mathcal{M} \to \mathbf{Psd}(C, C)$ in **2Cat**. And again this is a direct corollary of existence of Gray tensor product giving **2Cat** a symmetric monoidal closed structure by the Proposition 2.3.

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