The Gray tensor product for 2-quasi-categories

By

Yuki Maehara

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Contents

Summary

The content of this thesis is intended as a steppingstone towards reconstructing Street's *formal theory of monads* [Str72] in the $(\infty, 2)$-context. Although it is not explicitly mentioned in Street's original paper, the formal theory makes use of the monoidal closed structure on the category 2-$\underline{\mathrm{Cat}}$ given by the (*lax*) *Gray tensor product* [Gra74]. More specifically, it requires the 2-category of 2-functors, lax natural transformations and modifications (which is the left closed part of this structure) since Street characterises the familiar Eilenberg-Moore category of algebras as the *lax limit* of the monad in an appropriate sense. This thesis demonstrates that a homotopical counterpart of this monoidal closed structure exists. A more precise formulation is given at the end of this summary.

We adopt *2-quasi-categories*, which are the fibrant objects in $[\Theta_2^{\mathrm{op}}, \underline{\mathrm{Set}}]$ with respect to a model structure due to Ara [Ara14], for modelling $(\infty, 2)$-categories. In that paper, Ara characterised not only the 2-quasi-categories, but also the fibrations into them. More precisely, he proved them to be exactly those maps with the right lifting property with respect to a set $\mathcal{J}_A$ of monomorphisms. The purpose of Chapter 3 is to provide an alternative to $\mathcal{J}_A$ that is better suited for our purposes, *i.e.* combinatorics. More precisely, we prove that the set $\mathcal{J}_O$ consisting of Oury's *inner horn inclusions* and *equivalence extensions* [Our10] can be used in place of $\mathcal{J}_A$.

In Chapter 4, we construct the *2-quasi-categorical Gray tensor product* extending the 2-categorical one in an appropriate sense. Although this tensor product is not associative up to isomorphism, we can define the n-ary tensor product for each $n \geq 0$ and organise them into a *lax monoidal structure* on $[\Theta_2^{\mathrm{op}}, \underline{\mathrm{Set}}]$. That is, there exist appropriately coherent, but not necessarily invertible, comparison maps from nested tensor products to the corresponding total tensor products, *e.g.* $\otimes_2(\otimes_2(X, Y), Z) \to \otimes_3(X, Y, Z)$. We then use the combinatorial tool developed in Chapter 3 to prove that this lax monoidal structure may be regarded as a genuine monoidal (closed) structure in a homotopical sense. More precisely, each n-ary tensor product functor is shown to be left Quillen with respect to Ara's model structure, and also the (relative) comparison maps are shown to be trivial cofibrations.

Declaration

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Yuki Maehara

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Chapter 3 of this thesis has been submitted for publication, and I owe a debt of gratitude to the anonymous referees for helping me improve the presentation of the material and spot an error in the original proof of Lemma 3.1.5.

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1
Introduction

Many authors, most notably Joyal [Joy02, Joy] and Lurie [Lur09, Lur], have shown that one can "do category theory" in quasi-categories. In a similar vein, our ultimate goal is to "do 2-category theory" in 2-quasi-categories. The current thesis develops necessary tools for achieving this goal.

1.1 $(\infty, 1)$-categories

The notion of *2-quasi-category* is central to this thesis. In this section, we recall and motivate its 1-dimensional (and much more famous) cousin, namely the notion of *quasi-category*, and more generally the notion of $(\infty, 1)$-*category*.

1.1.1 What is an $(\infty, 1)$-category?

When one is dealing with mathematical objects with a geometric flavour, often one has not only a natural notion of morphism between the objects but also a natural notion of *homotopy* between such morphisms. These homotopies typically serve as witnesses for an appropriate notion of equivalence between morphisms, *e.g.* homotopies between continuous functions, chain homotopies between chain maps, and natural isomorphisms between functors. The term $(\infty, 1)$-*category* refers to the schematic concept (and not a mathematically rigorous definition) of a category-like structure equipped with such homotopies, homotopies between homotopies, homotopies between homotopies between homotopies, *ad infinitum*. More generally, an (∞, n)-*category* is an n-category-like structure equipped with an infinite hierarchy of homotopies. A variety of *models* realising this abstract idea of (∞, n)-category have been proposed by different authors, each with its advantages and disadvantages.

1.1.2 $(\infty, 1)$-categories as space-enriched categories

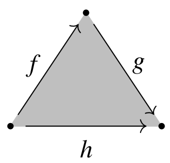
Categories enriched in an appropriate category of *spaces* provide an example of such a model for $(\infty, 1)$-categories. In this setting, morphisms, homotopies, homotopies between homotopies and so on in an $(\infty, 1)$-category correspond respectively to points, paths, homotopies between paths and so on in the hom-space. This is conceptually very simple, but this model is too "strict" for some purposes in the following sense. For mathematical structures in which there is a weaker notion of equivalence than equality, the "correct" kind of morphisms tend to be those that preserve the structure up to that equivalence rather than up to equality. Functors preserving certain (co)limits, strong monoidal functors and pseudo-functors are examples. Similarly, we would like for morphisms between $(\infty, 1)$-categories to preserve composition only up to homotopy, but usual enriched functors between space-enriched categories preserve composition strictly.

1.1.3 $(\infty, 1)$-categories as quasi-categories

In contrast, *quasi-categories* (née *weak Kan complexes* [CP86]) and similar models such as *complete Segal spaces* [Rez01] are "weak" in the sense that, informally speaking, they only remember homotopies and not equalities so that even the strictest kind of morphisms can only preserve composition up to homotopy. Formally, a quasi-category X is a simplicial set in which any *inner horn* admits a filler:

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "\forall"] \arrow[d, hook] & X \\
\Delta[n] \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

where $0 < k < n$ and $\Lambda^k[n]$ is the boundary of $\Delta[n]$ with the k-th face removed. We think of 0- and 1-simplices in X as objects and morphisms respectively, 2-simplices



as homotopies $gf \sim h$ (as opposed to equality $gf = h$) and similarly for higher dimensional simplices. Under this interpretation, the horn-filling condition may be thought of as a combinatorially convenient encoding of a composition that is well-defined, unital and associative up to coherent homotopy.

1.2 2-category theory

Generally $(\infty, 1)$-category theory is developed by imitating ordinary category theory while taking care of relevant homotopical information. Thus it is reasonable to expect to gain a better understanding of certain aspects of $(\infty, 1)$-category theory by first developing $(\infty, 2)$-category theory and then imitating *formal category theory* therein. In particular, the content of this thesis is intended as a steppingstone towards developing the theory of monads in the $(\infty, 1)$-context. This section reviews Street's *formal theory of monads* [Str72], which we aim to eventually reconstruct in 2-quasi-categories, and how it is related to the (*lax*) *Gray tensor product* [Gra74, Theorem I.4.9], whose 2-quasi-categorical analogue is the subject of Chapter 4.

1.2.1 Formal category theory

It is commonly accepted that the totality of (small) categories $\underline{\mathrm{Cat}}$ is better regarded as a *2-category* rather than a mere category. We can see that $\underline{\mathrm{Cat}}$ should be *at least* a 2-category since formulating such fundamental notions to category theory as those of *equivalence*, *adjunction* and *monad* all require natural transformations. The necessity of 2-cells is also supported by the following famous observation of Eilenberg and Mac Lane (quoted from [ML98, §I.4]):

"category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation".

On the other hand, how do we know that the 2-category structure on $\underline{\mathrm{Cat}}$ is (for many purposes) "enough" and we do not need to seek for a more elaborate structure? A practical justification would be to develop *formal category theory*, *i.e.* to exhibit that whatever piece of category theory we are interested in can be obtained by specialising some general 2-category theory to the 2-category $\underline{\mathrm{Cat}}$. In addition to having the obvious bonus of being applicable to a variety of other contexts including enriched and internal category theory, often such general theory also provides a more conceptual understanding of the subject.

1.2.2 The formal theory of monads

Street's *formal theory of monads* [Str72] is a seminal paper in formal category theory. A key observation in this paper is that the Eilenberg-Moore category of algebras is the *lax limit* of the monad in the following sense. A *monad* in a 2-category $\mathscr{A}$ consists of an object $x \in \mathscr{A}$, a 1-cell $t : x \to x$ and 2-cells $\eta : \mathrm{id}_x \to t$ and $\mu : tt \to t$ satisfying the usual unit and associativity axioms. The suspension $\mathtt{Mnd}$ of Δ_+ (*i.e.* the one-object 2-category whose only hom-category is Δ_+ and whose horizontal composition is given by the join operation) is the free 2-category containing a monad in the sense that a 2-functor $\mathtt{Mnd} \to \mathscr{A}$ amounts precisely to a monad in $\mathscr{A}$. Such 2-functors form a 2-category $[\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$ where we take lax natural transformations (recalled in the next subsection) as 1-cells and modifications as 2-cells. Sending each $x \in A$ to the obvious identity monad at x defines an inclusion 2-functor $\mathscr{A} \hookrightarrow [\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$, and when $\mathscr{A} = \underline{\mathrm{Cat}}$ this inclusion admits a right 2-adjoint $[\mathtt{Mnd}, \underline{\mathrm{Cat}}]_{\mathrm{lax}} \to \underline{\mathrm{Cat}}$ sending each monad to the Eilenberg-Moore category of algebras.

The lax limit of a monad in a general 2-category is commonly called the *Eilenberg-Moore object* of that monad. Street's formal theory of monads is the study of the 2-category $[\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$ and the universal property of Eilenberg-Moore objects. In addition to the Eilenberg-Moore category of algebras, the theory also covers such familiar monad-related notions as the Kleisli category (which is the Eilenberg-Moore object of an ordinary monad regarded as a monad in $\underline{\mathrm{Cat}}^{\mathrm{op}}$) and distributive laws (which are monads in $[\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$).

1.2.3 (Op)lax natural transformations and the Gray tensor product

A *lax natural transformation* σ between 2-functors $F, G : \mathscr{A} \to \mathscr{B}$ consists of a family of 1-cells $\sigma_x : Fx \to Gx$ in $\mathscr{B}$ indexed by $x \in \mathscr{A}$ and a family of 2-cells

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\sigma_x"'] & Fy \arrow[d, "\sigma_y"] \\ Gx \arrow[r, "Gf"'] \arrow[ur, Rightarrow, "\sigma_f"] & Gy \end{tikzcd}$$

indexed by 1-cells f in $\mathscr{A}$, satisfying appropriate coherence conditions. An *oplax natural transformation* is similar but the 2-cell σ_f points in the other direction. A *modification* θ between a parallel pair of lax natural transformations σ, τ is a family of 2-cells

$$\begin{tikzcd} Fx \arrow[d, bend right, "\sigma_x"'] \arrow[d, bend left, "\tau_x"] \arrow[d, phantom, "\overset{\theta_x}{\Rrightarrow}"] \\ Gx \end{tikzcd}$$

indexed by $x \in \mathscr{A}$ satisfying an appropriate condition. As we mentioned in the previous subsection, 2-functors $\mathscr{A} \to \mathscr{B}$, lax natural transformations and modifications form a 2-category $[\mathscr{A}, \mathscr{B}]_{\text{lax}}$, and there is also an oplax version $[\mathscr{A}, \mathscr{B}]_{\text{oplax}}$. The *Gray tensor product* provides the category 2-$\underline{\text{Cat}}$ with a monoidal structure for which $[-,-]_{\text{lax}}$ and $[-,-]_{\text{oplax}}$ are part of the associated closed structure, *i.e.* we have bijections

$$2\text{-}\underline{\text{Cat}}(\mathscr{B}, [\mathscr{A}, \mathscr{C}]_{\text{lax}}) \cong 2\text{-}\underline{\text{Cat}}(\mathscr{A} \boxtimes \mathscr{B}, \mathscr{C}) \cong 2\text{-}\underline{\text{Cat}}(\mathscr{A}, [\mathscr{B}, \mathscr{C}]_{\text{oplax}})$$

natural in all three variables. Gray originally defined lax natural transformations explicitly and then defined the tensor product as the corresponding monoidal structure. In contrast, we will construct the 2-quasi-categorical Gray tensor product first and then define $[-,-]_{\text{lax}}$ and $[-,-]_{\text{oplax}}$ as the corresponding closed structure.

1.3 2-quasi-categories

This section provides a rough definition of a 2-quasi-category and explains the sense in which this definition is analogous to that of a quasi-category.

1.3.1 Model categories

The notion of *model category* is due to Quillen [Qui67], and the term is a shorthand for "a category of models for a homotopy theory". A model category is an ordinary category equipped with distinguished classes of morphisms satisfying certain conditions, which induce a well-behaved notion of homotopy between morphisms. Thus model categories themselves may be thought of as presenting $(\infty, 1)$-categories, but they are also often used as ambient categories for constructing other models for $(\infty, 1)$- and more generally (∞, n)-categories. The *cofibrancy* and *fibrancy* conditions formalise "homotopical well-behavedness" of objects in a model category, and often those (∞, n)-categories are defined or characterised as the fibrant objects in a model category in which every object is cofibrant. In particular, this is the case for quasi-categories and 2-quasi-categories.

1.3.2 Quasi-categories and 2-quasi-categories as fibrant objects

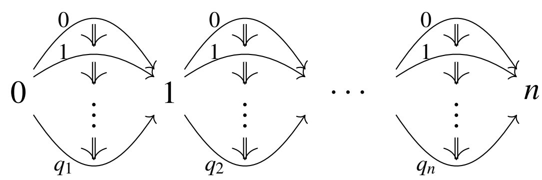
Quasi-categories are usually defined using the horn-filling condition in Section 1.1.3, but equivalently they are the fibrant objects with respect a model structure on $\widehat{\Delta} = [\Delta^{\mathrm{op}}, \underline{\mathrm{Set}}]$ due to Joyal. In [Ara14], Ara constructed for each $n \geq 1$ a model structure on $\widehat{\Theta_n}$ which presents (∞, n)-categories. In the case $n = 1$, (we have $\Theta_1 = \Delta$ and) Ara's model structure coincides with Joyal's. The *n-quasi-categories* are the fibrant objects in $\widehat{\Theta_n}$ with respect to this structure.

In the case $n = 1$, the indexing category Δ for simplicial sets is the category of free categories $[n]$ generated by linear graphs:

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow n$$

Thus we may think of Δ as "controling" composition of general arity. When spelled out using Ara's description of the model structure, the fibrancy condition essentially states that there is always a unique-up-to-homotopy way to compose any composable sequence of morphisms (of finite length).

Analogously, Θ_2 is the category of free 2-categories $[n; q_1, \dots, q_n]$ generated by "linear-graph-enriched linear graphs":



The fibrancy condition states that there is always a unique-up-to-homotopy way to compose any horizontally composable sequence of vertically composable sequences of 2-cells.

1.4 Overview of the thesis

The conclusion of this thesis may be paraphrased as:

the 2-quasi-categorical Gray tensor product is
part of an up-to-homotopy monoidal closed structure.

In this section, we make this statement more precise and explain how it is proved.

1.4.1 Inner horns for 2-quasi-categories

In [Ara14], Ara characterised not only the n-quasi-categories, but also the fibrations into them. More precisely, he proved them to be exactly those objects or maps with the right lifting property with respect to a set $\mathcal{J}_A$ of monomorphisms. Thus to prove, for example, that the 2-quasi-categorical Gray tensor product is left Quillen, it would suffice to check that it interacts nicely with the maps in $\mathcal{J}_A$. However, the definition of $\mathcal{J}_A$ is complicated and not very easy to deal with. The purpose of Chapter 3 is to provide, in the case $n = 2$, an alternative set which is combinatorially more tractable.

More specifically, we show the set $\mathcal{J}_O$ of *inner horn inclusions* and *equivalence extensions*, introduced by Oury in his PhD thesis [Our10], can be used in place of $\mathcal{J}_A$. These maps are constructed from their simplicial counterparts using the *box product* $\square : \widehat{\Delta} \wr \widehat{\Delta} \to \widehat{\Theta_2}$, analogously to how the bisimplicial horns may be constructed from the simplicial ones using the functor $\square : \widehat{\Delta} \times \widehat{\Delta} \to \widehat{\Delta \times \Delta}$. The precise construction and other background material will be reviewed in Chapter 2.

The most technical (and also the longest) section of Chapter 3 is Section 3.1 where we compare the sets $\mathcal{J}_A$ and $\mathcal{J}_O$ and the class of trivial cofibrations. In Section 3.2 we consider a different notion of inner horn, namely the sub-Θ_2-sets of the representables generated by all but one codimension-one faces. Section 3.3 is very short and devoted to proving that the infinite family of horizontal equivalences (contained in both $\mathcal{J}_A$ and $\mathcal{J}_O$) can in fact be replaced by a single map as long as we keep the inner horn inclusions in the defining set of monomorphisms. In Section 3.4 we prove that the set $\mathcal{J}_O$ may be used to characterise Ara's model structure (Theorem 3.4.1) and in particular to detect left Quillen functors out of $\widehat{\Theta_2}$ (Corollary 3.4.4). Section 3.5 discusses *special outer horns* which will be used in Chapter 4.

1.4.2 The Gray tensor product for 2-quasi-categories

In Chapter 4, we analyse a 2-quasi-categorical version of the Gray tensor product. For each $a \geq 0$, we define the *a-ary Gray tensor product* of presheaves over Θ_2

$$\otimes_a : \underbrace{\widehat{\Theta_2} \times \cdots \times \widehat{\Theta_2}}_{a \text{ times}} \to \widehat{\Theta_2}$$

by extending the composite

$$\Theta_2 \times \cdots \times \Theta_2 \hookrightarrow 2\text{-}\underline{\mathrm{Cat}} \times \cdots \times 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{\boxtimes_a} 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{N} \widehat{\Theta_2}$$

cocontinuously in each variable, where the second map $\boxtimes_a$ is the a-ary Gray tensor product of 2-categories and N is the nerve functor induced by the inclusion $\Theta_2 \hookrightarrow 2\text{-}\underline{\mathrm{Cat}}$.

It can be seen from this definition that it is crucial to have a good understanding of the 2-categorical Gray tensor products of objects in Θ_2. Indeed we analyse these 2-categories in Section 4.1, and in particular we provide a combinatorial description for them using the theory of *braid monoids with zero* reviewed in Appendix A. We then prove in Section 4.2 that the Leibniz/relative version of $\otimes_a$ preserves monomorphisms. The rest of the proof that $\otimes_a$ is left Quillen is divided into several cases, most of which follow a common combinatorial strategy. This strategy is illustrated in Section 4.3. In Section 4.4, we utilise this strategy and prove that the binary Gray tensor product $\otimes_2$ is left Quillen. Section 4.5 makes precise the statement that the Gray tensor product is associative up to homotopy. A few consequences of this associativity are investigated in Section 4.6, and in parcitular we prove that $\otimes_a$ is left Quillen for arbitrary $a \geq 1$.

1.5 Future work: the formal theory of homotopy coherent monads

As we mentioned in Section 1.2, the results in this thesis are intended as a steppingstone towards reconstructing Street's formal theory of monads in the homotopy coherent context.

In this section, we discuss how this project may be continued in future.

1.5.1 Eilenberg-Moore objects as lax limits

The 2-quasi-categorical Gray tensor product allows us to construct an analogue of the 2-category $[\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$, namely the object $A \leftarrowtriangle N\mathtt{Mnd}$ as defined in Definition 4.6.2. It follows from Theorem 4.6.1 (or Theorem 4.4.1) that $A \leftarrowtriangle N\mathtt{Mnd}$ is a 2-quasi-category whenever A is so. Since the totality of 2-quasi-categories forms an *∞-cosmos* (*i.e.* a well-behaved quasi-categorically enriched category), we may apply the theory developed in [RV16] to make sense of what it means for the canonical inclusion $A \hookrightarrow (A \leftarrowtriangle N\mathtt{Mnd})$ to have a right 2-adjoint. More generally, even if such a total right 2-adjoint does not exist, their framework allows us to define the *Eilenberg-Moore object* of a particular homotopy coherent monad $\mathbb{T} : N\mathtt{Mnd} \to A$ to be a terminal object in an appropriate comma 2-quasi-category.

This is a conceptually simple, but combinatorially complicated, encoding of the universal property. (Unwinding the definitions, one can check that the universal property involves maps of the form $N\mathtt{Mnd} \otimes (\mathbb{2} \times \mathbb{2} \times X) \to A$.) In order to mimick Street's construction of *e.g.* the free/forgetful adjunction, it is desirable to find a more tractable encoding of the same universal property. This may be done purely combinatorially, for instance by replacing various objects by weakly equivalent ones that admit simpler descriptions. A more conceptual approach, which is still likely to be combinatorially heavy, would be to identify Eilenberg-Moore objects as *weighted limits* (a theory of which in 2-quasi-categories is yet to be developed).

1.5.2 Eilenberg-Moore objects as weighted limits

The lax limit of a diagram in a 2-category may be computed as the *weighted limit* of the same diagram for an appropriate weight [BKPS89, §2]. In particular, Eilenberg-Moore objects may be regarded as weighted limits, and this is in fact the view taken in Lack and Street's follow-up paper [LS02] to Street's original formal theory of monads.

The free/forgetful adjunction is induced by an adjunction between weights, and in this sense the weighted limit approach is well-suited for studying the free/forgetful adjunction. On the other hand, it does not provide us immediate access to *monad functors/transformations* as defined in [Str72] (*i.e.* 1-cells and 2-cells in $[\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$) since the natural habitat of monads from this viewpoint is the "strict" hom-object $[\mathtt{Mnd}, \mathscr{A}]$. This is an issue if one is interested in *e.g.* distributive laws (see the next subsection). Thus to reconstruct the full-fledged formal theory of monads for 2-quasi-categories, it would be convenient to be able to switch between the two definitions of the Eilenberg-Moore object. In the 2-categorical case the two universal properties can be checked to be equivalent by hand, but it will require much more work in the 2-quasi-categorical case.

Riehl and Verity [RV16] adopted the weighted limit approach and proved an analogue of Beck's *Monadicity Theorem* for a different model of (well-behaved) $(\infty, 2)$-categories, namely ∞-cosmoi. More precisely, a *homotopy coherent monad* in an ∞-cosmos $\mathscr{A}$ is a simplicial functor $\mathtt{Mnd} \to \mathscr{A}$, and they defined its *Eilenberg-Moore object* to be the (enriched) limit weighted by the same weight as that used in the 2-categorical case; here the $\underline{\mathrm{Cat}}$-enriched gadgets (*i.e.* $\mathtt{Mnd}$ and the weight) are made into simplicially enriched ones by taking appropriate nerves.

Although this is not included in the current thesis, it is relatively easy to prove using Theorem 3.4.1 (or [Cam, Proposition 4.13]) that the appropriate homotopy coherent nerve of a quasi-categorically enriched category is a 2-quasi-category. Thus the following question makes sense: *is the Eilenberg-Moore object (in the sense of Riehl and Verity) of a homotopy coherent monad in an ∞-cosmos the same thing as the Eilenberg-Moore object (in our sense) of the corresponding monad in its nerve?*

1.5.3 Distributive laws

Since a distributive law between monads in an ordinary 2-category $\mathscr{A}$ amounts precisely to a monad in $[\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$, it is reasonable to define a *distributive law* in a 2-quasi-category A to be an object in $(A \mathrel{\triangleleft\!\!-} N\mathtt{Mnd}) \mathrel{\triangleleft\!\!-} N\mathtt{Mnd}$. Corollary 4.6.3 implies that this latter 2-quasi-category is equivalent to $A \mathrel{\triangleleft\!\!-} (N\mathtt{Mnd} \otimes N\mathtt{Mnd})$. We conjecture that the canonical comparison map $N\mathtt{Mnd} \otimes N\mathtt{Mnd} \to N(\mathtt{Mnd} \boxtimes \mathtt{Mnd})$ is a weak equivalence. If this is the case, then the 2-quasi-category of distributive laws in A may be constructed as $A \mathrel{\triangleleft\!\!-} N(\mathtt{Mnd} \boxtimes \mathtt{Mnd})$. The 2-category $\mathtt{Mnd} \boxtimes \mathtt{Mnd}$ is precisely the free 2-category containing a distributive law, and it admits a combinatorial description [Nik18, §A.1.2]. This description can facilitate computations involving distributive laws in a 2-quasi-category.

Also, if $N\mathtt{Mnd} \otimes N\mathtt{Mnd} \to N(\mathtt{Mnd} \boxtimes \mathtt{Mnd})$ is a weak equivalence then the composition of two homotopy coherent monads related by a distributive law admits the following simple description. In the 2-categorical case, the composite monad is obtained by composing the distributive law regarded as a 2-functor $\mathtt{Mnd} \boxtimes \mathtt{Mnd} \to \mathscr{A}$ with an appropriate 2-functor $\mathtt{Mnd} \to \mathtt{Mnd} \boxtimes \mathtt{Mnd}$. Analogously, homotopy coherent monads may be composed using the nerve of this latter 2-functor. In fact, this 2-functor is the comultiplication for a $\boxtimes$-comonoid structure on $\mathtt{Mnd}$, which induces a monad structure on the endofunctor $[\mathtt{Mnd}, -]_{\mathrm{lax}} : 2\text{-}\underline{\mathrm{Cat}} \to 2\text{-}\underline{\mathrm{Cat}}$. Street observed that this monad is in fact a (strict) 3-monad. We conjecture that an analogous result holds for 2-quasi-categories, but describing the homotopy coherent monad structure on $(-) \mathrel{\triangleleft\!\!-} N\mathtt{Mnd}$, even when it is regarded as an endomorphism on the $(\infty, 1)$-category (as opposed to the $(\infty, 3)$-category) of 2-quasi-categories, seems to be fairly non-trivial.

1.6 Related work

Certain aspects of the combinatorics of Θ_2, including notions of boundary and horn, were studied by Watson [Wat13] and we make use of a result proved in his thesis (see Proposition 2.1.9). Horns for a larger category Θ was also investigated by Berger [Ber02]. These horns are the "alternative" ones in our terminology (analysed in Section 3.2) and they differ from Oury's horns in the "horizontal" case (see Section 2.2.4).

In their book on derived algebraic geometry [GR17] Gaitsgory and Rozenblyum listed and exploited various properties the Gray tensor product of $(\infty, 2)$-categories should have, but they did not prove that such a tensor product indeed exists. Our main results from Chapter 4 correspond to some of the unproven statements of in that book, namely Propositions 3.2.6 and 3.2.9.

The theory of monads in the homotopy coherent context admits several existing approaches. For example, Lurie [Lur] defined and analysed monads on quasi-categories using his theory of ∞-*operads*. In particular, he gave an explicit definition of the quasi-category of algebras for such a monad, and proved an analogue of Beck's Monadicity Theorem. As mentioned in Section 1.5.2, Riehl and Verity [RV16] proved the Monadicity Theorem more generally for arbitrary ∞-*cosmoi* (including the ∞-cosmos of quasi-categories) where they define the Eilenberg-Moore object as an appropriate *weighted limit* of the monad rather than the lax limit.

Zaganidis [Zag17] has constructed an analogue of $[\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$ for a different model of $(\infty, 2)$-categories. More precisely, he gave a combinatorial description of a *stratified set* (simplicial set with extra data) of homotopy coherent monads in a given ∞-cosmos, and conjectured that it is always a 2-trivial, saturated weak complicial set (another model of $(\infty, 2)$-categories).

Another approach to homotopy coherent monads in the ∞-cosmological framework was presented by Verity at the Australian Category Seminar in April to May 2020. For an ordinary 2-category $\mathscr{A}$, a 1-cell in the 2-category $[\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$ may be identified with a 2-functor $\mathbb{2} \to [\mathtt{Mnd}, \mathscr{A}]_{\mathrm{lax}}$ or equivalently a 2-functor $\mathtt{Mnd} \to [\mathbb{2}, \mathscr{A}]_{\mathrm{oplax}}$. If $\mathscr{A}$ has comma objects, the 2-category $[\mathbb{2}, \mathscr{A}]_{\mathrm{oplax}}$ may be described as the 2-category of *representable modules* in an appropriate sense. Verity constructed, for an arbitrary ∞-cosmos $\mathscr{A}$, an analogous simplicial category of representable modules in $\mathscr{A}$, and used it to prove a version of Dubuc's *Adjoint Triangle Theorem*.

2
Background

In this chapter, we review the necessary background material which will be used in the main body of the thesis. We claim no originality for the content of this chapter.

2.1 Basic combinatorics of Θ_2

2-quasi-categories are certain presheaves over the category Θ_2. The current section reviews this category and analyses its basic combinatorics.

2.1.1 Simplicial sets and shuffles

As usual, we denote by Δ the category of non-empty finite ordinals $[n] \stackrel{\text{def}}{=} \{0, \ldots, n\}$ and order-preserving maps. The morphisms in Δ will be called *simplicial operators*. We often denote a simplicial operator $\alpha : [m] \to [n]$ by its "image" $\{\alpha(0), \ldots, \alpha(m)\}$; for instance, we write $\{0, 2\} = \delta^1 : [1] \to [2]$ for the 1st elementary face operator.

We will write $\widehat{\Delta}$ for the category $[\Delta^{\mathrm{op}}, \underline{\mathrm{Set}}]$ of *simplicial sets*, and write $\Delta[n]$ for the presheaf represented by $[n] \in \Delta$. If $X \in \widehat{\Delta}$ is a simplicial set, $x \in X_n$ and $\alpha : [m] \to [n]$ is a simplicial operator, then we will write $x \cdot \alpha$ for the image of x under $X(\alpha)$.

Definition 2.1.1. An (m, n)-*shuffle* is a non-degenerate $(m + n)$-simplex in the product $\Delta[m] \times \Delta[n]$.

Equivalently, an (m, n)-shuffle $\langle \alpha, \alpha' \rangle$ consists of two surjections

$$\begin{aligned} \alpha &: [m + n] \to [m], \\ \alpha' &: [m + n] \to [n] \end{aligned}$$

in Δ such that $\alpha(i) + \alpha'(i) = i$ for all $i \in [m + n]$. We write $\mathbf{Shfl}(m, n)$ for the set of (m, n)-shuffles. Note that an (m, n)-shuffle $\langle \alpha, \alpha' \rangle$ is uniquely determined by the surjection $\alpha : [m + n] \to [m]$ since α' can be recovered as $\alpha'(i) = i - \alpha(i)$. Thus the pointwise order on $\Delta([m + n], [m])$ induces a partial order $\leq$ on $\mathbf{Shfl}(m, n)$. We have drawn in Fig. 2.1 two

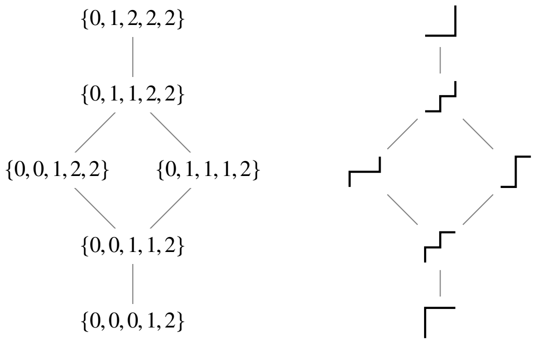


Figure 2.1: **Shfl**$(2, 2)$

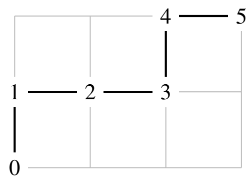


Figure 2.2: $\langle\{0, 0, 1, 2, 2, 3\}, \{0, 1, 1, 1, 2, 2\}\rangle$

copies of **Shfl**$(2, 2)$, where each vertex $\langle\alpha, \alpha'\rangle$ is labelled with α (left) or the corresponding grid-path (right) which we describe now.

We can visualise (m, n)-shuffles as paths on the $m \times n$ grid from the lower-left corner to the upper-right corner. For example, the path in Fig. 2.2 corresponds to the $(3, 2)$-shuffle $\langle\{0, 0, 1, 2, 2, 3\}, \{0, 1, 1, 1, 2, 2\}\rangle$. (If either $m = 0$ or $n = 0$ then the "grid" becomes a line segment. In this case we have a unique path connecting the two endpoints, which corresponds to having a unique (m, n)-shuffle.) This motivates the following notation.

Definition 2.1.2. Given an (m, n)-shuffle $\langle\alpha, \alpha'\rangle$, we will write:

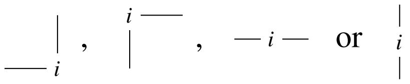
- $\lrcorner\langle\alpha, \alpha'\rangle$ for the set of all $1 \leq i \leq m + n - 1$ such that
$$\alpha(i + 1) = \alpha(i) = \alpha(i - 1) + 1$$
(or equivalently $\alpha'(i + 1) = \alpha'(i) + 1 = \alpha'(i - 1) + 1$) holds; and
- $\ulcorner\langle\alpha, \alpha'\rangle$ for the set of all $1 \leq i \leq m + n - 1$ such that
$$\alpha(i + 1) = \alpha(i) + 1 = \alpha(i - 1) + 1$$
(or equivalently $\alpha'(i + 1) = \alpha'(i) = \alpha'(i - 1) + 1$) holds.

For example, if $\langle\alpha, \alpha'\rangle$ is the $(3, 2)$-shuffle depicted in Fig. 2.2, then $\lrcorner\langle\alpha, \alpha'\rangle = \{3\}$ and $\ulcorner\langle\alpha, \alpha'\rangle = \{1, 4\}$. The following propositions are straightforward to prove.

Proposition 2.1.3. *Let $\langle\alpha, \alpha'\rangle, \langle\beta, \beta'\rangle$ be (m, n)-shuffles. Suppose $\alpha(i) = \beta(i)$ (and so $\alpha'(i) = \beta'(i)$) for each $i \in \lrcorner\langle\alpha, \alpha'\rangle$. Then $\langle\alpha, \alpha'\rangle \leq \langle\beta, \beta'\rangle$.*

Proposition 2.1.4. *Let $\langle\alpha, \alpha'\rangle$ be an (m, n)-shuffle and suppose $i \in \lrcorner\langle\alpha, \alpha'\rangle$. Then $\langle\alpha, \alpha'\rangle$ has an immediate predecessor $\langle\beta, \beta'\rangle$ such that $\langle\alpha, \alpha'\rangle \circ \delta^i = \langle\beta, \beta'\rangle \circ \delta^i$. Moreover, this condition determines $\langle\beta, \beta'\rangle$ uniquely and induces a bijection between $\lrcorner\langle\alpha, \alpha'\rangle$ and the set of immediate predecessors of $\langle\alpha, \alpha'\rangle$. Similarly, there is a bijection between $\ulcorner\langle\alpha, \alpha'\rangle$ and the set of immediate successors of $\langle\alpha, \alpha'\rangle$.*

For $1 \leq i \leq m+n-1$, the grid-path corresponding to $\langle \alpha, \alpha' \rangle \in \mathbf{Shfl}(m,n)$ locally looks like:



This observation can be formalised as follows.

Proposition 2.1.5. *Let $\langle \alpha, \alpha' \rangle$ be an (m,n)-shuffle. Then for any $1 \leq i \leq m+n-1$, precisely one of the following holds:*

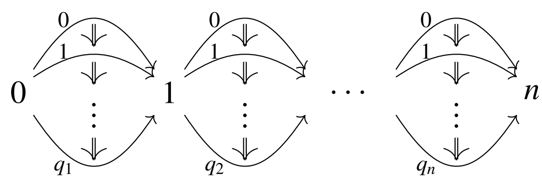
- $i \in {\lrcorner}\langle \alpha, \alpha' \rangle$*;*
- $i \in {\ulcorner}\langle \alpha, \alpha' \rangle$*;*
- $\alpha^{-1}(\alpha(i)) = \{i\}$*; or*
- $(\alpha')^{-1}(\alpha'(i)) = \{i\}$.

2.1.2 The category Θ_2

The category Δ can be seen as the full subcategory of $\underline{\mathrm{Cat}}$ spanned by the free categories $[n]$ generated by linear graphs:

$$0 \longrightarrow 1 \longrightarrow \dots \longrightarrow n$$

Similarly, Joyal's *2-cell category* Θ_2 is the full subcategory of 2-$\underline{\mathrm{Cat}}$ spanned by the free 2-categories $[n; q_1, \dots, q_n]$ generated by "linear-graph-enriched linear graphs":



whose hom-categories are given by

$$\hom(k,\ell) = \begin{cases} [q_{k+1}] \times \cdots \times [q_\ell] & \text{if } k \leq \ell, \\ \varnothing & \text{if } k > \ell. \end{cases}$$

More precisely, Θ_2 has objects $[n; \mathbf{q}] = [n; q_1, \dots, q_n]$ where $n, q_k \in \mathbb{N}$ for each k. A morphism $[\alpha; \boldsymbol{\alpha}] = [\alpha; \alpha_{\alpha(0)+1}, \dots, \alpha_{\alpha(m)}] : [m; \mathbf{p}] \to [n; \mathbf{q}]$ consists of simplicial operators $\alpha : [m] \to [n]$ and $\alpha_k : [p_\ell] \to [q_k]$ for each $k \in [n]$ such that there exists (necessarily unique) $\ell \in [m]$ with $\alpha(\ell - 1) < k \leq \alpha(\ell)$. By a *cellular operator* we mean a morphism in Θ_2. Clearly $[0]$ is a terminal object in Θ_2, and we will write $! : [n; \mathbf{q}] \to [0]$ for any cellular operator into $[0]$.

Remark. Here we are describing $\Theta_2 = \Delta \wr \Delta$ as an instance of Berger's *wreath product* construction. For any given category $\mathscr{C}$, the wreath product $\Delta \wr \mathscr{C}$ may be thought of as the category of free $\mathscr{C}$-enriched (or more accurately $\widehat{\mathscr{C}}$-enriched) categories generated by linear $\mathscr{C}$-enriched graphs. The precise definition can be found in [Ber07, Definition 3.1].

Remark. The notation for objects (and maps) in Θ_2 varies from author to author. (This is partly because some authors introduce a notation for objects in a general wreath product category which can be specialised to $\Theta_2 = \Delta \wr \Delta$ while others are interested in the particular category Θ_2 and hence able to adopt a more economical notation.) For example, the object we denote by $[n;\mathbf{q}] = [n; q_1, \ldots, q_n]$ would be denoted as:

- $\big([q_1], \ldots, [q_n]\big)$ in [Ber07];
- $[n;\mathbf{q}] = \big([n], [-1], [q_1], \ldots, [q_n], [-1]\big)$ in [Our10];
- $\big([n], [q_1], \ldots, [q_n]\big)$ in [Rez10]; and
- $\langle q_1, \ldots, q_n \rangle$ in [Wat13].

In [Ara14] an object in Θ_2 (or more generally in Θ_n) is specified using the *table of dimensions*; see *loc. cit.* for details.

The category Δ has an automorphism $(-)^{\mathrm{op}}$ which is the identity on objects and sends $\alpha : [m] \to [n]$ to $\alpha^{\mathrm{op}} : [m] \to [n]$ given by $\alpha^{\mathrm{op}}(i) = n - \alpha(m-i)$. This induces two automorphisms on Θ_2, namely:

- $(-)^{\mathrm{co}} : \Theta_2 \to \Theta_2$, which sends $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ to
$$[\alpha; \alpha^{\mathrm{op}}_{\alpha(0)+1}, \ldots, \alpha^{\mathrm{op}}_{\alpha(m)}] : [m;\mathbf{p}] \to [n;\mathbf{q}];$$
and
- $(-)^{\mathrm{op}} : \Theta_2 \to \Theta_2$, which sends $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ to
$$[\alpha^{\mathrm{op}}; \alpha_{\alpha(m)}, \ldots, \alpha_{\alpha(0)+1}] : [m; p_m, \ldots, p_1] \to [n; q_n, \ldots, q_1].$$

2.1.3 Face maps in Θ_2

There is a Reedy category structure on Θ_2 defined as follows; see [BR13, Proposition 2.11] or [Ber02, Lemma 2.4] for a proof.

Definition 2.1.6. The *dimension* of $[n;\mathbf{q}]$ is $\dim[n;\mathbf{q}] \overset{\text{def}}{=} n + \sum_{k=1}^{n} q_k$. A cellular operator $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ is a *face operator* if α is monic and $\{\alpha_k : \alpha(\ell-1) < k \leq \alpha(\ell)\}$ is jointly monic for each $1 \leq \ell \leq m$. It is a *degeneracy operator* if α and all α_k are surjective.

Definition 2.1.7. A simplicial operator $\alpha : [m] \to [n]$ is *inert* if it is a subinterval inclusion, that is, if $\alpha(i+1) = \alpha(i) + 1$ for $0 \leq i \leq m-1$.

Definition 2.1.8. We say a face map $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ is:

- *inner* if α and all α_k preserve the top and bottom elements, and otherwise *outer*;
- *horizontal* if each α_k is surjective;
- *vertical* if $\alpha = \mathrm{id}$; and
- *inert* if α and all α_k are inert.

(Examples of each kind can be found in Table 2.1.) A horizontal face map of the form $[\delta^k;\boldsymbol{\alpha}]$ will be called a *k-th horizontal face*.

By the *codimension* of a face map $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$, we mean the difference $\dim[n;\mathbf{q}] - \dim[m;\mathbf{p}]$. We will in particular be interested in the face maps of codimension 1, which we call *hyperfaces*. Such a map $[\alpha;\boldsymbol{\alpha}]$ has precisely one of the following forms:

- for $n \geq 1$, $[n;\mathbf{q}]$ always has a unique *0-th horizontal face*
$$\delta_h^0 \overset{\text{def}}{=} [\delta^0;\mathbf{id}] : [n-1;q_2,\dots,q_n] \to [n;\mathbf{q}]$$
which has codimension 1 if and only if $q_1 = 0$;
- similarly, if $q_n = 0$ then the unique *n-th horizontal face*
$$\delta_h^n \overset{\text{def}}{=} [\delta^n;\mathbf{id}] : [n-1;q_1,\dots,q_{n-1}] \to [n;\mathbf{q}]$$
has codimension 1;
- for each $1 \leq k \leq n-1$, there is a family of *k-th horizontal hyperfaces*
$$\delta_h^{k;\langle\beta,\beta'\rangle} \overset{\text{def}}{=} [\delta^k;\boldsymbol{\alpha}] : [n-1;q_1,\dots,q_{k-1},q_k+q_{k+1},q_{k+2},\dots,q_n] \to [n;\mathbf{q}]$$
indexed by $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_k,q_{k+1})$ where $\alpha_\ell = \mathrm{id}$ for $k \neq \ell \neq k+1$, $\alpha_k = \beta$ and $\alpha_{k+1} = \beta'$; and
- for each $1 \leq k \leq n$ satisfying $q_k \geq 1$ and for each $0 \leq i \leq q_k$, the *$(k;i)$-th vertical hyperface*
$$\delta_v^{k;i} \overset{\text{def}}{=} [\mathrm{id};\boldsymbol{\alpha}] : [n;q_1,\dots,q_{k-1},q_k-1,q_{k+1},\dots,q_n] \to [n;\mathbf{q}]$$
is given by $\alpha_k = \delta^i$ and $\alpha_\ell = \mathrm{id}$ for $\ell \neq k$.

Convention. Strictly speaking, we are giving the same name to different cellular operators, and this can lead to confusion. So in the rest of this paper, we will assume the codomain of any cellular operator denoted by δ (with some decoration) is always *whatever is called* $[n;\mathbf{q}]$ *at that point* (or some cellular subset of $\Theta_2[n;\mathbf{q}]$ as described in Section 2.1.4). When this is not the case, we will indicate the codomain $[m;\mathbf{p}]$ either by writing $\delta[m;\mathbf{p}]$ instead of δ, or by drawing δ as an arrow $[m';\mathbf{p}'] \xrightarrow{\delta} [m;\mathbf{p}]$.

In Table 2.1, we have listed various faces of $[2;0,2]$. We will briefly describe how to read the pictures. In the first row is the "standard picture" of $[2;0,2]$, in which we have nicely placed its objects ($\bullet$), generating 1-cells ($\longrightarrow$) and generating 2-cells ($\Longrightarrow$). In the rest of the table, a face operator $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [2;0,2]$ is illustrated as the standard picture of $[m;\mathbf{p}]$ appropriately distorted so that the ℓ-th object appears in the $\alpha(\ell)$-th position and each generating 1-cell lies roughly where the factors of its image used to. In the third row (where α_1 is not injective), we have left small gaps between the generating 1-cells so that they do not intersect with each other.

The hyperfaces of $[n;\mathbf{q}]$ are precisely the maximal faces of $[n;\mathbf{q}]$ in the following sense.

Proposition 2.1.9 ([Wat13, Proposition 6.2.4])**.** *Any face map* $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ *of positive codimension factors through a hyperface of* $[n;\mathbf{q}]$.

We will also need the following outer version of this proposition.

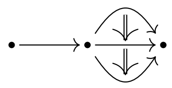
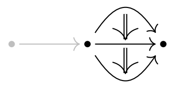
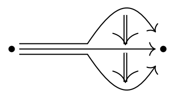
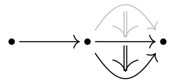
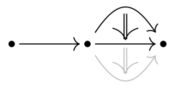
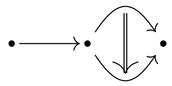
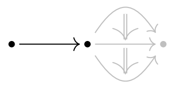
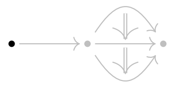
	picture	domain	inner/outer	horizontal	vertical	inert
id		$[2;0,2]$	inner	✓	✓	✓
$\delta_h^0 = [\delta^0;\mathrm{id}]$		$[1;2]$	outer	✓	×	✓
$\delta_h^{1;\langle !,\mathrm{id}\rangle} = [\delta^1;!,\mathrm{id}]$		$[1;2]$	inner	✓	×	×
$\delta_v^{2;0} = [\mathrm{id};\mathrm{id},\delta^0]$ $\delta_v^{2;2} = [\mathrm{id};\mathrm{id},\delta^2]$	 	$[2;0,1]$	outer	×	✓	✓
$\delta_v^{2;1} = [\mathrm{id};\mathrm{id},\delta^1]$		$[2;0,1]$	inner	×	✓	×
$\delta_h^2 = [\delta^2;\mathrm{id}]$		$[1;0]$	outer	✓	×	✓
$[\{0\}]$		$[0]$	outer	✓	×	✓

Table 2.1: Some faces of $[2;0,2]$

Proposition 2.1.10. *Any outer face map* $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ *factors through an outer hyperface of* $[n;\mathbf{q}]$.

Proof. Recall that $[\alpha;\boldsymbol{\alpha}]$ is inner (= non-outer) if and only if α and all α_k preserve the top and bottom elements. We will consider the cases where either α or some α_k does not preserve the top elements; the other cases can be treated dually.

(i) If $\alpha(m) \neq n$ and $q_n = 0$ then we can factorise $[\alpha;\boldsymbol{\alpha}]$ as

$$[m;\mathbf{p}] \xrightarrow{[\beta;\boldsymbol{\alpha}]} [n-1;q_1,\ldots,q_{n-1}] \xrightarrow{\delta_h^n} [n;\mathbf{q}]$$

where $\beta : [m] \to [n-1]$ is given by $\beta(i) = \alpha(i)$.

(ii) If $\alpha(m) \neq n$ and $q_n \geq 1$ then we can factorise $[\alpha;\boldsymbol{\alpha}]$ as

$$[m;\mathbf{p}] \xrightarrow{[\alpha;\boldsymbol{\alpha}]} [n;q_1,\ldots,q_{n-1},q_n-1] \xrightarrow{\delta_v^{n;0}} [n;\mathbf{q}].$$

(iii) If $\alpha_k(p_\ell) \neq q_k$ for some $\alpha(\ell-1) < k \leq \alpha(\ell)$ then we can factorise $[\alpha;\boldsymbol{\alpha}]$ as

$$[m;\mathbf{p}] \xrightarrow{[\alpha;\boldsymbol{\beta}]} [n;q_1,\ldots,q_k-1,\ldots,q_n] \xrightarrow{\delta_v^{k;q_k}} [n;\mathbf{q}]$$

where $\beta_k : [p_\ell] \to [q_k-1]$ is given by $\beta_k(i) = \alpha_k(i)$ and $\beta_{k'} = \alpha_{k'}$ for $k' \neq k$.

□

We will also introduce the following notations for later use.

Definition 2.1.11. For any $[n;\mathbf{q}] \in \Theta_2$ and any $1 \leq k \leq n$, we denote by η_h^k the face map

$$\eta_h^k \overset{\text{def}}{=} \big[\{k-1,k\};\mathrm{id}\big] : [1;q_k] \to [n;\mathbf{q}].$$

Definition 2.1.12. For any $0 \leq i \leq q$, we denote by η_v^i the face map

$$\eta_v^i \overset{\text{def}}{=} \big[\mathrm{id};\{i\}\big] : [1;0] \to [1;q].$$

2.1.4 Cellular sets

We will write $\widehat{\Theta_2}$ for the category $[\Theta_2^{\mathrm{op}}, \underline{\mathrm{Set}}]$ of *cellular sets*. If X is a cellular set, $x \in X_{n;\boldsymbol{q}} \overset{\text{def}}{=} X([n;\mathbf{q}])$ and $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ is a cellular operator, then we will write $x \cdot [\alpha;\boldsymbol{\alpha}]$ for the image of x under $X([\alpha;\boldsymbol{\alpha}])$. The Reedy structure on Θ_2 is (*EZ* and hence) *elegant*, which means the following.

Theorem 2.1.13 ([BR13, Corollary 4.5])**.** *For any cellular set X and for any $x \in X_{m;\boldsymbol{p}}$, there is a unique way to express x as $x = y \cdot [\alpha;\boldsymbol{\alpha}]$ where $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ is a degeneracy operator and $y \in X_{n;\boldsymbol{q}}$ is non-degenerate.*

Definition 2.1.14. A *cellular subset* of $X \in \widehat{\Theta_2}$ is a subfunctor of X. If S is a set of cells in $X \in \widehat{\Theta_2}$ (not necessarily closed under the action of cellular operators), the smallest cellular subset $\overline{S}$ of X containing S is given by

$$\overline{S}_{m;\boldsymbol{p}} = \Big\{ s \cdot [\alpha;\boldsymbol{\alpha}] : s \in S_{n;\boldsymbol{q}},\ \ [m;\mathbf{p}] \xrightarrow{[\alpha;\boldsymbol{\alpha}]} [n;\mathbf{q}]\ \Big\}.$$

We call $\overline{S}$ the cellular subset of X *generated by S*.

(Abuse of) notation. We will write Θ_2^θ or $\Theta_2[n;\mathbf{q}]$ for the presheaf represented by the object $\theta = [n;\mathbf{q}] \in \Theta_2$. If $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ is a cellular operator, then the corresponding map $\Theta_2[m;\mathbf{p}] \to \Theta_2[n;\mathbf{q}]$ will also be denoted by $[\alpha;\boldsymbol{\alpha}]$. Moreover, if $X \subset \Theta_2[n;\mathbf{q}]$ is a cellular subset and there exists a (necessarily unique) factorisation

$$\begin{tikzcd} & X \arrow[d, hook, "\subset"] \\ \Theta_2[m;\mathbf{p}] \arrow[ur, dashed] \arrow[r, "{[\alpha;\boldsymbol{\alpha}]}"'] & \Theta_2[n;\mathbf{q}] \end{tikzcd}$$

then we abuse the notation and write $[\alpha;\boldsymbol{\alpha}]$ for the dashed map too. Note that the domain of $[\alpha;\boldsymbol{\alpha}]$ is still the representable one and so $[\alpha;\boldsymbol{\alpha}]$ *always corresponds to a single cell in its codomain.* The convention introduced in Section 2.1.2 extends to this context in the sense that any map in $\widehat{\Theta_2}$ denoted by δ (with some decoration) will always have as codomain *some cellular subset of* $\Theta_2[n;\mathbf{q}]$ unless indicated otherwise.

There is a functor $\Theta_2 \to \Delta$ given by sending $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ to $\alpha : [m] \to [n]$. We will regard $\widehat{\Delta}$ as a full subcategory of $\widehat{\Theta_2}$ via the embedding $\widehat{\Delta} \to \widehat{\Theta_2}$ induced by this functor. Hence the square

$$\begin{tikzcd} \underline{\mathrm{Cat}} \arrow[r] \arrow[d, "N"'] & 2\text{-}\underline{\mathrm{Cat}} \arrow[d, "N"] \\ \widehat{\Delta} \arrow[r, hook, "\subset"] & \widehat{\Theta_2} \end{tikzcd}$$

commutes up to isomorphism, where the upper horizontal map sends each category to the obvious locally discrete 2-category, and the vertical maps are the nerve functors induced by the inclusions $\Delta \hookrightarrow \underline{\mathrm{Cat}}$ and $\Theta_2 \hookrightarrow 2\text{-}\underline{\mathrm{Cat}}$.

2.2 Oury's anodyne extensions

Most content of this section is taken from Oury's PhD thesis [Our10].

2.2.1 The category $\widehat{\Delta} \wr \widehat{\Delta}$

In this subsection, we will describe Oury's *generalised wreath product* $\widehat{\Delta} \wr \widehat{\Delta}$ which should be thought of as a category of presentations of certain cellular sets in terms of their "horizontal" and "vertical" components. The *box product* $\square : \widehat{\Delta} \wr \widehat{\Delta} \to \widehat{\Delta \wr \Delta} = \widehat{\Theta_2}$ defined in Section 2.2.2 then realises such presentations into actual cellular sets. This functor should be thought of as analogous to the box product functor $\square : \widehat{\Delta} \times \widehat{\Delta} \to \widehat{\Delta \times \Delta}$ for bisimplicial sets, hence the name. In Section 2.2.4 we will use these tools to turn simplicial inner horns into cellular ones.

We start by going back to the representable cellular sets and "decomposing" them into simplicial sets, to motivate the definition of $\widehat{\Delta} \wr \widehat{\Delta}$.

Since the "length" of $[n;\mathbf{q}]$ is n, the horizontal component of $\Theta_2[n;\mathbf{q}]$ should be $\Delta[n]$. The description of the hom-categories of $[n;\mathbf{q}]$ tells us that the vertical component of $\Theta_2[n;\mathbf{q}]$ should assign the product $\Delta[q_{k+1}] \times \cdots \times \Delta[q_\ell]$ to each 1-simplex $\{k,\ell\}$ in $\Delta[n]$. The resulting functor $\chi_1 : \Delta[n]_1 \to \widehat{\Delta}$ (where $\Delta[n]_m$ is the set of m-simplices in $\Delta[n]$ regarded as a discrete category) then encodes the $\underline{\mathrm{Cat}}$-enriched graph structure of $[n;\mathbf{q}]$. The (free) horizontal composition is witnessed by the canonical isomorphism

$$\chi_1(\alpha \cdot \{0,1\}) \times \chi_1(\alpha \cdot \{1,2\}) \to \chi_1(\alpha \cdot \{0,2\})$$

for each 2-simplex $\alpha : [2] \to [n]$. These isomorphisms can be organised into a single natural isomorphism

$$\begin{tikzcd} \Delta[n]_2 \arrow[r, "\chi_2"] \arrow[d, "-\cdot\{0,2\}"'] \arrow[dr, phantom, "\cong"] & \widehat{\Delta} \times \widehat{\Delta} \arrow[d, "\times"] \\ \Delta[n]_1 \arrow[r, "\chi_1"'] & \widehat{\Delta} \end{tikzcd}$$

where the right vertical map is the binary product functor and χ_2 is the unique functor induced by the universal property as in:

$$\begin{tikzcd} \Delta[n]_1 \arrow[r, "\chi_1"] & \widehat{\Delta} \\ \Delta[n]_2 \arrow[r, dashed, "\chi_2"] \arrow[u, "-\cdot\{0,1\}"] \arrow[d, "-\cdot\{1,2\}"'] & \widehat{\Delta} \times \widehat{\Delta} \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ \Delta[n]_1 \arrow[r, "\chi_1"'] & \widehat{\Delta} \end{tikzcd}$$

These three squares can be seen as part of a pseudo-natural transformation

$$\begin{tikzcd} & \underline{\mathrm{Set}} \arrow[dr] \arrow[d, phantom, "\Downarrow\chi"] & \\ \Delta^{\mathrm{op}} \arrow[ur, "{\Delta[n]}"] \arrow[rr, "\widehat{\Delta}^{(-)}"'] & & \underline{\mathrm{CAT}} \end{tikzcd}$$

into the pseudo-functor $\widehat{\Delta}^{(-)}$ which we now describe. (Here $\underline{\mathrm{CAT}}$ must be large enough to contain $\widehat{\Delta}$ and its powers as objects.)

The object part of $\widehat{\Delta}^{(-)}$ assigns to each $[m] \in \Delta$ the product $\widehat{\Delta}^m = \widehat{\Delta} \times \cdots \times \widehat{\Delta}$ of m copies of the category $\widehat{\Delta}$. If $\beta : [k] \to [m]$ is a simplicial operator, then its image $\widehat{\Delta}^\beta : \widehat{\Delta}^m \to \widehat{\Delta}^k$ acts by

$$\{S_j\}_{1\le j\le m} \mapsto \left\{ \prod_{\beta(i-1)<j\le\beta(i)} S_j \right\}_{1\le i\le k}.$$

Since $\widehat{\Delta}^\gamma \widehat{\Delta}^\beta$ is only naturally isomorphic (via suitably coherent isomorphisms) and not equal to $\widehat{\Delta}^{\beta\gamma}$, we obtain a pseudo-functor $\Delta^{\mathrm{op}} \to \underline{\mathrm{CAT}}$ instead of a strict (2-)functor.

We define the $[m]$-component $\chi_m : \Delta[n]_m \to \widehat{\Delta}^m$ of the pseudo-natural transformation χ by

$$\chi_m(\alpha) = \left\{ \prod_{\alpha(i-1)<j\le\alpha(i)} \Delta[q_j] \right\}_{1\le i\le m}$$

for each $\alpha : [m] \to [n]$. To complete the description of χ, we need to specify an appropriately coherent family of natural isomorphisms

$$\begin{tikzcd} \Delta[n]_m \arrow[r, "\chi_m"] \arrow[d, "-\cdot\beta"'] \arrow[dr, phantom, "\cong"] & \widehat{\Delta}^m \arrow[d, "\widehat{\Delta}^\beta"] \\ \Delta[n]_k \arrow[r, "\chi_k"'] & \widehat{\Delta}^k \end{tikzcd}$$

indexed by the simplicial operators $\beta : [k] \to [m]$. But this amounts to giving an isomorphism

$$\prod_{0<j\le m} \chi_1(\alpha \cdot \{j-1, j\}) \cong \chi_1(\alpha \cdot \{0, m\})$$

for each $\alpha \in \Delta[n]_m$ compatible with the simplicial structure of $\Delta[n]$, and one can check that the canonical isomorphisms indeed form such a compatible family. As we mentioned above for the case $m = 2$, this isomorphism can be thought of as witnessing the m-ary horizontal composition. The invertibility of this map says that $[n; \mathbf{q}]$ is horizontally free, and the compatibility with the simplicial structure says that the horizontal composition is coherent in the sense that it is associative, the witnesses to associativity satisfy the pentagon law, and so on.

This "decomposition" provides a motivation for thinking of the objects in the following category as presentations of certain cellular sets.

Definition 2.2.1. For any simplicial set W, let $\left(\widehat{\Delta} \wr \widehat{\Delta}\right)_W$ denote the category of pseudo-natural transformations

$$\begin{tikzcd}
 & \underline{\mathrm{Set}} \arrow[dr] \arrow[d, phantom, "\Downarrow \chi"] & \\
\Delta^{\mathrm{op}} \arrow[ur, "W"] \arrow[rr, "\widehat{\Delta}^{(-)}"'] & & \underline{\mathrm{CAT}}
\end{tikzcd}$$

and modifications between them.

A morphism $\chi \to \chi'$ in the category $\left(\widehat{\Delta} \wr \widehat{\Delta}\right)_W$ essentially amounts to a family of simplicial maps $\chi_1(\alpha) \to \chi'_1(\alpha)$ indexed by $\alpha \in W_1$ that is compatible with the pseudo-naturality isomorphisms in an appropriate sense. In particular, we have the following proposition.

Proposition 2.2.2. *There is an equivalence of categories*

$$\left(\widehat{\Delta} \wr \widehat{\Delta}\right)_{\Delta[n]} \simeq \widehat{\Delta}^n$$

whose object part is given by evaluating each pseudo-natural transformation at the unique non-degenerate n-simplex in $\Delta[n]$.

Proof. This is an instance of the bicategorical Yoneda lemma [Str80, §1.9]. □

If $f : W \to W'$ is a map in $\widehat{\Delta}$, then there is a functor $f^* : \left(\widehat{\Delta} \wr \widehat{\Delta}\right)_{W'} \to \left(\widehat{\Delta} \wr \widehat{\Delta}\right)_W$ given by composing with f, *i.e.* $f^*(\chi)$ is the pseudo-natural transformation:

$$\begin{tikzcd}
 & \underline{\mathrm{Set}} \arrow[dr] \arrow[d, phantom, "\Downarrow \chi"] & \\
\Delta^{\mathrm{op}} \arrow[ur, bend left, "W"] \arrow[ur, bend right, "W'"'] \arrow[ur, phantom, "\Downarrow f"] \arrow[rr, "\widehat{\Delta}^{(-)}"'] & & \underline{\mathrm{CAT}}
\end{tikzcd}$$

Moreover, sending each f to f^* defines a (strict) functor $\left(\widehat{\Delta} \wr \widehat{\Delta}\right)_{(-)} : \widehat{\Delta}^{\mathrm{op}} \to \underline{\mathrm{CAT}}$.

Definition 2.2.3. The *generalised wreath product* $\widehat{\Delta} \wr \widehat{\Delta}$ is the total category of the Grothendieck construction of the functor $\left(\widehat{\Delta} \wr \widehat{\Delta}\right)_{(-)}$.

More explicitly, the category $\widehat{\Delta} \wr \widehat{\Delta}$ has as objects the pairs (W, χ) as above and as morphisms pairs $(f, \omega) : (W, \chi) \to (W', \chi')$ where $f : W \to W'$ is a morphism of simplicial sets and $\omega : \chi \to f^*(\chi')$ is a modification between the pseudo-natural transformations.

Remark. For any monoidal category $\mathscr{V}$, one can construct a similar category $\widehat{\Delta} \wr \mathscr{V}$ by replacing the pseudo-functor $\widehat{\Delta}^{(-)}$ with $\mathscr{V}^{(-)}$ (whose morphism part is defined using the monoidal structure). In fact, Oury originally described $\widehat{\Delta} \wr \widehat{\Delta}$ as a particular instance of this general construction.

2.2.2 The functors $\square$ and $\square_n$

We start by making precise the "decomposition" of representable cellular sets discussed in the previous subsection.

Proposition 2.2.4 ([Our10, Observation 3.53 and Lemma 3.60])**.** *Sending each* $[n;\mathbf{q}]$ *to the image of*

$$\big(\Delta[q_1],\dots,\Delta[q_n]\big) \in \widehat{\Delta}^n$$

under the equivalence $\widehat{\Delta}^n \simeq \big(\widehat{\Delta}\wr\widehat{\Delta}\big)_{\Delta[n]}$ *of Proposition 2.2.2 defines the object part of a full embedding* $\Theta_2 \hookrightarrow \widehat{\Delta}\wr\widehat{\Delta}$.

Definition 2.2.5. The *box product* $\square : \widehat{\Delta}\wr\widehat{\Delta} \to \widehat{\Theta_2}$ is the nerve functor induced by this embedding.

Note that the embedding being full is equivalent to the composite

$$\Theta_2 \lhook\joinrel\longrightarrow \widehat{\Delta}\wr\widehat{\Delta} \xrightarrow{\ \square\ } \widehat{\Theta_2}$$

being naturally isomorphic to the Yoneda embedding.

Remark. We will briefly describe how Oury's box product functor is related to Rezk's *intertwining functor* [Rez10, §4.4]

$$V : \Delta\wr\big[\mathscr{C}^{\mathrm{op}},\widehat{\Delta}\big] \to \big[(\Delta\wr\mathscr{C})^{\mathrm{op}},\widehat{\Delta}\big].$$

(If the reader is not familiar with Rezk's work on Θ_n-spaces, they may safely ignore this remark.) One can check that restricting the intertwining functor to the obvious "discrete" objects yields

$$V : \Delta\wr\big[\mathscr{C}^{\mathrm{op}},\underline{\mathrm{Set}}\big] \to \big[(\Delta\wr\mathscr{C})^{\mathrm{op}},\underline{\mathrm{Set}}\big]$$

and so in particular we obtain $V : \Delta\wr\widehat{\Delta} \to \widehat{\Theta_2}$ for $\mathscr{C} = \Delta$. The domain of this functor is equivalent to the full subcategory of $\widehat{\Delta}\wr\widehat{\Delta}$ spanned by the objects of the form $(\Delta[n],\chi)$, and $\Delta\wr\widehat{\Delta} \lhook\joinrel\longrightarrow \widehat{\Delta}\wr\widehat{\Delta} \xrightarrow{\ \square\ } \widehat{\Theta_2}$ is naturally isomorphic to V.

Given any cartesian fibration $P : \mathscr{E} \to \mathscr{B}$ and $B \in \mathscr{B}$, let $\mathscr{B}_{/B}$ and $\mathscr{E}_B$ denote the slice and the fibre over B respectively. Then there is a functor

$$H : \mathscr{B}_{/B} \times \mathscr{E}_B \to \mathscr{E}$$

whose object part is given by sending each pair (f,E) to the domain f^*E of a cartesian lift $\tilde{f} : f^*E \to E$ of f. For any map $g : A_1 \to A_2$ over B and any map $e : E_1 \to E_2$ in $\mathscr{E}_B$, we can factor $e \circ \tilde{f}_1$ uniquely through the cartesian lift $\tilde{f}_2$ as in

$$\boxed{\begin{tikzcd} f_1^*E_1 \arrow[rr, "\tilde{f}_1"] \arrow[dr, dashed] & & E_1 \arrow[d, "e"] \\ & f_2^*E_2 \arrow[r, "\tilde{f}_2"'] & E_2 \end{tikzcd}} \quad \overset{P}{\longmapsto} \quad \boxed{\begin{tikzcd} A_1 \arrow[rr, "f_1"] \arrow[dr, "g"'] & & B \\ & A_2 \arrow[ur, "f_2"'] & \end{tikzcd}}$$

and this defines the morphism part of H.

Definition 2.2.6. Let $\square_n$ denote the composite functor

$$\square_n :\ \widehat{\Delta}_{/\Delta[n]} \times \underbrace{\widehat{\Delta}\times\cdots\times\widehat{\Delta}}_{n\text{ times}} \longrightarrow \widehat{\Delta}_{/\Delta[n]} \times \big(\widehat{\Delta}\wr\widehat{\Delta}\big)_{\Delta[n]} \xrightarrow{\ H\ } \widehat{\Delta}\wr\widehat{\Delta} \xrightarrow{\ \square\ } \widehat{\Theta_2}$$

where the first map is induced by the equivalence of Proposition 2.2.2 and the second map is an instance of the above construction.

Note that we have $\square_n\big(\mathrm{id}_{\Delta[n]};\Delta[q_1],\dots,\Delta[q_n]\big) \cong \Theta_2[n;\mathbf{q}]$.

Proposition 2.2.7 ([Our10, Lemmas 3.74 and 3.77])**.** *The functor $\square_n$ preserves:*

- *small colimits in the first variable; and*
- *small connected colimits in each of the other n variables.*

Definition 2.2.8. If $f : X \to Y$ is a map in $\widehat{\Delta}$, then we will write

$$[\mathrm{id}; f] : \Theta_2[1; X] \to \Theta_2[1; Y]$$

for its image under the functor $\square_1(\mathrm{id}_{\Delta[1]}; -) : \widehat{\Delta} \to \widehat{\Theta_2}$.

This notation is motivated by the fact that $\square_1(\mathrm{id}_{\Delta[1]}; -)$ extends the functor $\Delta \to \widehat{\Theta_2}$ given by sending $\alpha : [m] \to [n]$ to $[\mathrm{id}; \alpha] : \Theta_2[1; m] \to \Theta_2[1; n]$. It takes a simplicial set X to its "suspension", *i.e.* the nerve of the following simplicially enriched category:

$$\begin{tikzcd} 0 \arrow[loop left, "{\Delta[0]}"] \arrow[r, bend left, "X"] & 1 \arrow[l, bend left, "\varnothing"] \arrow[loop right, "{\Delta[0]}"] \end{tikzcd}$$

2.2.3 Leibniz construction

We describe the (*n-ary*) *Leibniz construction*. Suppose $F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \to \mathscr{D}$ is a functor and $\mathscr{D}$ has finite colimits. Then the (*n-ary*) *Leibniz construction*

$$\hat{F} : \mathscr{C}_1^{\mathbb{2}} \times \cdots \times \mathscr{C}_n^{\mathbb{2}} \to \mathscr{D}^{\mathbb{2}}$$

of F, where $\mathbb{2} = \{0 \to 1\}$ is the "walking arrow" category, is defined as follows. Let $f_i : X_i^0 \to X_i^1$ be an object in $\mathscr{C}_i^{\mathbb{2}}$ for each i. Then the assignment $(\epsilon_1,\dots,\epsilon_n) \mapsto F(X_1^{\epsilon_1},\dots,X_n^{\epsilon_n})$ defines a functor $G : \mathbb{2}^n \to \mathscr{D}$. Denote by I the inclusion of the full subcategory of $\mathbb{2}^n$ spanned by all non-terminal objects. Then G defines a cone under the diagram GI, so we obtain an induced morphism $\operatorname{colim} GI \to F(X_1^1,\dots,X_n^1)$. Sending $(f_1,\dots,f_n)$ to this morphism defines the object part of $\hat{F}$, and the morphism part is defined in the obvious way by the universal property.

Lemma 2.2.9. *Let $F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \to \mathscr{D}$ be a functor into a presheaf category $\mathscr{D}$. Fix $f_i : X_i^0 \to X_i^1$ in each $\mathscr{C}_i$ and suppose G (as above) sends each square of the form*

$$\begin{tikzcd} (1,\dots,1,\overset{i\text{-th}}{\overset{\downarrow}{0}},1,\dots,1,\overset{j\text{-th}}{\overset{\downarrow}{0}},1,\dots,1) \arrow[r] \arrow[d] & (1,\dots,1,\overset{i\text{-th}}{\overset{\downarrow}{0}},1,\dots,1) \arrow[d] \\ (1,\dots,1,\underset{j\text{-th}}{\underset{\uparrow}{0}},1,\dots,1) \arrow[r] & (1,\dots,1) \end{tikzcd} \tag{2.1}$$

to a pullback square of monomorphisms. Then $\hat{F}(f_1,\dots,f_n)$ is a monomorphism.

Proof. This is straightforward to check when $\mathscr{D} = \underline{\mathrm{Set}}$, and the general result follows from this special instance since limits and colimits in presheaf categories are computed pointwise. $\square$

Definition 2.2.10. For any set $\mathcal{S}$ of morphisms in a category with pushouts and transfinite compositions, let $\mathrm{cell}(\mathcal{S})$ denote the closure of $\mathcal{S}$ under transfinite composition and taking pushouts along arbitrary maps.

Lemma 2.2.11. *Suppose that a functor $F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \to \mathscr{D}$ preserves pushouts and transfinite compositions in each variable. Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be collections of morphisms in $\mathscr{C}_1, \ldots, \mathscr{C}_n$ respectively. Then*

$$\hat{F}\big(\mathrm{cell}(\mathcal{S}_1), \ldots, \mathrm{cell}(\mathcal{S}_n)\big) \subset \mathrm{cell}\big(\hat{F}(\mathcal{S}_1, \ldots, \mathcal{S}_n)\big).$$

Proof. A proof can be found in [Our10, Corollary 3.11]. The case $n = 2$ is also proved in [RV14, Proposition 5.12]. □

2.2.4 Oury's anodyne extensions

Joyal's model structure for quasi-categories on $\widehat{\Delta}$ can be characterised using:

- the *boundary inclusions* $\partial\Delta[n] \hookrightarrow \Delta[n]$;
- the (*inner*) *horn inclusions* $\Lambda^k[n] \hookrightarrow \Delta[n]$; and
- the *equivalence extension* $e : \Delta[0] \hookrightarrow J$ which is the nerve of the inclusion $\{\lozenge\} \hookrightarrow \{\lozenge \cong \blacklozenge\}$ into the chaotic category on two objects.

Oury constructs the Θ_2-version of those morphisms using the *Leibniz box product* $\hat{\Box}_n$ as follows.

Definition 2.2.12. The *boundary inclusion* $\partial\Theta_2[n; \mathbf{q}] \hookrightarrow \Theta_2[n; \mathbf{q}]$ is defined by the $(n+1)$-ary Leibniz construction

$$\hat{\Box}_n \left(\begin{array}{c} \partial\Delta[n] \\ \big\downarrow \\ \Delta[n] \end{array} ; \begin{array}{c} \partial\Delta[q_1] \\ \big\downarrow \\ \Delta[q_1] \end{array}, \ldots, \begin{array}{c} \partial\Delta[q_n] \\ \big\downarrow \\ \Delta[q_n] \end{array} \right)$$

where the first argument $\partial\Delta[n] \hookrightarrow \Delta[n]$ is regarded as a map over $\Delta[n]$ in the obvious way.

As its name suggests, this map is the "usual" boundary inclusion.

Proposition 2.2.13 ([Our10, Observation 3.84])**.** *The map $\partial\Theta_2[n; \mathbf{q}] \hookrightarrow \Theta_2[n; \mathbf{q}]$ is (isomorphic to) the inclusion of the cellular subset consisting precisely of those maps into $[n; \mathbf{q}]$ that factor through objects of lower dimension.*

Proposition 2.2.14. *The cellular subset $\partial\Theta_2[n; \mathbf{q}] \subset \Theta_2[n; \mathbf{q}]$ is generated by the hyperfaces of $\Theta_2[n; \mathbf{q}]$.*

Proof. This follows from Propositions 2.1.9 and 2.2.13. □

For example, when $[n; \mathbf{q}] = [2; 0, 2]$ (see Table 2.1):

- $\Box_2\big(\partial\Delta[2]; \Delta[0], \Delta[2]\big) \subset \Theta_2[2; 0, 2]$ is generated by δ_h^0, $\delta_h^{1;\langle !, \mathrm{id}\rangle}$ and δ_h^2;
- $\Box_2\big(\Delta[2]; \partial\Delta[0], \Delta[2]\big)$ is generated by δ_h^0 and $[\{0\}]$; and

- $\square_2\big(\Delta[2]; \Delta[0], \partial\Delta[2]\big)$ is generated by $\delta_v^{2;0}$, $\delta_v^{2;1}$ and $\delta_v^{2;2}$.

It can be seen from the defining colimit diagram that $\partial\Theta_2[2;0,2]$ is the union of these three cellular subsets. Thus $\partial\Theta_2[2;0,2]$ is indeed generated by the hyperfaces of $\Theta_2[2;0,2]$.

Definition 2.2.15. We write $\mathcal{I}$ for the set of boundary inclusions, *i.e.*

$$\mathcal{I} \overset{\text{def}}{=} \big\{\partial\Theta_2[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}] : [n;\mathbf{q}] \in \Theta_2\big\}.$$

The following proposition follows from Theorem 2.1.13.

Proposition 2.2.16. *The class* $\operatorname{cell}(\mathcal{I})$ *consists precisely of the monomorphisms in* $\widehat{\Theta_2}$.

Definition 2.2.17. The *k-th horizontal horn inclusion* $\Lambda_h^k[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$, where $0 \le k \le n$, is

$$\hat{\square}_n\left(\begin{array}{c}\Lambda^k[n]\\ \big\downarrow\\ \Delta[n]\end{array};\ \begin{array}{c}\partial\Delta[q_1]\\ \big\downarrow\\ \Delta[q_1]\end{array},\ldots,\begin{array}{c}\partial\Delta[q_n]\\ \big\downarrow\\ \Delta[q_n]\end{array}\right).$$

It is called *inner* if $1 \le k \le n-1$.

Proposition 2.2.18. *The map* $\Lambda_h^k[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ *is (isomorphic to) the inclusion of the cellular subset generated by all hyperfaces except for the k-th horizontal ones.*

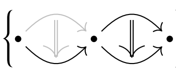
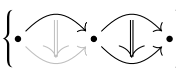
Proof. It follows from Lemma 2.2.11 and Proposition 2.2.13 that this map is a monomorphism. Thus it suffices to check that it has the correct image, which can be done by considering the defining colimit diagram for $\Lambda_h^k[n;\mathbf{q}]$. $\square$

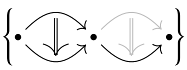
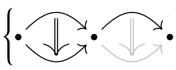
For example, when $[n;\mathbf{q}] = [2;0,2]$ and $k = 1$:

- $\square_2\big(\Lambda^1[2]; \Delta[0], \Delta[2]\big)$ is generated by δ_h^0 and δ_h^2;
- $\square_2\big(\Delta[2]; \partial\Delta[0], \Delta[2]\big)$ is generated by δ_h^0 and $[\{0\}]$; and
- $\square_2\big(\Delta[2]; \Delta[0], \partial\Delta[2]\big)$ is generated by $\delta_v^{2;0}$, $\delta_v^{2;1}$ and $\delta_v^{2;2}$.

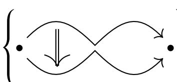
Thus their union $\Lambda_h^1[2;0,2]$ is indeed generated by all hyperfaces except $\delta_h^{1;\langle !,\mathrm{id}\rangle}$.

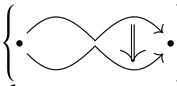
Remark. The faces $[\alpha;\boldsymbol{\alpha}] : \Theta_2[m;\mathbf{p}] \to \Theta_2[n;\mathbf{q}]$ not contained in the horizontal horn $\Lambda_h^k[n;\mathbf{q}]$ are precisely the k-th horizontal ones. In particular, $\Lambda_h^k[n;\mathbf{q}]$ may be missing faces of $\Theta_2[n;\mathbf{q}]$ that have codimension greater than 1. For example, one can check that $\Lambda_h^1[2;1,1]$ is generated by the vertical hyperfaces

$\delta_v^{1;0} = \Big\{$  $\Big\}$, $\delta_v^{1;1} = \Big\{$  $\Big\}$,

$\delta_v^{2;0} = \Big\{$  $\Big\}$ and $\delta_v^{2;1} = \Big\{$  $\Big\}$

and so it does not contain the face

$[\delta^1; \mathrm{id}, \mathrm{id}] = \Big\{$  $\Big\}$

of codimension 2. (The last face may equally well be depicted as $\Big\{$$\Big\}$; the position of the double arrow has no significance.) This differs from the more commonly found definition of a horn (*e.g.* [Ber02, Wat13]) as "boundary with one hyperface removed". In Section 3.2, we show that for our purposes such alternative horns may be used in place of Oury's ones.

Definition 2.2.19. The $(k;i)$*-th vertical horn inclusion* $\Lambda_v^{k;i}[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$, where $0 \leq k \leq n$ satisfies $q_k \geq 1$ and $0 \leq i \leq q_k$, is

$$\hat{\square}_n\left(\begin{array}{c} \partial\Delta[n] \\ \big\downarrow \\ \Delta[n] \end{array};\ \begin{array}{c} \partial\Delta[q_1] \\ \big\downarrow \\ \Delta[q_1] \end{array},\ldots,\begin{array}{c} \partial\Delta[q_{k-1}] \\ \big\downarrow \\ \Delta[q_{k-1}] \end{array},\begin{array}{c} \Lambda^i[q_k] \\ \big\downarrow \\ \Delta[q_k] \end{array},\begin{array}{c} \partial\Delta[q_{k+1}] \\ \big\downarrow \\ \Delta[q_{k+1}] \end{array},\ldots,\begin{array}{c} \partial\Delta[q_n] \\ \big\downarrow \\ \Delta[q_n] \end{array}\right).$$

It is called *inner* if $1 \leq i \leq q_k - 1$.

The following proposition can be proved similarly to Proposition 2.2.18.

Proposition 2.2.20. *The map* $\Lambda_v^{k;i}[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ *is (isomorphic to) the inclusion of the cellular subset generated by all hyperfaces except for the* $(k;i)$*-th vertical ones.*

For example, when $[n;\mathbf{q}] = [2;0,2]$, $k = 2$ and $i = 1$:

- $\square_2\big(\partial\Delta[2];\Delta[0],\Delta[2]\big)$ is generated by δ_h^0, $\delta_h^{1;\langle !,\mathrm{id}\rangle}$ and δ_h^2;
- $\square_2\big(\Delta[2];\partial\Delta[0],\Delta[2]\big)$ is generated by δ_h^0 and $[\{0\}]$; and
- $\square_2\big(\Delta[2];\Delta[0],\Lambda^1[2]\big)$ is generated by $\delta_v^{2;0}$ and $\delta_v^{2;2}$.

Thus their union $\Lambda_v^{2;1}[2;0,2]$ is indeed generated by all hyperfaces except $\delta_v^{2;1}$.

Definition 2.2.21. A *horizontal equivalence extension* is a map of the form

$$\big(\Theta_2[0] \xhookrightarrow{e} J\big)\hat{\times}\big(\partial\Theta_2[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]\big)$$

where $\hat{\times}$ is the Leibniz construction of the usual binary product functor. Here the simplicial set J is regarded as a cellular set via the inclusion $\widehat{\Delta} \hookrightarrow \widehat{\Theta_2}$ described in Section 2.1.4.

Definition 2.2.22. If $[n;\mathbf{q}] \in \Theta_2$ has $q_k = 0$ for some $1 \leq k \leq n$ then we denote by $\Psi^k[n;\mathbf{q}] \hookrightarrow \Phi^k[n;\mathbf{q}]$ the *vertical equivalence extension*

$$\hat{\square}_n\left(\begin{array}{c} \partial\Delta[n] \\ \big\downarrow \\ \Delta[n] \end{array};\ \begin{array}{c} \partial\Delta[q_1] \\ \big\downarrow \\ \Delta[q_1] \end{array},\ldots,\begin{array}{c} \partial\Delta[q_{k-1}] \\ \big\downarrow \\ \Delta[q_{k-1}] \end{array},\begin{array}{c} \Delta[0] \\ \big\downarrow{\scriptstyle e} \\ J \end{array},\begin{array}{c} \partial\Delta[q_{k+1}] \\ \big\downarrow \\ \Delta[q_{k+1}] \end{array},\ldots,\begin{array}{c} \partial\Delta[q_n] \\ \big\downarrow \\ \Delta[q_n] \end{array}\right).$$

Definition 2.2.23. Let $\mathcal{H}_h$, $\mathcal{H}_v$, $\mathcal{E}_h$, and $\mathcal{E}_v$ denote the sets of inner horizontal horn inclusions, inner vertical horn inclusions, horizontal equivalence extensions, and vertical equivalence extensions respectively. We write $\mathcal{J}_O$ for the union

$$\mathcal{J}_O \overset{\text{def}}{=} \mathcal{H}_h \cup \mathcal{H}_v \cup \mathcal{E}_h \cup \mathcal{E}_v.$$

By an *O-anodyne extension* we mean an element f of $\mathrm{cell}(\mathcal{J}_O)$, which is *elementary* if $f \in \mathcal{J}_O$.

One of Oury's main results is the following.

Theorem 2.2.24 ([Our10, Corollary 3.11 and Theorem 4.22])**.** *The O-anodyne extensions are stable under taking Leibniz products with arbitrary monomorphisms.*

2.3 Ara's model structure for 2-quasi-categories

By definition, a *2-quasi-category* is a fibrant object in $\widehat{\Theta_2}$ with respect to Ara's model structure. We review this model structure in this section.

2.3.1 Model categories

We recall the definition of a model category and related notions in this subsection.

Definition 2.3.1. Let ℓ, r be morphisms in a category $\mathscr{C}$. We say ℓ has the *left lifting property* with respect to r, or equivalently r has the *right lifting property* with respect to ℓ, if any commutative square of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\ell"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

admits a diagonal lift as indicated, making the two triangles commutative.

Definition 2.3.2. A *weak factorisation system* $(\mathcal{L}, \mathcal{R})$ on a category $\mathscr{C}$ consists of two classes of morphisms $\mathcal{L}, \mathcal{R}$ such that:

- any morphism f in $\mathscr{C}$ admits a factorisation of the form

$$\begin{tikzcd} & \cdot \arrow[dr, "r"] & \\ \cdot \arrow[ur, "\ell"] \arrow[rr, "f"'] & & \cdot \end{tikzcd}$$

  with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$;

- $\mathcal{L}$ is precisely the class of morphisms that have the left lifting property with respect to every member of $\mathcal{R}$; and

- $\mathcal{R}$ is precisely the class of morphisms that have the right lifting property with respect to every member of $\mathcal{L}$.

Remark. Note that the second clause in this definition implies $\mathrm{cell}(\mathcal{L}) = \mathcal{L}$.

Definition 2.3.3. A *model category* $\mathscr{M}$ is a category with finite limits and finite colimits equipped with a *model structure*, that is, three classes of morphisms $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ such that:

- (*2-out-of-3 property*) for any composable pair f, g in $\mathscr{M}$, if any two of f, g and gf are in $\mathcal{W}$ then so is the third; and

- $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorisation systems on $\mathscr{M}$.

The members of $\mathcal{C}$, $\mathcal{C} \cap \mathcal{W}$, $\mathcal{F}$, $\mathcal{F} \cap \mathcal{W}$ and $\mathcal{W}$ are called *cofibrations*, *trivial cofibrations*, *fibrations*, *trivial fibrations* and *weak equivalences* respectively. An object X in $\mathscr{M}$ is called *cofibrant* if the unique map from the initial object to X is a cofibration. Dually X is *fibrant* if the unique map from X to the terminal object is a fibration.

Definition 2.3.4. Let $\mathscr{M}_1, \ldots, \mathscr{M}_n, \mathscr{N}$ be model categories. An n-ary functor

$$F : \mathscr{M}_1 \times \cdots \times \mathscr{M}_n \to \mathscr{N}$$

is said to be *left Quillen* if:

(1) for any $1 \leq k \leq n$ and for any choice of objects $X_i \in \mathscr{M}_i$ for $i \neq k$, the functor

$$F(X_1, \ldots, X_{k-1}, -, X_{k+1}, \ldots, X_n) : \mathscr{M}_k \to \mathscr{N}$$

admits a right adjoint; and

(2) the Leibniz construction $\hat{F}(f_1, \ldots, f_n)$ is a cofibration for any cofibrations $f_1, \ldots, f_n$, and it is moreover trivial if f_k is so for some $1 \leq k \leq n$.

Let $F : \mathscr{M}_1 \times \cdots \times \mathscr{M}_n \to \mathscr{N}$ be an n-ary functor satisfying (1) above, and fix $1 \leq k \leq n$. Then the right adjoint functors for all possible choices of $X_i \in \mathscr{M}_i$ for $i \neq k$ assemble into a single functor

$$R_k : \mathscr{M}_1^{\mathrm{op}} \times \cdots \times \mathscr{M}_{k-1}^{\mathrm{op}} \times \mathscr{M}_{k+1}^{\mathrm{op}} \times \cdots \times \mathscr{M}_n^{\mathrm{op}} \times \mathscr{N} \to \mathscr{M}_k.$$

In this situation, we write $\check{R}_k$ for the Leibniz construction applied to

$$R_k^{\mathrm{op}} : \mathscr{M}_1 \times \cdots \times \mathscr{M}_{k-1} \times \mathscr{M}_{k+1} \times \cdots \times \mathscr{M}_n \times \mathscr{N}^{\mathrm{op}} \to \mathscr{M}_k^{\mathrm{op}}$$

so that the codomain of $\check{R}_k(f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_n, g)$ is the limit of a cube-like-shaped diagram in $\mathscr{M}_k$.

Proposition 2.3.5. *Let F and R_k be as above. Let f_i be a morphism in $\mathscr{M}_i$ for $1 \leq i \leq n$, and let g be a morphism in $\mathscr{N}$. Then $\hat{F}(f_1, \ldots, f_n)$ has the left lifting property with respect to g if and only if f_k has the left lifting property with respect to $\check{R}_k(f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_n, g)$.*

Proof. Write $f_i : X_i^0 \to X_i^1$ and $g : Y^0 \to Y^1$ for the domains and codomains of these maps. Let $G : \mathbb{2}^{n+1} \to \underline{\mathrm{Set}}$ be the functor whose object part is given by

$$G(\epsilon_1, \ldots, \epsilon_n, \epsilon) = \mathscr{N}\left(F(X_1^{1-\epsilon_1}, \ldots, X_n^{1-\epsilon_n}), Y^{\epsilon}\right)$$

and whose morphism part is the obvious one. Denote by I the inclusion of the full subcategory of $\mathbb{2}^{n+1}$ spanned by all non-initial objects. Then G defines a cone over the diagram GI, so we obtain an induced morphism

$$\mathscr{N}\left(F(X_1^1, \ldots, X_n^1), Y^0\right) \to \lim GI.$$

One can check that $\hat{F}(f_1, \ldots, f_n)$ has the left lifting property with respect to g if and only if this induced morphism is a surjection.

By the definition of R_k, the functor G is naturally isomorphic to G' given by

$$G'(\epsilon_1, \ldots, \epsilon_n, \epsilon) = \mathscr{M}_k\left(X_k^{1-\epsilon_k}, R_k(X_1^{1-\epsilon_1}, \ldots, X_{k-1}^{1-\epsilon_{k-1}}, X_{k+1}^{1-\epsilon_{k+1}}, \ldots, X_n^{1-\epsilon_n}, Y^{\epsilon})\right)$$

One can check that f_k has the left lifting property with respect to $\check{R}_k(f_1, \ldots, f_{k-1}, f_{k+1}, \ldots, f_n, g)$ if and only if the map

$$\mathscr{M}_k\left(X_k^1, R_k(X_1^1, \ldots, X_{k-1}^1, X_{k+1}^1, \ldots, X_n^1, Y^0\right) \to \lim G'I$$

induced by G' (regarded as a cone over $G'I$) is a surjection. The desired equivalence now follows. ◻

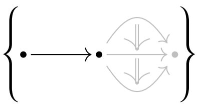
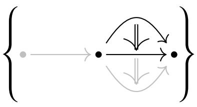
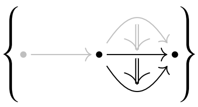
2.3.2 Vertebrae and spines

Here we introduce the notions of *vertebra* and of *spine*.

Definition 2.3.6. The only *vertebra* of $\Theta_2[0]$ is the identity map $\mathrm{id} : \Theta_2[0] \to \Theta_2[0]$. For $[n; \mathbf{q}] \in \Theta_2$ with $n \geq 1$:

- if $1 \leq k \leq n$ and $q_k = 0$, then
$$[\{k-1, k\}; \mathrm{id}] : \Theta_2[1;0] \to \Theta_2[n;\mathbf{q}]$$
is a *vertebra*; and
- if $1 \leq k \leq n$ and $q_k \geq 1$, then for each $1 \leq i \leq q_k$,
$$[\{k-1, k\}; \{i-1, i\}] : \Theta_2[1;1] \to \Theta_2[n;\mathbf{q}]$$
is a *vertebra*.

For example, $\Theta_2[2;0,2]$ has three vertebrae

, , and .

Definition 2.3.7. Let $\Xi[n;\mathbf{q}] \subset \Theta_2[n;\mathbf{q}]$ denote the cellular subset generated by the vertebrae of $\Theta_2[n;\mathbf{q}]$, and call it the *spine* of $\Theta_2[n;\mathbf{q}]$.

If $[n;\mathbf{q}]$ is $[0]$, $[1;0]$ or $[1;1]$, then $\Theta_2[n;\mathbf{q}]$ has a unique vertebra and $\Xi[n;\mathbf{q}] = \Theta_2[n;\mathbf{q}]$. We will call these cells *mono-vertebral*; otherwise $[n;\mathbf{q}]$ is *poly-vertebral*.

Note that if $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ is inert then it restricts to a map between the spines as in

$$\begin{tikzcd}
\Xi[m;\mathbf{p}] \arrow[r, hook, "\subset"] \arrow[d] & \Theta_2[m;\mathbf{p}] \arrow[d, "{[\alpha;\boldsymbol{\alpha}]}"] \\
\Xi[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Theta_2[n;\mathbf{q}]
\end{tikzcd}$$

and moreover this square is a pullback.

Observe that we left the map $\Xi[m;\mathbf{p}] \to \Xi[n;\mathbf{q}]$ unlabelled in the above square. In general, we adopt the following convention.

Convention. Whenever we draw a square of the form

$$\begin{tikzcd}
\cdot \arrow[r, hook, "\subset"] \arrow[d] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, hook, "\subset"] & \cdot
\end{tikzcd}$$

the unlabelled map is assumed to be the appropriate restriction of f. Typically the square is a gluing square (defined in Section 2.4.1) and f is a map of the form $\delta : \Theta_2[m;\mathbf{p}] \to X$ where $X \subset \Theta_2[n;\mathbf{q}]$, but this convention is not restricted to such situations.

2.3.3 Ara's model structure for 2-quasi-categories

In [Ara14], Ara defines a model structure on $\widehat{\Theta_n}$ whose fibrant objects (called *n-quasi-categories*) model (∞, n)-categories. Here we review Ara's characterisation of this model structure, but specialise to the case $n = 2$.

Recall that e denotes the nerve of the inclusion $\{\lozenge\} \hookrightarrow \{\lozenge \cong \blacklozenge\}$ so that its suspension $[\mathrm{id}; e] : \Theta_2[1; 0] \to \Theta_2[1; J]$ is (isomorphic to) the nerve of the 2-functor

$$\left\{\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \end{tikzcd}\right\} \hookrightarrow \left\{\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\cong"] & \bullet \end{tikzcd}\right\}$$

whose codomain is locally chaotic.

Definition 2.3.8. Let $\mathcal{J}_A$ denote the union of $\mathcal{E}_h$ and the closure of

$$\left\{\Xi[n; \mathbf{q}] \hookrightarrow \Theta_2[n; \mathbf{q}] : [n; \mathbf{q}] \in \Theta_2\right\} \cup \left\{[\mathrm{id}; e]\right\}$$

under taking Leibniz products

$$(-)\hat{\times}\big(\Theta_2[0] \amalg \Theta_2[0] \hookrightarrow J\big)$$

with the nerve of $\{\lozenge\} \amalg \{\blacklozenge\} \hookrightarrow \{\lozenge \cong \blacklozenge\}$. We will call elements of $\mathcal{J}_A$ *elementary A-anodyne extensions*.

Theorem 2.3.9 ([Ara14, §2.10 and §5.17])**.** *There is a model structure on $\widehat{\Theta_2}$ characterised by the following properties:*

- *the cofibrations are precisely the monomorphisms; and*
- *a map $f : X \to Y$ into a fibrant cellular set Y is a fibration if and only if it has the right lifting property with respect to all maps in $\mathcal{J}_A$.*

In particular, the fibrant objects, called *2-quasi-categories*, are precisely those objects with the right lifting property with respect to all elementary A-anodyne extensions.

This is the only model structure on $\widehat{\Theta_2}$ with which we are concerned in this thesis, and hence no confusion should arise in the following when we simply refer to "(trivial) cofibrations" without further qualification.

2.4 Proof strategy

Almost all of the proofs in this thesis use *gluing* in the following sense.

2.4.1 Gluing

Many of the results in this thesis are of the form

(i) the inclusion $\mathcal{J} \subset \mathrm{cell}(\mathcal{J}')$ holds for certain sets $\mathcal{J}$ and $\mathcal{J}'$ of maps in $\widehat{\Theta_2}$; or

(ii) a certain set $\mathcal{J}$ of monomorphisms (= cofibrations) in $\widehat{\Theta_2}$ is contained in the class of trivial cofibrations.

We prove the results of the first kind by directly expressing each map in $\mathcal{J}$ as a transfinite composite of pushouts of maps in $\mathrm{cell}(\mathcal{J}')$. For those of the second kind, we make use of the *right cancellation property*, *i.e.* we show that f and gf are trivial cofibrations and then deduce that the cofibration g must also be trivial. In each case, the proof reduces to checking the existence of certain *gluing squares*, as defined below.

Suppose we have a pullback square

$$\begin{tikzcd} W \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, hook, "\subset"] & Z \end{tikzcd}$$

in $\widehat{\Theta_2}$ such that $Z = f(X) \cup Y$, and f is injective on $f^{-1}(Z \setminus Y) = X \setminus W$. Then the square is also a pushout, and we will say Z is obtained from Y by *gluing X along W*. Note that if Y is generated by a set S of cells in Z, then W is generated by the pullbacks of $\Theta_2[n; \mathbf{q}] \xrightarrow{s} Z$ along f for all $s \in S$.

3

Inner horns for 2-quasi-categories

The set $\mathcal{J}_A$ appearing in Ara's characterisation of the model structure is complicated and difficult to deal with. In this chapter, we prove that (a subset of) Oury's anodyne extensions $\mathcal{J}_O$ may be used in place of $\mathcal{J}_A$ for characterising this model structure. This alternative characterisation will play a crucial role in Chapter 4.

3.1 O-anodyne extensions and Ara's model structure

Here we prove that elementary A-anodyne extensions are O-anodyne extensions, and also (elementary) O-anodyne extensions are trivial cofibrations.

3.1.1 Elementary A-anodyne extensions are O-anodyne extensions

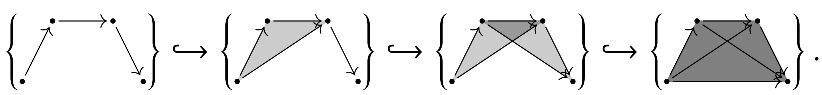
In this subsection, we prove the following lemma.

Lemma 3.1.1. *Every map in $\mathcal{J}_A$ is an O-anodyne extension.*

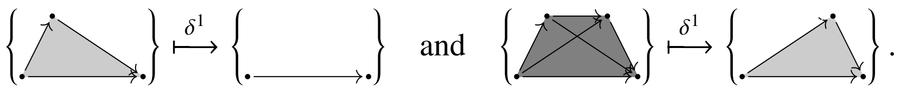
Proof. Since the O-anodyne extensions are closed under taking Leibniz products with arbitrary monomorphisms (Theorem 2.2.24), and $[\mathrm{id}; e] : \Theta_2[1;0] \to \Theta_2[1;J]$ is isomorphic to the elementary O-anodyne extension $\Psi^1[1;0] \hookrightarrow \Phi^1[1;0]$, it suffices to show that the spine inclusions $\Xi[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ (which are the remaining "generating" elements of $\mathcal{J}_A$) are O-anodyne extensions. This is done in Lemma 3.1.2 below. $\square$

Lemma 3.1.2. *The spine inclusion $\Xi[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ is in $\mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v)$ for any $[n;\mathbf{q}] \in \Theta_2$.*

The corresponding result for quasi-categories has been proved by Joyal [Joy, Proposition 2.13]. Our proof presented below is essentially Joyal's proof repeated twice, first in the vertical direction and then in the horizontal direction. In each step, we decompose the spine inclusion $\Xi[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ into three inclusions which, when $[n;\mathbf{q}] = [3;\mathbf{0}]$, look like



In general, the first two maps glue the outer faces along lower dimensional spine(-like) inclusions. The remaining non-degenerate cells are precisely those containing both of the "endpoints" (*i.e.* $0, q_k \in [q_k]$ in the vertical case and $0, n \in [n]$ in the horizontal case). We can group such cells into pairs $\{x, y\}$ so that the only difference between x and y is whether they contain 1 (meaning $1 \in [q_k]$ in the vertical case and $1 \in [n]$ in the horizontal case). Such a pair necessarily satisfies $y = x \cdot \delta^1$ (up to interchanging x and y), *e.g.*



Thus the last inclusion can be obtained by gluing the x's along Λ^1.

Definition 3.1.3. If S is any set of faces of $\Theta_2[n;\mathbf{q}]$, we will write $\Xi^S[n;\mathbf{q}] \subset \Theta_2[n;\mathbf{q}]$ for the cellular subset generated by $\Xi[n;\mathbf{q}]$ and S.

Proof of Lemma 3.1.2. Recall that $\Xi[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ for mono-vertebral $[n;\mathbf{q}]$ (*i.e.* for $[n;\mathbf{q}] = [0], [1;0]$ or $[1;1]$) is the identity and hence trivially O-anodyne. These serve as the base cases for our induction.

We first consider the case where $n = 1$. For any $q \geq 1$, let $\Xi^\dagger[1;q] = \Xi^{\{\delta_v^{1;q}\}}[1;q]$ and let $\Xi^\ddagger[1;q] = \Xi^{\{\delta_v^{1;0},\delta_v^{1;q}\}}[1;q]$. We prove by induction on q that each of the inclusions

$$\Xi[1;q] \hookrightarrow \Xi^\dagger[1;q] \hookrightarrow \Xi^\ddagger[1;q] \hookrightarrow \Theta_2[1;q]$$

is in $\mathrm{cell}(\mathcal{H}_v)$.

Assuming $q \geq 2$, the first inclusion fits into the gluing square

$$\begin{tikzcd}
\Xi[1;q-1] \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[1;q-1] \arrow[d, "\delta_v^{1;q}"] \\
\Xi[1;q] \arrow[r, hook, "\subset"] \arrow[ur, phantom, "\ulcorner", very near end] & \Xi^\dagger[1;q]
\end{tikzcd}$$

where the upper horizontal map is in $\mathrm{cell}(\mathcal{H}_v)$ by the inductive hypothesis. Similarly, the second inclusion fits into the following gluing square:

$$\begin{tikzcd}
\Xi^\dagger[1;q-1] \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[1;q-1] \arrow[d, "\delta_v^{1;0}"] \\
\Xi^\dagger[1;q] \arrow[r, hook, "\subset"] \arrow[ur, phantom, "\ulcorner", very near end] & \Xi^\ddagger[1;q]
\end{tikzcd}$$

Then a face map $[\mathrm{id};\alpha] : [1;p] \to [1;q]$ corresponds to a cell in $\Theta_2[1;q] \setminus \Xi^\ddagger[1;q]$ if and only if $0, q \in \operatorname{im}\alpha$. Thus the last inclusion can be obtained by gluing the faces corresponding to those α with $0, 1, q \in \operatorname{im}\alpha$ along $\Lambda_v^{1;1}[1;p]$ in increasing order of p. This completes the proof for the special case $n = 1$.

Now consider the general case. For any $[n;\mathbf{q}] \in \Theta_2$, let $\Xi'[n;\mathbf{q}] = \Xi^{\{\delta_h^n\}}[n;\mathbf{q}]$ and let $\Xi''[n;\mathbf{q}] = \Xi^{\{\delta_h^0,\delta_h^n\}}[n;\mathbf{q}]$. We prove by induction on $\dim[n;\mathbf{q}]$ that each of the inclusions

$$\Xi[n;\mathbf{q}] \hookrightarrow \Xi'[n;\mathbf{q}] \hookrightarrow \Xi''[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$$

is in $\mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v)$.

If $n = 1$ then the first two inclusions are the identity and the last inclusion was treated above. So we may assume $n \geq 2$, in which case the first inclusion fits into the gluing square

$$\begin{tikzcd} \Xi[n-1;\mathbf{q}'] \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[n-1;\mathbf{q}'] \arrow[d, "\delta_h^n"] \\ \Xi[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Xi'[n;\mathbf{q}] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $\mathbf{q}' = (q_1, \ldots, q_{n-1})$. The upper horizontal map is in cell$(\mathcal{H}_h \cup \mathcal{H}_v)$ by the inductive hypothesis, and so the lower map is also in cell$(\mathcal{H}_h \cup \mathcal{H}_v)$. Similarly, the second inclusion fits into the gluing square

$$\begin{tikzcd} \Xi'[n-1;\mathbf{q}''] \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[n-1;\mathbf{q}''] \arrow[d, "\delta_h^0"] \\ \Xi'[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Xi''[n;\mathbf{q}] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $\mathbf{q}'' = (q_2, \ldots, q_n)$.

Then a face map $[\alpha;\boldsymbol{\alpha}] : [m;\mathbf{p}] \to [n;\mathbf{q}]$ corresponds to a cell in $\Theta_2[n;\mathbf{q}] \setminus \Xi''[n;\mathbf{q}]$ if and only if $0, n \in \operatorname{im}\alpha$. Thus the last inclusion $\Xi''[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ can be obtained by gluing the faces corresponding to those $[\alpha;\boldsymbol{\alpha}]$ with $0, 1, n \in \operatorname{im}\alpha$ along $\Lambda_h^1[m;\mathbf{p}]$ in increasing order of dim $[m;\mathbf{p}]$. This completes the proof for the general case. □

3.1.2 Oury's inner horn inclusions are trivial cofibrations

The aim of this subsection is to prove the following lemma.

Lemma 3.1.4. *Every map in $\mathcal{H}_h \cup \mathcal{H}_v$ is a trivial cofibration.*

In fact, we will prove a wider class of "generalised inner horn inclusions" is contained in the trivial cofibrations. These horns are constructed from the spines by filling lower dimensional horns. Then the right cancellation property applied to $\Xi[n;\mathbf{q}] \hookrightarrow \Lambda[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ implies the second factor is a trivial cofibration. This general strategy is the same as that adopted by Joyal and Tierney to prove the corresponding result for quasi-categories [JT07, Lemma 3.5] although the combinatorics here is much more involved.

We start by gluing the outer hyperfaces of $\Theta_2[n;\mathbf{q}]$ to $\Xi[n;\mathbf{q}]$ according to the following total order $\prec$:

$$\delta_v^{1;0} \prec \delta_v^{2;0} \prec \cdots \prec \delta_v^{n;0} \prec \delta_h^0 \prec \delta_h^n \prec \delta_v^{1;q_1} \prec \delta_v^{2;q_2} \prec \cdots \prec \delta_v^{n;q_n}.$$

(Note that not all of these hyperfaces may exist. The face δ_h^0 (respectively δ_h^n) is a hyperface only if $q_0 = 0$ (resp. if $q_n = 0$), and the hyperfaces $\delta_v^{k;0}$ and $\delta_v^{k;q_k}$ exist only if $q_k \geq 1$.)

Lemma 3.1.5. *The inclusion $\Xi^S[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ is a trivial cofibration for any $[n;\mathbf{q}] \in \Theta_2$ and for any set S of outer hyperfaces of $\Theta_2[n;\mathbf{q}]$ that is downward closed with respect to $\prec$.*

Proof. We proceed by induction on $|S|$. Fix $[n;\mathbf{q}] \in \Theta_2$ and a downward closed set S of outer hyperfaces of $\Theta_2[n;\mathbf{q}]$. If S is empty then $\Xi^S[n;\mathbf{q}] = \Xi[n;\mathbf{q}]$ and so the result follows trivially. So suppose $|S| \geq 1$. Let $\delta : \Theta_2[m;\mathbf{p}] \to \Theta_2[n;\mathbf{q}]$ be the $\prec$-maximum element in

S and let $S' = S \setminus \{\delta\}$. Then $\Xi^{S'}[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ is a trivial cofibration by the inductive hypothesis, and hence it suffices to show $\Xi^{S'}[n;\mathbf{q}] \hookrightarrow \Xi^{S}[n;\mathbf{q}]$ is also a trivial cofibration. Since $\Xi^{S}[n;\mathbf{q}]$ can be obtained by gluing $\Theta_2[m;\mathbf{p}]$ to $\Xi^{S'}[n;\mathbf{q}]$ along the pullback X in the gluing square

$$\begin{tikzcd} X \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[m;\mathbf{p}] \arrow[d, "\delta"] \\ \Xi^{S'}[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Xi^{S}[n;\mathbf{q}] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

this reduces to showing we have $X = \Xi^{T}[m;\mathbf{p}]$ for some downward closed set T of outer hyperfaces of $[m;\mathbf{p}]$ with $|T| < |S|$. Since δ is an outer hyperface and hence inert, pulling back $\Xi[n;\mathbf{q}]$ along δ yields $\Xi[m;\mathbf{p}]$. To describe the remaining cells in X, we have to consider the following cases separately.

(1) $\delta = \delta_v^{k;0}$: In this case $S' = \{\delta_v^{\ell;0} : \ell < k,\ q_\ell \geq 1\}$. Thus X is generated by $\Xi[n;\mathbf{p}]$ (where $\boldsymbol{p} = (q_1,\ldots,q_k - 1,\ldots,q_n)$) and the pullbacks of these faces $\delta_v^{\ell;0} \in S'$ along $\delta_v^{k;0}$. For any $\ell < k$ with $q_\ell \geq 1$, the pullback of $\delta_v^{\ell;0}$ along $\delta_v^{k;0}$ is $\delta_v^{\ell;0}[n;\boldsymbol{p}]$, *i.e.* the square

$$\begin{tikzcd} \Theta_2[n;q_1,\ldots,q_\ell - 1,\ldots,q_k - 1,\ldots,q_n] \arrow[r, "\delta_v^{\ell;0}"] \arrow[d, "\delta_v^{k;0}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[n;q_1,\ldots,q_k - 1,\ldots,q_n] \arrow[d, "\delta_v^{k;0}"] \\ \Theta_2[n;q_1,\ldots,q_\ell - 1,\ldots,q_n] \arrow[r, "\delta_v^{\ell;0}"'] & \Xi^{S}[n;\mathbf{q}] \end{tikzcd}$$

is a pullback. Hence $X = \Xi^{T}[n;\mathbf{p}]$ where

$$T = \left\{\delta_v^{\ell;0}[n;\mathbf{p}] : \ell < k,\ q_\ell \geq 1\right\} = \left\{\delta_v^{\ell;0}[n;\mathbf{p}] : \ell < k,\ p_\ell \geq 1\right\}.$$

(2) $\delta = \delta_h^{0}$: In this case $S' = \{\delta_v^{k;0} : q_k \geq 1\}$. Note that since δ_h^0 is a hyperface, we must have $q_1 = 0$ and hence $k \neq 1$ for all $\delta_v^{k;0} \in S'$. It then follows that the pullback of $\delta_v^{k;0}$ along δ_h^0 is $\delta_v^{k-1;0}[n-1;\boldsymbol{p}]$ where $\boldsymbol{p} = (q_2,\ldots,q_n)$, *i.e.* the square

$$\begin{tikzcd} \Theta_2[n-1;q_2,\ldots,q_k - 1,\ldots,q_n] \arrow[r, "\delta_v^{k-1;0}"] \arrow[d, "\delta_h^{0}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[n-1;q_2,\ldots,q_n] \arrow[d, "\delta_h^{0}"] \\ \Theta_2[n;q_1,\ldots,q_k - 1,\ldots,q_n] \arrow[r, "\delta_v^{k;0}"'] & \Xi^{S}[n;\mathbf{q}] \end{tikzcd}$$

is a pullback. Therefore $X = \Xi^{T}[n-1;\boldsymbol{p}]$ and

$$T = \left\{\delta_v^{k-1;0}[n-1;\boldsymbol{p}] : q_k \geq 1\right\} = \left\{\delta_v^{k;0}[n-1;\boldsymbol{p}] : p_k \geq 1\right\}.$$

(The second equality holds because $p_{k-1} = q_k$.)

(3) $\delta = \delta_h^{n}$: This case can be treated similarly to the previous one except we may have $\delta_h^0 \in S'$. If this is the case, the pullback of δ_h^0 along δ_h^n is $\delta_h^0[n-1;\boldsymbol{p}]$ where $\boldsymbol{p} = (q_1,\ldots,q_{n-1})$, hence $X = \Xi^{T}[n-1;\boldsymbol{p}]$ where

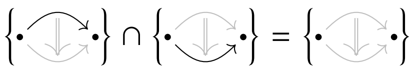
$$T = \left\{\delta_v^{k;0}[n-1;\boldsymbol{p}] : p_k \geq 1\right\} \cup \left\{\delta_h^{0}[n-1;\boldsymbol{p}]\right\}.$$

Since $p_1 = q_1 = 0$ (where the second equality follows from our assumption that $\delta_h^0 \in S'$), $\delta_h^0[n-1;\boldsymbol{p}]$ is indeed a hyperface of $\Theta_2[n-1;\boldsymbol{p}]$.

(4a) $\delta = \delta_v^{k;q_k}$ and $q_k \geq 2$: The pullback of $\delta_v^{\ell;0}$ along $\delta_v^{k;q_k}$ is $\delta_v^{\ell;0}[n;\mathbf{p}]$ (where $\boldsymbol{p} = (q_1,\ldots,q_k-1,\ldots,q_n)$) for all ℓ, and similarly for $\delta_v^{\ell;q_\ell}$. If $q_1 = 0$, then we know $k \neq 1$ and the pullback of $\delta_h^0 \in S'$ along $\delta_v^{k;q_k}$ is $\delta_h^0[n;\mathbf{p}]$. Note in this case $\delta_h^0[n;\mathbf{p}]$ is a hyperface of $\Theta_2[n;\mathbf{p}]$ since $p_1 = q_1 = 0$. Conversely, if $p_1 = 0$ then we must have $q_1 = 0$ and so $\delta_h^0 \in S'$. Similarly, $p_n = 0$ if and only if $q_n = 0$, in which case the pullback of $\delta_h^n \in S'$ along $\delta_v^{k;q_k}$ is the hyperface $\delta_h^n[n;\mathbf{p}]$. Therefore $X = \Xi^T[n;\mathbf{p}]$ where:

- $\delta_v^{\ell;0}[n;\mathbf{p}] \in T$ iff $p_\ell \geq 1$;
- $\delta_v^{\ell;q_\ell}[n;\mathbf{p}] = \delta_v^{\ell;p_\ell}[n;\mathbf{p}] \in T$ iff $\ell < k$ and $p_\ell \geq 1$;
- $\delta_h^0[n;\mathbf{p}] \in T$ iff $p_1 = 0$; and
- $\delta_h^n[n;\mathbf{p}] \in T$ iff $p_n = 0$.

(4b) $\delta = \delta_v^{1;q_1}$ and $q_1 = 1$: The difference between this case and the previous one is that the pullback of $\delta_v^{1;0}$ along $\delta_v^{1;q_1} = \delta_v^{1;1}$ is generated by the horizontal hyperface $\delta_h^0[n;\mathbf{p}]$ of $\Theta_2[n;\mathbf{p}] = \Theta_2[n;0,q_2,\ldots,q_n]$ and the point $[\{0\}]$. (This is essentially the intersection of two semicircles



horizontally composed with $[n-1;q_2,\ldots,q_n]$.) Hence $X = \Xi^T[n;\mathbf{p}]$ where:

- $\delta_v^{\ell;0}[n;\mathbf{p}] \in T$ iff $p_\ell \geq 1$;
- $\delta_h^0[n;\mathbf{p}] \in T$; and
- $\delta_h^n[n;\mathbf{p}] \in T$ iff $p_n = 0$.

(4c) $\delta = \delta_v^{n;q_n}$ and $q_n = 1$: This case is similar to the previous one, and we can deduce $X = \Xi^T[n;\mathbf{p}]$ where $\boldsymbol{p} = (q_1,\ldots,q_{n-1},0)$ and:

- $\delta_v^{\ell;0}[n;\mathbf{p}] \in T$ iff $p_\ell \geq 1$;
- $\delta_v^{\ell;q_\ell}[n;\mathbf{p}] = \delta_v^{\ell;p_\ell}[n;\mathbf{p}] \in T$ iff $\ell < n$ and $p_\ell \geq 1$;
- $\delta_h^0[n;\mathbf{p}] \in T$ iff $p_1 = 0$; and
- $\delta_h^n[n;\mathbf{p}] \in T$.

(4d) $\delta = \delta_v^{k;q_k}$ for some $2 \leq k \leq n-1$ and $q_k = 1$: In this case, we have $\boldsymbol{p} = (q_1,\ldots,q_k - 1,\ldots,q_n)$ and the pullback of $\delta_v^{k;0}$ along $\delta_v^{k;q_k}$ is generated by

$$[\{0,\ldots,k-1\};\mathbf{id}] : \Theta_2[k-1;p_1,\ldots,p_{k-1}] \to \Theta_2[n;\mathbf{p}]$$

and

$$[\{k,\ldots,n\};\mathbf{id}] : \Theta_2[n-k;p_{k+1},\ldots,p_n] \to \Theta_2[n;\mathbf{p}].$$

Observe that $[\{0,\ldots,k-1\};\mathbf{id}]$ is contained in the hyperface $\delta_h^n[n;\mathbf{p}]$ if $p_n = 0$, and in the hyperface $\delta_v^{n;0}[n;\mathbf{p}]$ if $p_n \geq 1$. Similarly, $[\{k,\ldots,n\};\mathbf{id}]$ is contained in $\delta_h^0[n;\mathbf{p}]$ or $\delta_v^{1;0}[n;\mathbf{p}]$. Therefore $X = \Xi^T[n;\mathbf{p}]$ where:

- $\delta_v^{\ell;0}[n;\mathbf{p}] \in T$ iff $\ell \neq k$ and $p_\ell \geq 1$;
- $\delta_v^{\ell;q_\ell}[n;\mathbf{p}] = \delta_v^{\ell;p_\ell}[n;\mathbf{p}] \in T$ iff $\ell < k$ and $p_\ell \geq 1$;
- $\delta_h^0[n;\mathbf{p}] \in T$ iff $p_1 = 0$; and
- $\delta_h^n[n;\mathbf{p}] \in T$ iff $p_n = 0$.

In each of these cases, it is straightforward to check that T is a downward closed set of outer hyperfaces of $\Theta_2[m;\mathbf{p}]$. Moreover, since the elements of T are obtained by pulling back the elements in S', we have $|T| \leq |S'| < |S|$. This completes the proof of Lemma 3.1.5. □

We are particularly interested in the instance of Lemma 3.1.5 where S is the set of all outer hyperfaces of $\Theta_2[n;\mathbf{q}]$. Note that if $[n;\mathbf{q}]$ is poly-vertebral (*i.e.* $[n;\mathbf{q}]$ is not $[0]$, $[1;0]$ or $[1;1]$) then each vertebra of $[n;\mathbf{q}]$ is an outer face. Thus in this case it follows from Proposition 2.1.10 that $\Xi^S[n;\mathbf{q}]$ is generated by the outer hyperfaces of $\Theta_2[n;\mathbf{q}]$ alone. This is why the following definition does not mention the spine.

Definition 3.1.6. For any set S of faces of $\Theta_2[n;\mathbf{q}]$, let $\Upsilon^S[n;\mathbf{q}] \subset \Theta_2[n;\mathbf{q}]$ denote the cellular subset generated by all outer hyperfaces of $\Theta_2[n;\mathbf{q}]$ and the faces in S.

We first consider the case where S is some set of inner vertical hyperfaces.

Definition 3.1.7. A set S of inner vertical hyperfaces of $\Theta_2[n;\mathbf{q}]$ is called *admissible* if it is not the set of all inner hyperfaces.

Note that if S is a non-admissible set of inner vertical hyperfaces of $\Theta_2[n;\mathbf{q}]$, then all inner hyperfaces of $\Theta_2[n;\mathbf{q}]$ must be vertical. Therefore we must have $n = 1$ and $S = \{\delta_v^{1;k} : 1 \leq k \leq q_1 - 1\}$.

Lemma 3.1.8. *The inclusion* $\Upsilon^S[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ *is a trivial cofibration for any poly-vertebral* $[n;\mathbf{q}] \in \Theta_2$ *and for any admissible set S of inner vertical hyperfaces of* $\Theta_2[n;\mathbf{q}]$.

Proof. Again, we proceed by induction on $|S|$. If $S = \varnothing$ then the lemma follows from Lemma 3.1.5. So we may assume $|S| \geq 1$. Choose an element $\delta_v^{k;i} \in S$, which then necessarily satisfies $1 \leq k \leq n$ and $1 \leq i \leq q_k - 1$. Let $S' = S \setminus \{\delta_v^{k;i}\}$. By a similar argument to that presented above for Lemma 3.1.5, what we must prove reduces to showing that X in the gluing square

$$\begin{tikzcd} X \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[n;\boldsymbol{p}] \arrow[d, "\delta_v^{k;i}"] \\ \Upsilon^{S'}[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Upsilon^S[n;\mathbf{q}] \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is of the form $X = \Upsilon^T[n;\mathbf{p}]$ (where $\boldsymbol{p} = (q_1, \ldots, q_k - 1, \ldots, q_n)$) for some admissible set T of inner vertical hyperfaces of $\Theta_2[n;\mathbf{p}]$ with $|T| < |S|$. Note that $[n;\mathbf{p}]$ must be poly-vertebral as the only cell with an inner vertical hyperface of mono-vertebral shape is $\Theta_2[1;2]$, and for $[n;\mathbf{q}] = [1;2]$ the only admissible S is the empty set.

We first show that the square

$$\begin{tikzcd} \Upsilon^{\varnothing}[n;\mathbf{p}] \arrow[r, hook, "\subset"] \arrow[d] & \Theta_2[n;\mathbf{p}] \arrow[d, "\delta_v^{k;i}"] \\ \Upsilon^{\varnothing}[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Upsilon^S[n;\mathbf{q}] \end{tikzcd}$$

is a pullback. Since $q_k \geq 2$ and $p_\ell = q_\ell$ for $\ell \neq k$, we have $p_1 = 0$ if and only if $q_1 = 0$. Moreover, if $p_1 = q_1 = 0$ then the pullback of δ_h^0 along $\delta_v^{k;i}$ is $\delta_h^0[n;\mathbf{p}]$. Similarly, $p_n = 0$ if and only if $q_n = 0$, in which case the pullback of δ_h^n along $\delta_v^{k;i}$ is $\delta_h^n[n;\mathbf{p}]$. For the outer vertical hyperfaces, if $q_\ell \geq 1$ and either $j = 0$ or $j = q_\ell$ then the pullback of $\delta_v^{\ell;j}$ along $\delta_v^{k;i}$ is $\delta_v^{\ell;j}[n;\mathbf{p}]$ except when $(\ell, j) = (k, q_k)$, in which case the pullback is $\delta_v^{k;q_k-1}[m;\mathbf{p}] = \delta_v^{k;p_k}[m;\mathbf{p}]$. Thus the above square is indeed a pullback.

It then follows that $X = \Upsilon^T[n;\mathbf{p}]$ where T consists of the pullbacks of elements of S' along $\delta_v^{k;i}$. Similarly to the outer case considered above, the pullback of $\delta_v^{\ell;j} \in S'$ along $\delta_v^{k;i}$ is $\delta_v^{\ell;j}[n;\mathbf{p}]$ except when $\ell = k$ and $j > i$, in which case the pullback is $\delta_v^{k;j-1}[n;\mathbf{p}]$. Hence T is a set of inner vertical hyperfaces of $\Theta_2[n;\mathbf{p}]$. Moreover pulling back along $\delta_v^{k;i}$ gives a bijection between S' and T and hence $|T| = |S| - 1$. Thus it remains to show that T is admissible. Suppose otherwise, then as we mentioned before the statement of Lemma 3.1.8, we must have $[n;\mathbf{p}] = [1;p_1]$ and

$$|T| = \left|\{\delta_v^{1;\ell}[1;p_1] : 1 \leq \ell \leq p_1 - 1\}\right| = p_1 - 1.$$

This implies $|S| = |T| + 1 = p_1 = q_1 - 1$. But then S contains all of the inner hyperfaces of $\Theta_2[n;\mathbf{q}] = \Theta_2[1;q_1]$, which contradicts our assumption that S is admissible. This completes the proof of Lemma 3.1.8. □

Now we consider the inner horizontal hyperfaces of $\Theta_2[n;\mathbf{q}]$. Recall that for each $1 \leq k \leq n-1$, we have a family of k-th horizontal hyperfaces $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ indexed by $\langle\alpha,\alpha'\rangle \in \mathbf{Shfl}(q_k, q_{k+1})$.

Definition 3.1.9. If S is a set of faces of $\Theta_2[n;\mathbf{q}]$, we define

$$\mathbf{Shfl}_S(q_k, q_{k+1}) \overset{\text{def}}{=} \left\{\langle\alpha,\alpha'\rangle \in \mathbf{Shfl}(q_k, q_{k+1}) : \delta_h^{k;\langle\alpha,\alpha'\rangle} \in S\right\}.$$

Definition 3.1.10. A set S of inner hyperfaces of $\Theta_2[n;\mathbf{q}]$ is called *admissible* if:

(i) S is not the set of all inner hyperfaces of $\Theta_2[n;\mathbf{q}]$;

(ii) there is at most one $1 \leq k \leq n-1$ such that

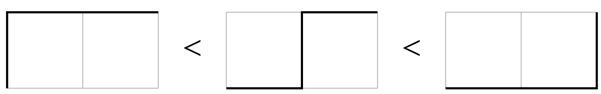
$$\varnothing \neq \mathbf{Shfl}_S(q_k, q_{k+1}) \neq \mathbf{Shfl}(q_k, q_{k+1})$$

(we will write k_S for such k if it exists); and

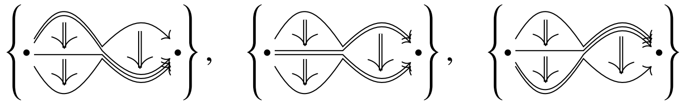
(iii) if k_S exists, then $\mathbf{Shfl}_S(q_{k_S}, q_{k_S+1})$ is downward closed with respect to the order described in Section 2.1.1.

Note that Definition 3.1.10 reduces to Definition 3.1.7 if S contains no horizontal hyperfaces.

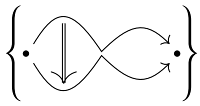
Remark. The role of Definition 3.1.10(iii) is to ensure that the intersections (meaning pullbacks) of the hyperfaces in S are well-behaved so that we do not have to worry about faces of $\Theta_2[n;\mathbf{q}]$ of codimension larger than 2. For example, consider the case $[n;\mathbf{q}] = [2;2,1]$. There are three inner horizontal hyperfaces in this case, corresponding to the three $(2,1)$-shuffles $\langle\alpha,\alpha'\rangle < \langle\beta,\beta'\rangle < \langle\gamma,\gamma'\rangle$; graphically, the shuffles



correspond to the hyperfaces



respectively. The intersection of $\delta_h^{1;\langle\alpha,\alpha'\rangle}$ and $\delta_h^{1;\langle\gamma,\gamma'\rangle}$ is then the face



of codimension 3, which is "too small". If S is an admissible set containing $\delta_h^{1;\langle\alpha,\alpha'\rangle}$ and $\delta_h^{1;\langle\gamma,\gamma'\rangle}$, then (iii) implies that S also contains $\delta_h^{1;\langle\beta,\beta'\rangle}$. Since this "too small" face is contained in the intersection of $\delta_h^{1;\langle\alpha,\alpha'\rangle}$ (or $\delta_h^{1;\langle\gamma,\gamma'\rangle}$) and $\delta_h^{1;\langle\beta,\beta'\rangle}$, we may essentially disregard it.

There are two obviously downward closed subsets of $\mathbf{Shfl}(q_k, q_{k+1})$, namely $\varnothing$ and $\mathbf{Shfl}(q_k, q_{k+1})$. Definition 3.1.10(ii) asks that we always have one of these two subsets for any value of k, with a possible exception of $k = k_S$. This simplifies the proof and in particular the descriptions of the sets T_1 and T_1' defined below, but it is not essential. Indeed, it seems possible to prove a variant of Lemma 3.1.11 where (ii) is removed from Definition 3.1.10 and (iii) is replaced by:

(iii') $\mathbf{Shfl}_S(q_k, q_{k+1})$ is downward closed for all $1 \leq k \leq n-1$.

Although this modification makes Lemma 3.1.11 slightly more general, we see no use in this extra generality.

Lemma 3.1.11. *The inclusion $\Upsilon^S[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ is a trivial cofibration for any poly-vertebral $[n;\mathbf{q}] \in \Theta_2$ and for any admissible set S of inner hyperfaces of $\Theta_2[n;\mathbf{q}]$.*

Proof. Let $S_h \subset S$ denote the set of horizontal hyperfaces in S. We proceed by induction on $\dim[n;\mathbf{q}]$ and $|S_h|$. If $S_h = \varnothing$ then the result follows from Lemma 3.1.8, so we may assume $|S_h| \geq 1$. Choose $1 \leq k \leq n-1$ so that S contains a k-th horizontal hyperface, where we take $k = k_S$ if the latter exists. Let $\langle\alpha,\alpha'\rangle \in \mathbf{Shfl}_S(q_k, q_{k+1})$ be a maximal one. Then $S' = S \setminus \{\delta_h^{k;\langle\alpha,\alpha'\rangle}\}$ is admissible, and so once again it suffices to prove that X in the gluing square

$$\begin{tikzcd} X \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[n-1;\boldsymbol{p}] \arrow[d, "\delta_h^{k;\langle\alpha,\alpha'\rangle}"] \arrow[dl, phantom, "\ulcorner", very near end] \\ \Upsilon^{S'}[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Upsilon^S[n;\mathbf{q}] \end{tikzcd}$$

(where $\boldsymbol{p} = (q_1, \ldots, q_{k-1}, q_k + q_{k+1}, q_{k+2}, \ldots, q_n)$) is of the form $X = \Upsilon^T[n-1;\boldsymbol{p}]$ for some admissible T. By a similar argument to that presented in the proof of Lemma 3.1.8, $[n-1;\boldsymbol{p}]$ must be poly-vertebral.

Claim 0. Let $Y \subset \Theta_2[n-1;\boldsymbol{p}]$ be the cellular subset defined by the following pullback square:

$$\begin{tikzcd} Y \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[n-1;\boldsymbol{p}] \arrow[d, "\delta_h^{k;\langle\alpha,\alpha'\rangle}"] \\ \Upsilon^{\varnothing}[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Upsilon^S[n;\mathbf{q}] \end{tikzcd}$$

Then Y is generated by the outer hyperfaces of $\Theta_2[n-1;\boldsymbol{p}]$, *i.e.* $Y = \Upsilon^{\varnothing}[n-1;\boldsymbol{p}]$.

Proof. We first show the containment $Y \subset \Upsilon^{\varnothing}[n-1;\boldsymbol{p}]$. If $q_1 = 0$, then the pullback of the hyperface δ_h^0 along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is $\delta_h^0[n-1;\boldsymbol{p}]$. Since $\delta_h^0[n-1;\boldsymbol{p}]$ is an outer face of $\Theta_2[n-1;\boldsymbol{p}]$ (of codimension q_2+1 if $k=1$ and of codimension 1 otherwise), it is contained in $\Upsilon^{\varnothing}[n-1;\boldsymbol{p}]$ by Proposition 2.1.10. The hyperface δ_h^n (if it exists) can be treated dually.

Next we consider the vertical hyperfaces of $\Theta_2[n;\mathbf{q}]$. Fix $1 \leq \ell \leq n$ with $q_\ell \geq 1$. Then the pullback of $\delta_v^{\ell;0}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is:

- $\delta_v^{\ell;0}[n-1;\boldsymbol{p}]$ if $\ell < k$;
- $\delta_v^{\ell-1;0}[n-1;\boldsymbol{p}]$ if $\ell > k+1$; and
- contained in $\delta_v^{k;0}[n-1;\boldsymbol{p}]$ if $\ell = k$ or $\ell = k+1$.

The hyperfaces $\delta_v^{\ell;q_\ell}$ can be treated dually. This proves $Y \subset \Upsilon^{\varnothing}[n-1;\boldsymbol{p}]$.

For the other containment $\Upsilon^{\varnothing}[n-1;\boldsymbol{p}] \subset Y$, we must show that any outer hyperface of $\Theta_2[n-1;\boldsymbol{p}]$ can be obtained by pulling back some outer hyperface of $\Theta_2[n;\mathbf{q}]$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$. If $p_1 = 0$, then $q_1 = 0$ (because $p_1 = q_1$ if $k \neq 1$ and $p_1 = q_1 + q_2$ if $k = 1$) and the hyperface $\delta_h^0[n-1;\boldsymbol{p}]$ is precisely the pullback of δ_h^0 along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$. The other horizontal hyperface $\delta_h^{n-1}[n-1;\boldsymbol{p}]$ (if it exists) can be treated dually.

Now we consider the vertical hyperfaces of $\Theta_2[n-1;\boldsymbol{p}]$. Fix $1 \leq \ell \leq m = n-1$ with $p_\ell \geq 1$. Then the hyperface $\delta_v^{\ell;0}[n-1;\boldsymbol{p}]$ is the pullback (along $\delta_v^{k;\langle\alpha,\alpha'\rangle}$) of:

- $\delta_v^{\ell;0}$ if $\ell < k$;
- $\delta_v^{\ell+1;0}$ if $\ell > k$;
- $\delta_v^{k;0}$ if $\ell = k$ and $\alpha(1) = 1$; and
- $\delta_v^{k+1;0}$ if $\ell = k$ and $\alpha'(1) = 1$.

Note that if $\delta_v^{k;0}[n-1;\boldsymbol{p}]$ exists then $p_k \geq 1$ so $\alpha(1) \in [q_k]$ and $\alpha'(1) \in [q_{k+1}]$ are well-defined. Moreover, $\langle\alpha,\alpha'\rangle \in \mathbf{Shfl}(q_k, q_{k+1})$ implies that we must have either $\alpha(1) = 1$ or $\alpha'(1) = 1$. Thus the above list indeed covers all possible cases.

The remaining hyperfaces $\delta_v^{\ell;p_\ell}[n-1;\boldsymbol{p}]$ can be treated dually, and this completes the proof of Claim 0. □

It now follows from the following claims that $X = \Upsilon^{T}[n-1;\boldsymbol{p}]$ holds for

$$T = T_1 \cup T_1' \cup T_2 \cup T_3 \cup T_3' \cup T_4 \cup T_4'$$

where

$$
\begin{aligned}
T_1 &= \big\{\delta_h^{\ell;\langle\gamma,\gamma'\rangle}[n-1;\boldsymbol{p}] : 1 \le \ell \le k-1,\ \mathbf{Shfl}_S(q_\ell,q_{\ell+1}) = \mathbf{Shfl}(q_\ell,q_{\ell+1}),\\
&\qquad\qquad \langle\gamma,\gamma'\rangle \in \mathbf{Shfl}(p_\ell,p_{\ell+1})\big\},\\
T_1' &= \big\{\delta_h^{\ell-1;\langle\gamma,\gamma'\rangle}[n-1;\boldsymbol{p}] : k+1 \le \ell \le n-1,\ \mathbf{Shfl}_S(q_\ell,q_{\ell+1}) = \mathbf{Shfl}(q_\ell,q_{\ell+1}),\\
&\qquad\qquad \langle\gamma,\gamma'\rangle \in \mathbf{Shfl}(p_{\ell-1},p_\ell)\big\},\\
T_2 &= \left\{\delta_v^{k;j}[n-1;\boldsymbol{p}] : j \in \lrcorner\langle\alpha,\alpha'\rangle\right\},\\
T_3 &= \left\{\delta_v^{\ell;j}[n-1;\boldsymbol{p}] : 1 \le \ell < k,\ \delta_v^{\ell;j} \in S'\right\},\\
T_3' &= \left\{\delta_v^{\ell-1;j}[n-1;\boldsymbol{p}] : k+1 < \ell \le n,\ \delta_v^{\ell;j} \in S'\right\},\\
T_4 &= \left\{\delta_v^{k;j}[n-1;\boldsymbol{p}] : (\exists i \in [q_k])\left[\delta_v^{k;i} \in S',\ \alpha^{-1}(i) = \{j\}\right]\right\}, \text{and}\\
T_4' &= \left\{\delta_v^{k;j}[n-1;\boldsymbol{p}] : (\exists i \in [q_{k+1}])\left[\delta_v^{k+1;i} \in S',\ (\alpha')^{-1}(i) = \{j\}\right]\right\}.
\end{aligned}
$$

(See Definition 2.1.2 for the definition of $\lrcorner\langle\alpha,\alpha'\rangle$.) For each $1 \le m \le 4$, Claim m below relates the elements in T_m (and T_m') to appropriate inner hyperfaces in S'.

Claim 1. Fix $1 \le \ell \le k-1$. Then:

(i) for any $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_\ell,q_{\ell+1})$, each cell in the pullback of $\delta_h^{\ell;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is contained in some $\delta_h^{\ell;\langle\gamma,\gamma'\rangle}[n-1;\boldsymbol{p}]$; and

(ii) for any $\langle\gamma,\gamma'\rangle \in \mathbf{Shfl}(p_\ell,p_{\ell+1})$, the hyperface $\delta_h^{\ell;\langle\gamma,\gamma'\rangle}[n-1;\boldsymbol{p}]$ is contained in the pullback of some $\delta_h^{\ell;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$.

The dual version of this claim relates, for $k+1 \le \ell \le n$, the ℓ-th horizontal hyperfaces of $\Theta_2[n;\mathbf{q}]$ to the $(\ell-1)$-th horizontal hyperfaces of $\Theta_2[n-1;\boldsymbol{p}]$.

Proof. If $1 \le \ell < k-1$ (note the strict inequality) then both (i) and (ii) are straightforward since

$$\mathbf{Shfl}(p_\ell,p_{\ell+1}) = \mathbf{Shfl}(q_\ell,q_{\ell+1})$$

and the pullback of $\delta_h^{\ell;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is precisely $\delta_h^{\ell;\langle\beta,\beta'\rangle}[n-1;\boldsymbol{p}]$ for any $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_\ell,q_{\ell+1})$.

Now we prove (i) for the case $\ell = k-1$. Let $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_{k-1},q_k)$ and suppose we are given a commutative square

$$
\begin{tikzcd}
{[u;\mathbf{r}]} \arrow[r, "{[\zeta;\boldsymbol{\zeta}]}"] \arrow[d, "{[\xi;\boldsymbol{\xi}]}"'] & {[n-1;\boldsymbol{p}]} \arrow[d, "\delta_h^{k;\langle\alpha,\alpha'\rangle}"] \\
{[n-1;q_1,\ldots,q_{k-1}+q_k,\ldots,q_n]} \arrow[r, "\delta_h^{k-1;\langle\beta,\beta'\rangle}"'] & {[n;\mathbf{q}]}
\end{tikzcd}
$$

in Θ_2. Then the square

$$
\begin{tikzcd}
{[u]} \arrow[r, "\zeta"] \arrow[d, "\xi"'] & {[n-1]} \arrow[d, "\delta^k"] \\
{[n-1]} \arrow[r, "\delta^{k-1}"'] & {[n]}
\end{tikzcd}
$$

in Δ commutes so $k-1 \notin \operatorname{im}(\zeta)$. We will assume there is some $1 \le v \le u$ such that $\zeta(v-1) \le k-2$ and $\zeta(v) \ge k$. (Otherwise either $\zeta(v) \le k-2$ for all v or $\zeta(v) \ge k$ for all v, and in either case $[\zeta; \boldsymbol{\zeta}]$ obviously factors through $\delta_h^{k-1;\langle\gamma,\gamma'\rangle}[n-1;\boldsymbol{p}]$ for any $\langle\gamma,\gamma'\rangle \in \mathbf{Shfl}(p_{k-1},p_k)$.) Since the (p_{k-1},p_k)-shuffles are the maximal non-degenerate simplices in $\Delta[p_{k-1}] \times \Delta[p_k]$, the map $\langle\zeta_{k-1},\zeta_k\rangle$ admits a factorisation

$$\Delta[r_v] \xrightarrow{\phi} \Delta[p_k + p_{k+1}] \xrightarrow{\langle\gamma,\gamma'\rangle} \Delta[p_{k-1}] \times \Delta[p_k]$$

such that $\langle\gamma,\gamma'\rangle$ is a (p_{k-1},p_k)-shuffle. Then $[\zeta;\boldsymbol{\zeta}]$ clearly factors through the hyperface $\delta_h^{k-1;\langle\gamma,\gamma'\rangle}[n-1;\boldsymbol{p}]$. This proves the first part of the claim for $\ell = k-1$.

For (ii), let $\langle\gamma,\gamma'\rangle \in \mathbf{Shfl}(p_{k-1},p_k)$. Since the (q_{k-1},q_k)-shuffles are the maximal non-degenerate simplices in $\Delta[q_{k-1}] \times \Delta[q_k]$, the composite

$$\Delta[p_{k-1}+p_k] \xrightarrow{\langle\gamma,\gamma'\rangle} \Delta[p_{k-1}] \times \Delta[p_k] = \Delta[q_{k-1}] \times \Delta[q_k + q_{k+1}] \xrightarrow{\mathrm{id}\times\alpha} \Delta[q_{k-1}] \times \Delta[q_k]$$

admits a factorisation

$$\Delta[p_{k-1}+p_k] = \Delta[q_{k-1}+q_k+q_{k+1}] \xrightarrow{\zeta} \Delta[q_{k-1}+q_k] \xrightarrow{\langle\beta,\beta'\rangle} \Delta[q_{k-1}] \times \Delta[q_k]$$

such that $\langle\beta,\beta'\rangle$ is a (q_{k-1},q_k)-shuffle. Then $\delta_h^{k-1;\langle\gamma,\gamma'\rangle}[n-1;\boldsymbol{p}]$ is contained in the pullback of $\delta_h^{k-1;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ since the square

$$\begin{tikzcd}
{[n-2; q_1,\dots,q_{k-1}+q_k+q_{k+1},\dots,q_n]} \arrow[r, "\delta_h^{k-1;\langle\gamma,\gamma'\rangle}"] \arrow[d, "\delta_h^{k-1;\langle\zeta,\alpha'\gamma'\rangle}"'] & {[n-1; q_1,\dots,q_k+q_{k+1},\dots,q_n]} \arrow[d, "\delta_h^{k;\langle\alpha,\alpha'\rangle}"] \\
{[n-1; q_1,\dots,q_{k-1}+q_k,\dots,q_n]} \arrow[r, "\delta_h^{k-1;\langle\beta,\beta'\rangle}"'] & {[n;\mathbf{q}]}
\end{tikzcd}$$

commutes. This completes the proof of Claim 1. □

Claim 2.

(i) For any $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_k,q_{k+1})$ with $\langle\beta,\beta'\rangle \ngeq \langle\alpha,\alpha'\rangle$, the pullback of $\delta_h^{k;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is contained in $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ for some $j \in_{\lrcorner} \langle\alpha,\alpha'\rangle$.

(ii) For any $j \in_{\lrcorner} \langle\alpha,\alpha'\rangle$, the hyperface $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ is the pullback of $\delta_h^{k;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ for some $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_k,q_{k+1})$ with $\langle\beta,\beta'\rangle < \langle\alpha,\alpha'\rangle$.

Proof. For (i), suppose $\langle\beta,\beta'\rangle$ is a (q_k,q_{k+1})-shuffle with $\langle\beta,\beta'\rangle \ngeq \langle\alpha,\alpha'\rangle$. Then by Proposition 2.1.3, we can choose $j \in_{\lrcorner} \langle\alpha,\alpha'\rangle$ such that $(\alpha(j),\alpha'(j)) \neq (\beta(j),\beta'(j))$. The pullback of $\delta_h^{k;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is contained in $\delta_v^{k;j}[n-1;\boldsymbol{p}]$.

To prove (ii), suppose $j \in_{\lrcorner} \langle\alpha,\alpha'\rangle$. Then the hyperface $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ is the pullback of $\delta_h^{k;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ where $\langle\beta,\beta'\rangle$ is the (q_k,q_{k+1})-shuffle corresponding to j under Proposition 2.1.4. Note that $\langle\beta,\beta'\rangle$ is an immediate predecessor of $\langle\alpha,\alpha'\rangle$ and so in particular $\langle\beta,\beta'\rangle < \langle\alpha,\alpha'\rangle$. □

Claim 3. For any $1 \le \ell < k$ (respectively $k+1 < \ell \le n$) and $1 \le i \le q_\ell - 1$, the pullback of $\delta_v^{\ell;i}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is $\delta_v^{\ell;i}[n-1;\boldsymbol{p}]$ (resp. $\delta_v^{\ell-1;i}[n-1;\boldsymbol{p}]$).

Proof. This is straightforward to check. □

Claim 4. Fix $1 \le i \le q_k - 1$ (respectively $1 \le i \le q_{k+1} - 1$). Then the pullback of $\delta_v^{k;i}$ (resp. $\delta_v^{k+1;i}$) along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is:

- precisely $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ if $\alpha^{-1}(i) = \{j\}$ (resp. $(\alpha')^{-1}(i) = \{j\}$) for some $1 \le j \le p_k - 1$; and
- contained in $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ for some $j \in \lrcorner\langle\alpha,\alpha'\rangle$ otherwise.

Proof. We will only consider the hyperfaces $\delta_v^{k;i}$ as $\delta_v^{k+1;i}$ can be treated dually. The first case is straightforward to check. In the second case, let $j = \min(\alpha^{-1}(i))$. Then clearly the pullback of $\delta_v^{k;i}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is contained in $\delta_v^{k;j}[n-1;\boldsymbol{p}]$, and so it remains to show that $j \in \lrcorner\langle\alpha,\alpha'\rangle$. Note that $1 \le i \le q_k - 1$ and $\alpha(j) = i$ imply $1 \le j \le p_k - 1$. Moreover, $\alpha(j-1) = \alpha(j) - 1$ by our choice of j, and $\alpha(j+1) = i = \alpha(j)$ since $|\alpha^{-1}(i)| \ge 2$. Therefore $j \in \lrcorner\langle\alpha,\alpha'\rangle$. □

Now we go back to the proof of Lemma 3.1.11. We can deduce from Claims 1 to 4 that $X = \Upsilon^T[n-1;\boldsymbol{p}]$. It thus remains to prove that T is an admissible set of inner hyperfaces of $\Theta_2[n-1;\boldsymbol{p}]$. It is clear from our definitions of T_1 and T_1' that, for any $1 \le \ell \le n-2$, either $\mathbf{Shfl}_T(p_\ell, p_{\ell+1}) = \varnothing$ or $\mathbf{Shfl}_T(p_\ell, p_{\ell+1}) = \mathbf{Shfl}(p_\ell, p_{\ell+1})$. Thus T satisfies Definition 3.1.10(ii) and (iii). To prove T also satisfies (i), we will assume otherwise (*i.e.* T contains all of the inner hyperfaces of $\Theta_2[n-1;\boldsymbol{p}]$) and deduce then S does not satisfy (i), which is a contradiction.

For any $1 \le \ell \le k-1$, we have $\mathbf{Shfl}_T(p_\ell, p_{\ell+1}) = \mathbf{Shfl}(p_\ell, p_{\ell+1})$ and so our definition of T_1 implies $\mathbf{Shfl}_S(q_\ell, q_{\ell+1}) = \mathbf{Shfl}(q_\ell, q_{\ell+1})$. Dually, we have $\mathbf{Shfl}_S(q_\ell, q_{\ell+1}) = \mathbf{Shfl}(q_\ell, q_{\ell+1})$ for all $k+1 \le \ell \le n$. Thus S contains all of the ℓ-th horizontal hyperfaces of $\Theta_2[n;\mathbf{q}]$ for all $1 \le \ell \le n-1$ with $\ell \ne k$.

Next we consider the k-th horizontal hyperfaces of $\Theta_2[n;\mathbf{q}]$. Note that since S is admissible, S contains all of the k-th horizontal hyperfaces if and only if $\langle\alpha,\alpha'\rangle$ is the maximum (q_k, q_{k+1})-shuffle. We will prove this latter statement. For any $1 \le j \le p_k - 1$, T contains $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ and so our definitions of T_2, T_4 and T_4' imply that one of the following must hold:

- $j \in \lrcorner\langle\alpha,\alpha'\rangle$;
- $\alpha(j') \ne \alpha(j)$ for all $j' \in [p_k]$ with $j' \ne j$; or
- $\alpha'(j') \ne \alpha'(j)$ for all $j' \in [p_k]$ with $j' \ne j$.

Therefore $\ulcorner\langle\alpha,\alpha'\rangle = \varnothing$, or equivalently, $\langle\alpha,\alpha'\rangle$ is the maximum (q_k, q_{k+1})-shuffle (by Proposition 2.1.4).

Lastly, we consider the inner vertical hyperfaces of $\Theta_2[n;\mathbf{q}]$. For any $1 \le \ell < k$ and for any $1 \le i \le q_\ell - 1$, T contains $\delta_v^{\ell;j}[n-1;\boldsymbol{p}]$ and so our definition of T_3 implies that $\delta_v^{\ell;j} \in S$. Dually, $\delta_v^{\ell;j} \in S$ for all $k+1 < \ell \le n$ and for all $1 \le j \le q_\ell - 1$. Note that $\langle\alpha,\alpha'\rangle \in \mathbf{Shfl}(q_k, q_{k+1})$ is the maximum one and so we have

$$\alpha = \{0, 1, \dots, q_k, \underbrace{q_k, \dots, q_k}_{q_{k+1}\text{ times}}\},$$

$$\alpha' = \{\underbrace{0, \dots, 0}_{q_k\text{ times}}, 0, 1, \dots, q_{k+1}\}.$$

Thus for each $1 \le i \le q_k - 1$, $\delta_v^{k;i}[n-1; \boldsymbol{p}] \in T$ and our definition of T_4 imply that $\delta_v^{k;i} \in S$. Similarly, for each $1 \le i \le q_{k+1} - 1$, $\delta_v^{k;q_k+i} \in T$ and our definition of T_4' imply that $\delta_v^{k+1;i} \in S$. This completes the proof of Lemma 3.1.11. □

Proof of Lemma 3.1.4. The desired result follows from Lemma 3.1.11 since setting

$$S = \left\{\text{all inner hyperfaces of } \Theta_2[n;\mathbf{q}] \text{ except for } \delta_v^{k;i}\right\}$$

yields $\Upsilon^S[n;\mathbf{q}] = \Lambda_v^{k;i}[n;\mathbf{q}]$ by Proposition 2.2.20 and setting

$$S = \{\text{all inner hyperfaces of } \Theta_2[n;\mathbf{q}] \text{ except for the } k\text{-th horizontal ones}\}$$

yields $\Upsilon^S[n;\mathbf{q}] = \Lambda_h^k[n;\mathbf{q}]$ by Proposition 2.2.18 for the appropriate ranges of k and i. □

3.1.3 Vertical equivalence extensions are trivial cofibrations

We will prove the following lemma in this subsection.

Lemma 3.1.12. *Every map in $\mathcal{E}_v$ is a trivial cofibration.*

Recall that for any $[n;\mathbf{q}] \in \Theta_2$ and $1 \le k \le n$ with $q_k = 0$, the map $\Psi^k[n;\mathbf{q}] \hookrightarrow \Phi^k[n;\mathbf{q}]$ is by definition the Leibniz box product

$$\hat{\square}_n\left(\begin{matrix} \partial\Delta[n] \\ \big\downarrow \\ \Delta[n] \end{matrix} ; \begin{matrix} \partial\Delta[q_1] \\ \big\downarrow \\ \Delta[q_1] \end{matrix}, \dots, \begin{matrix} \partial\Delta[q_{k-1}] \\ \big\downarrow \\ \Delta[q_{k-1}] \end{matrix}, \begin{matrix} \Delta[0] \\ \big\downarrow{\scriptstyle e} \\ J \end{matrix}, \begin{matrix} \partial\Delta[q_{k+1}] \\ \big\downarrow \\ \Delta[q_{k+1}] \end{matrix}, \dots, \begin{matrix} \partial\Delta[q_n] \\ \big\downarrow \\ \Delta[q_n] \end{matrix}\right)$$

where e is the nerve of the inclusion $\{\lozenge\} \hookrightarrow \{\lozenge \cong \blacklozenge\}$. Hence one of the legs in the defining colimit cone for $\Psi^k[n;\mathbf{q}]$ is the (monic) map

$$\Theta_2[n;\mathbf{q}] \cong \square_n(\Delta[n]; \Delta[q_1], \dots, \Delta[q_{k-1}], \Delta[0], \Delta[q_{k+1}], \dots, \Delta[q_n]) \to \Psi^k[n;\mathbf{q}].$$

In this subsection, we regard $\Theta_2[n;\mathbf{q}]$ as a cellular subset of $\Psi^k[n;\mathbf{q}]$ via this map.

Proof. We will prove Lemma 3.1.12 by induction on $\dim[n;\mathbf{q}]$. Note that the base case is trivial since $\Psi^1[1;0] \hookrightarrow \Phi^1[1;0]$ is isomorphic to the elementary A-anodyne extension $[\mathrm{id}; e] : \Theta_2[1;0] \hookrightarrow \Theta_2[1;J]$; indeed, both of these maps are isomorphic to the nerve of the 2-functor that looks like:

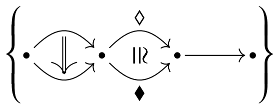
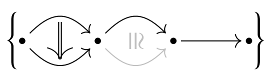
$$\left\{\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \end{tikzcd}\right\} \hookrightarrow \left\{\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\cong"] & \bullet \end{tikzcd}\right\}.$$

For the inductive step, it suffices to show that both $\Theta_2[n;\mathbf{q}] \hookrightarrow \Psi^k[n;\mathbf{q}]$ and $\Theta_2[n;\mathbf{q}] \hookrightarrow \Phi^k[n;\mathbf{q}]$ are trivial cofibrations. These facts follow from Lemmas 3.1.13 to 3.1.17 which concern intermediate cellular subsets

$$\Theta_2[n;\mathbf{q}] \subset X_0 \subset X_1 \subset X_2 \subset X_3 \subset \Psi^k[n;\mathbf{q}].$$

□

We will illustrate our argument below by providing pictures for the special case where $[n;\mathbf{q}] = [3;1,0,0]$ and $k = 2$. In this case $\Phi^k[n;\mathbf{q}]$ and $\Theta_2[n;\mathbf{q}]$ look like:

$\Phi^2[3;1,0,0] =$  , $\Theta_2[3;1,0,0] =$  .

Fix $[n;\mathbf{q}] \in \Theta_2$ and $1 \le k \le n$ such that $n \ge 2$ and $q_k = 0$. Note that an $(m;\boldsymbol{p})$-cell in $\Phi^k[n;\mathbf{q}]$ consists of $\alpha : [m] \to [n]$ in Δ and $\alpha_{k'} : \Delta[p_\ell] \to V_{k'}$ for $\alpha(\ell-1) < k' \le \alpha(\ell)$ where

$$(V_1,\dots,V_n) = \big(\Delta[q_1],\dots,\Delta[q_{k-1}], J, \Delta[q_{k+1}],\dots,\Delta[q_n]\big).$$

Such $[\alpha;\boldsymbol{\alpha}]$ factors through:

(∗) $\Theta_2[n;\mathbf{q}]$ unless there exists $1 \le \ell \le m$ such that $\alpha(\ell-1) < k \le \alpha(\ell)$ and $\blacklozenge \in \operatorname{im}\alpha_k$; and

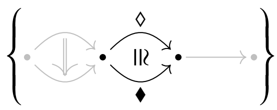
(∗∗) $\Psi^k[n;\mathbf{q}]$ unless α and all α_ℓ are surjective for $\ell \ne k$ and $\blacklozenge \in \operatorname{im}\alpha_k$.

We may assume $k \le n-1$ since the dual argument covers the case $k \ge 2$ and our assumption $n \ge 2$ implies that at least one of $k \le n-1$ and $k \ge 2$ must hold.

First, glue $\Theta_2[1;J]$ to $\Theta_2[n;\mathbf{q}]$ as in the square

$$\begin{tikzcd}
\Theta_2[1;0] \arrow[r, hook, "{[\mathrm{id};e]}"] \arrow[d, "{[\{k-1,k\};\mathrm{id}]}"'] & \Theta_2[1;J] \arrow[d] \arrow[ddr, bend left, "{[\{k-1,k\};\mathrm{id}]}"] & \\
\Theta_2[n;\mathbf{q}] \arrow[r, hook] \arrow[drr, hook, bend right, "\subset"'] & X_0 \arrow[dr, hook, dashed] & \\
& & \Psi^k[n;\mathbf{q}]
\end{tikzcd}$$

to obtain $X_0 \subset \Psi^k[n;\mathbf{q}]$. In our example, the image of $[\{1,2\};\mathrm{id}]$ looks like:



The following lemma records our construction of X_0.

Lemma 3.1.13. *The inclusion* $\Theta_2[n;\mathbf{q}] \hookrightarrow X_0$ *is a pushout of* $[\mathrm{id};e] : \Theta_2[1;0] \hookrightarrow \Theta_2[1;J]$.

Let $X_1 \subset \Phi^k[n;\mathbf{q}]$ be the cellular subset generated by X_0 and those $(m;\boldsymbol{p})$-cells $[\alpha;\boldsymbol{\alpha}]$ satisfying $\alpha(m) = k$. Since we are assuming $k \le n-1$, this condition $\alpha(m) = k$ implies $X_1 \subset \Psi^k[n;\mathbf{q}]$. Note that a non-degenerate $(m;\boldsymbol{p})$-cell $[\alpha;\boldsymbol{\alpha}]$ in $\Phi^k[n;\mathbf{q}]$ is contained in $X_1 \setminus X_0$ if and only if it satisfies:

(1a) $\alpha(0) < k-1$;

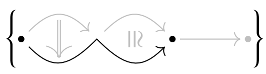
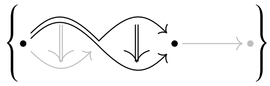
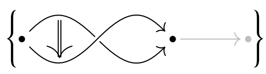
(1b) $\alpha(m) = k$; and

(1c) $\blacklozenge \in \operatorname{im}\alpha_k$.

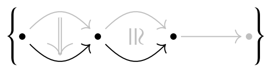
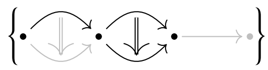
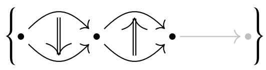
Observe that for any such $[\alpha; \boldsymbol{\alpha}]$, either it additionally satisfies

(1d) $\alpha(m-1) = k-1$

or there is a unique $(m'; \mathbf{p}')$-cell $[\beta; \boldsymbol{\beta}]$ in $X_1 \setminus X_0$ satisfying (1a-d) such that $[\alpha; \boldsymbol{\alpha}]$ is an $(m'-1)$-th horizontal face of $[\beta; \boldsymbol{\beta}]$ (not necessarily of codimension 1). *e.g.*

,  and 

are 1st horizontal faces of

,  and 

respectively.

Lemma 3.1.14. *The inclusion $X_0 \hookrightarrow X_1$ is in* $\mathrm{cell}(\mathcal{H}_h)$.

Proof. The discussion above shows that the set of non-degenerate cells in $X_1 \setminus X_0$ can be partitioned into subsets of the form

$$\big\{[\alpha; \boldsymbol{\alpha}] \text{ and all of its } (m-1)\text{-th horizontal faces}\big\}$$

where $[\alpha; \boldsymbol{\alpha}]$ is an $(m; \boldsymbol{p})$-cell satisfying (1a-d). We prove that X_1 may be obtained from X_0 by gluing such $[\alpha; \boldsymbol{\alpha}]$ along the horizontal horn $\Lambda_h^{m-1}[m; \mathbf{p}]$ in increasing order of $\dim [m; \mathbf{p}]$. Note that this horn is inner by (1a) and (1d).

Fix a non-degenerate $(m; \boldsymbol{p})$-cell $[\alpha; \boldsymbol{\alpha}]$ satisfying (1a-d). We must show that any cell in the image of the composite

$$\Lambda_h^{m-1}[m; \mathbf{p}] \hookrightarrow \Theta_2[m; \mathbf{p}] \xrightarrow{[\alpha;\boldsymbol{\alpha}]} \Phi^k[n; \mathbf{q}]$$

is contained either in X_0 or in some cell that satisfies (1a-d) and has dimension strictly smaller than $\dim [m; \mathbf{p}]$. It suffices to check this for the generating faces of $\Lambda_h^{m-1}[m; \mathbf{p}]$ described in Proposition 2.2.18:

- $[\alpha; \boldsymbol{\alpha}] \cdot \delta_h^0$:
  - is contained in X_0 if $\alpha(1) = k-1$; and
  - satisfies (1a-d) otherwise;
- $[\alpha; \boldsymbol{\alpha}] \cdot \delta_h^m$ is contained in $\Theta_2[n; \mathbf{q}]$;
- $[\alpha; \boldsymbol{\alpha}] \cdot \delta_h^{\ell;\langle\gamma,\gamma'\rangle}$ satisfies (1a-d) for any $\ell \leq m-2$ and for any $\langle\gamma,\gamma'\rangle$;
- $[\alpha; \boldsymbol{\alpha}] \cdot \delta_v^{\ell;j}$ satisfies (1a-d) for any $\ell \neq m$ and for any $j \in [p_\ell]$; and
- $[\alpha; \boldsymbol{\alpha}] \cdot \delta_v^{m;j}$ is:
  - contained in $\Theta_2[n; \mathbf{q}]$ if $\alpha_k^{-1}(\blacklozenge) = \{j\}$; and
  - a (possibly trivial) degeneracy of some cell that satisfies (1a-d) otherwise.

(By the trivial degeneracy of a cell, we mean the cell itself. Also, the codomain of any δ appearing in the form $[\alpha; \boldsymbol{\alpha}] \cdot \delta$ in this proof is assumed to be $[m; \mathbf{p}]$ so that $[\alpha; \boldsymbol{\alpha}] \cdot \delta$ is well-defined.) This completes the proof. $\square$

Next, let $X_2 \subset \Phi^k[n; \mathbf{q}]$ be the cellular subset generated by X_1 and those cells $[\alpha; \boldsymbol{\alpha}]$ such that $\delta^k : [n-1] \to [n]$ does not factor through α. Then clearly $X_2 \subset \Psi^k[n; \mathbf{q}]$. Note that a non-degenerate $(m; \boldsymbol{p})$-cell $[\alpha; \boldsymbol{\alpha}]$ in $\Phi^k[n; \mathbf{q}]$ is contained in $X_2 \setminus X_1$ if and only if it satisfies:

(2a) $\alpha(0) \le k-1$;

(2b) $\alpha(m) > k$;

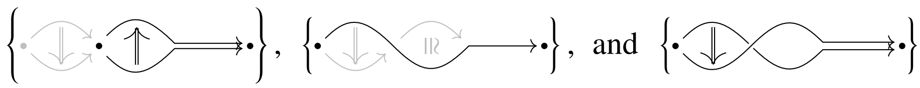
(2c) $\delta^k \ne \alpha \ne \mathrm{id}$; and

(2d) $\blacklozenge \in \operatorname{im} \alpha_k$.

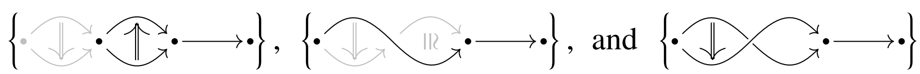
Observe that for any such $[\alpha; \boldsymbol{\alpha}]$, either it additionally satisfies

(2e) there exists $1 \le \ell_\alpha \le m-1$ such that $\alpha(\ell_\alpha) = k$

or there is a unique $(m'; \mathbf{p}')$-cell $[\beta; \boldsymbol{\beta}]$ in $X_2 \setminus X_1$ satisfying (2a-e) such that $[\alpha; \boldsymbol{\alpha}]$ is an ℓ_β-th horizontal face of $[\beta; \boldsymbol{\beta}]$. *e.g.*



are 1st horizontal faces of



respectively.

Lemma 3.1.15. *The inclusion* $X_1 \hookrightarrow X_2$ *is in* $\mathrm{cell}(\mathcal{H}_h)$.

Proof. The discussion above shows that the set of non-degenerate cells in $X_2 \setminus X_1$ can be partitioned into subsets of the form

$$\big\{[\alpha; \boldsymbol{\alpha}] \text{ and all of its } \ell_\alpha\text{-th horizontal faces}\big\}$$

where $[\alpha; \boldsymbol{\alpha}]$ is an $(m; \boldsymbol{p})$-cell satisfying (2a-e). We prove that X_2 may be obtained from X_1 by gluing such $[\alpha; \boldsymbol{\alpha}]$ along the horizontal horn $\Lambda_h^{\ell_\alpha}[m; \mathbf{p}]$ in increasing order of $\dim[m; \mathbf{p}]$. Note that this horn is inner by (2e).

Fix a non-degenerate $(m; \boldsymbol{p})$-cell $[\alpha; \boldsymbol{\alpha}]$ satisfying (2a-e). Similarly to the proof of Lemma 3.1.14, we must check that the following faces of $[\alpha; \boldsymbol{\alpha}]$ are contained either in X_1 or in some cell that satisfies (2a-e) and has dimension strictly smaller than $\dim[m; \mathbf{p}]$:

- $[\alpha; \boldsymbol{\alpha}] \cdot \delta_h^0$:
  - is contained in $\Theta_2[n; \mathbf{q}]$ if $\alpha(1) = k$; and
  - satisfies (2a-e) otherwise;
- $[\alpha; \boldsymbol{\alpha}] \cdot \delta_h^m$:
  - is contained in X_1 if $\alpha(m-1) = k$; and

- satisfies (2a-e) otherwise;

- $[\alpha;\boldsymbol{\alpha}]\cdot\delta_h^{\ell;\langle\gamma,\gamma'\rangle}$ satisfies (2a-e) for any $\ell\neq\ell_\alpha$ and for any $\langle\gamma,\gamma'\rangle$;
- $[\alpha;\boldsymbol{\alpha}]\cdot\delta_v^{\ell;j}$ satisfies (2a-e) for any $\ell\neq\ell_\alpha$ and for any $j\in[p_\ell]$; and
- $[\alpha;\boldsymbol{\alpha}]\cdot\delta_v^{\ell_\alpha;j}$, for any $j\in[p_\ell]$, is:
  - contained in $\Theta_2[n;\mathbf{q}]$ if $\alpha_k^{-1}(\blacklozenge)=\{j\}$; and
  - a (possibly trivial) degeneracy of some cell that satisfies (2a-e) otherwise.

This completes the proof. □

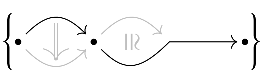
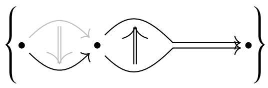
Now let $X_3\subset\Phi^k[n;\mathbf{q}]$ be the cellular subset generated by X_2 and those cells $[\alpha;\boldsymbol{\alpha}]$ such that α_ℓ is not surjective for some $\ell\neq k$. Then clearly $X_3\subset\Psi^k[n;\mathbf{q}]$. Note that a non-degenerate $(m;\boldsymbol{p})$-cell $[\alpha;\boldsymbol{\alpha}]$ in $\Phi^k[n;\mathbf{q}]$ is contained in $X_3\setminus X_2$ if and only if it satisfies:

(3a) $\alpha=\delta^k$ or $\alpha=\mathrm{id}$;

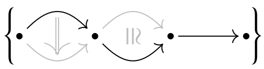
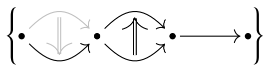
(3b) $\blacklozenge\in\operatorname{im}\alpha_k$; and

(3c) there exists $1\leq\ell\leq n$ such that $\ell\neq k$ and α_ℓ is not surjective.

Observe that if $[\delta^k;\boldsymbol{\alpha}]$ satisfies (3b) and (3c), then there is a unique cell $[\mathrm{id};\boldsymbol{\beta}]$ satisfying (3b) and (3c) such that $[\delta^k;\boldsymbol{\alpha}]$ is a k-th horizontal face of $[\mathrm{id};\boldsymbol{\beta}]$. *e.g.*

 and 

are 2nd horizontal faces of

 and 

respectively.

Lemma 3.1.16. *The inclusion $X_2\hookrightarrow X_3$ is in* cell($\mathcal{H}_h$).

Proof. The discussion above shows that the set of non-degenerate cells in $X_3\setminus X_2$ can be partitioned into subsets of the form

$$\big\{[\mathrm{id};\boldsymbol{\alpha}]\text{ and all of its }k\text{-th horizontal faces}\big\}$$

where $[\mathrm{id};\boldsymbol{\alpha}]$ is an $(n;\boldsymbol{p})$-cell satisfying (3b) and (3c). We prove that X_3 may be obtained from X_2 by gluing such $[\mathrm{id};\boldsymbol{\alpha}]$ along the horizontal horn $\Lambda_h^k[n;\mathbf{p}]$ in increasing order of $\dim[n;\mathbf{p}]$. Note that this horn is inner since we are assuming $1\leq k\leq n-1$.

Fix a non-degenerate $(n;\boldsymbol{p})$-cell $[\mathrm{id};\boldsymbol{\alpha}]$ satisfying (3b) and (3c). We must check that the following faces of $[\mathrm{id};\boldsymbol{\alpha}]$ are contained either in X_2 or some $[\mathrm{id};\boldsymbol{\beta}]$ that satisfies (3b) and (3c) and has dimension strictly smaller than $\dim[n;\mathbf{p}]$:

- $[\mathrm{id};\boldsymbol{\alpha}]\cdot\delta_h^0$ is contained in X_2;
- $[\mathrm{id};\boldsymbol{\alpha}]\cdot\delta_h^n$ is contained in X_2;
- $[\mathrm{id};\boldsymbol{\alpha}]\cdot\delta_h^{\ell;\langle\gamma,\gamma'\rangle}$ is contained in X_2 for any $\ell\neq k$ and for any $\langle\gamma,\gamma'\rangle$;

- $[\mathrm{id};\boldsymbol{\alpha}]\cdot\delta_v^{\ell;j}$ satisfies (3a-c) for any $\ell \neq k$ and for any $j \in [p_\ell]$; and
- $[\mathrm{id};\boldsymbol{\alpha}]\cdot\delta_v^{k;j}$ is:
  - contained in $\Theta_2[n;\mathbf{q}]$ if $\alpha_k^{-1}(\blacklozenge) = \{j\}$; and
  - a (possibly trivial) degeneracy of some cell that satisfies (3a-c) otherwise.

This completes the proof. □

Observe that the non-degenerate cells in $\Psi^k[n;\mathbf{q}] \setminus X_3$ are precisely those

$$[\delta^k;\boldsymbol{\alpha}] : \Theta_2[n-1;\boldsymbol{p}] \to \Phi^k[n;\mathbf{q}]$$

such that:

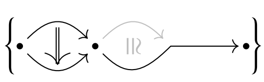
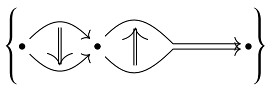
(4a) $\alpha_\ell = \mathrm{id}$ for $k \neq \ell \neq k+1$;

(4b) α_{k+1} is surjective;

(4c) $\blacklozenge \in \mathrm{im}\,\alpha_k$; and

(4d) $\langle \alpha_k, \alpha_{k+1}\rangle : \Delta[p_k] \to J \times \Delta[q_{k+1}]$ is non-degenerate.

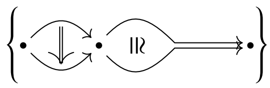
For our example $\Psi^2[3;1,0,0]$, these faces include

 and  .

In fact, there is a map

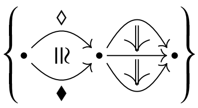
$$\Phi^2[2;1,0] \to \Phi^3[2;1,0,0]$$

which looks like



and $\Psi^2[3;1,0,0] \setminus X_3$ is precisely the image of $\Phi^2[2;1,0] \setminus \Psi^2[2;1,0]$ under this map.

The above observation can be generalised to include all cases where $q_{k+1} = 0$. However it does not hold if $q_{k+1} \geq 1$, *e.g.* $[n;\mathbf{q}] = [2;0,2]$ (with $k=1$) in which case $\Phi^k[n;\mathbf{q}]$ looks like

$\Phi^1[2;0,2] =$  .

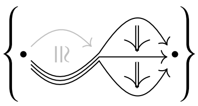
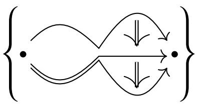
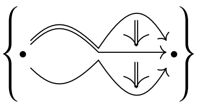
In this case, given a non-degenerate cell $[\delta^k;\boldsymbol{\alpha}]$ satisfying (4a-d), let $i_\alpha = \alpha_{k+1}\big(\min\big(\alpha_k^{-1}(\blacklozenge)\big)\big)$ and

$$j_\alpha = \begin{cases} \min\big(\alpha_{k+1}^{-1}(i_\alpha)\big) & \text{if } i_\alpha \geq 1, \\ \max\big(\alpha_{k+1}^{-1}(0)\big) & \text{if } i_\alpha = 0. \end{cases}$$

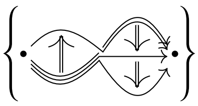
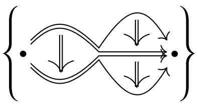
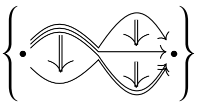
Then the cell $[\delta^k;\boldsymbol{\alpha}]$ either satisfies

(4e) $\alpha_k(j_\alpha) = \lozenge$

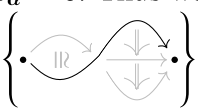
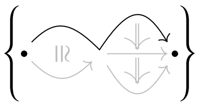
or there is a unique $(n-1;\mathbf{p}')$-cell $[\delta^k;\boldsymbol{\beta}]$ in $\Psi^k[n;\mathbf{q}] \setminus X_3$ satisfying (4a-e) such that $[\delta^k;\boldsymbol{\alpha}]$ is the $(k;j_{\boldsymbol{\beta}})$-th vertical hyperface of $[\delta^k;\boldsymbol{\beta}]$. *e.g.*

, , and 

have $i_{\boldsymbol{\alpha}} = 0, 1$, and 2 respectively, and they are moreover the appropriate vertical hyperfaces of

, , and 

respectively.

The motivation behind the definitions of $i_{\boldsymbol{\alpha}}$ and $j_{\boldsymbol{\alpha}}$ is as follows. Ideally, we would like to simply say "the first 1-cell involving $\blacklozenge$ is preceded by an otherwise identical 1-cell that involves $\lozenge$" in (4e) and use this extra $\lozenge$ to identify the interior/face pairs for the inner horns to be filled. However, this horn is outer if (and only if) $i_{\boldsymbol{\alpha}} = 0$. Thus we define $j_{\boldsymbol{\alpha}}$ differently in this case so that (4e) says "the *last* 1-cell of the form  is *followed* by one of the form " instead.

Lemma 3.1.17. *The inclusion* $X_3 \hookrightarrow \Phi^k[n;\mathbf{q}]$ *is in* $\mathrm{cell}(\mathcal{H}_h)$. *The inclusion* $X_3 \hookrightarrow \Psi^k[n;\mathbf{q}]$ *is:*

- *a pushout of* $\Psi^\ell[n-1;\boldsymbol{p}] \hookrightarrow \Phi^\ell[n-1;\boldsymbol{p}]$ *for some* $[n-1;\boldsymbol{p}] \in \Theta_2$ *with* $\dim[n-1;\boldsymbol{p}] < \dim[n;\mathbf{q}]$ *if* $q_{k+1} = 0$*; and*
- *in* $\mathrm{cell}(\mathcal{H}_v)$ *if* $q_{k+1} \geq 1$.

Proof. For the inclusion $X_3 \hookrightarrow \Phi^k[n;\mathbf{q}]$, we can simply continue gluing the remaining cells $[\mathrm{id};\boldsymbol{\alpha}] : \Theta_2[n;\mathbf{p}] \to \Phi^k[n;\mathbf{q}]$ satisfying (3b) (but not (3c)) along $\Lambda^k_h[n;\mathbf{p}]$ in increasing order of $\dim[n;\mathbf{p}]$.

Consider the inclusion $X_3 \hookrightarrow \Psi^k[n;\mathbf{q}]$. For the case $q_{k+1} = 0$, recall that the functor $\widehat{\Delta} \wr \widehat{\Delta} \to \widehat{\Delta}$ is a (split) cartesian fibration. Thus there is a cartesian lift of the map $\delta^k : \Delta[n-1] \to \Delta[n]$ at the object

$$\big(\Delta[q_1], \dots, \Delta[q_{k-1}], J, \Delta[q_{k+1}], \dots, \Delta[q_n]\big) \in \widehat{\Delta}^n \simeq \big(\widehat{\Delta} \wr \widehat{\Delta}\big)_{\Delta[n]}.$$

Applying the box product functor $\square$ to this lift yields a map

$$[\delta^k; \mathrm{id}, \dots, !, \dots, \mathrm{id}] : \Phi^k[n-1;\boldsymbol{p}] \to \Phi^k[n;\mathbf{q}]$$

where $\boldsymbol{p} = (q_1, \dots, q_{k-1}, 0, q_{k+2}, \dots, q_n)$. This map factors through $\Psi^k[n;\mathbf{q}]$ because its image is generated by the cells of the form $[\delta^k;\boldsymbol{\alpha}]$. Moreover, one can check by comparing (4a-d) and (∗∗) (the latter of which appeared in the second paragraph after the proof of Lemma 3.1.12) that this map fits into the following gluing square:

$$\begin{tikzcd}
\Psi^k[n-1;\boldsymbol{p}] \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi^k[n-1;\boldsymbol{p}] \arrow[d, "{[\delta^k;\mathrm{id},\dots,!,\dots,\mathrm{id}]}"] \arrow[dl, phantom, "\ulcorner", very near end] \\
X_3 \arrow[r, hook, "\subset"] & \Psi^k[n;\mathbf{q}]
\end{tikzcd}$$

This completes the proof for the first case.

Next consider the case $q_{k+1} \geq 2$. The discussion before Lemma 3.1.17 shows that the set of non-degenerate cells in $\Psi^k[n;\mathbf{q}] \setminus X_3$ can be partitioned into subsets of the form

$$\left\{[\delta^k;\boldsymbol{\alpha}], [\delta^k;\boldsymbol{\alpha}] \cdot \delta_v^{k;j_\alpha}\right\}$$

where $[\delta^k;\boldsymbol{\alpha}]$ is an $(n-1;\boldsymbol{p})$-cell satisfying (4a-e). We prove that $\Psi^k[n;\mathbf{q}]$ may be obtained from X_3 by gluing such $[\delta^k\boldsymbol{\alpha}]$ along the vertical horn $\Lambda_v^{k;j_\alpha}[n-1;\boldsymbol{p}]$ in lexicographically increasing order of $\dim[n-1;\boldsymbol{p}]$ and $|\alpha_k^{-1}(\blacklozenge)|$. Note that this horn is inner by the definition of j_α.

Fix a non-degenerate $(n-1;\boldsymbol{p})$-cell $[\delta^k;\boldsymbol{\alpha}]$ satisfying (4a-e). We must check that the appropriate faces of $[\delta^k;\boldsymbol{\alpha}]$ are contained either in X_3 or in some $(n-1;\mathbf{p}')$-cell $[\delta^k;\boldsymbol{\beta}]$ satisfying (4a-e) such that:

- $\dim[n-1;\mathbf{p}'] < \dim[n-1;\boldsymbol{p}]$; or
- $\dim[n-1;\mathbf{p}'] \leq \dim[n-1;\boldsymbol{p}]$ and $|\beta_k^{-1}(\blacklozenge)| < |\alpha_k^{-1}(\blacklozenge)|$.

If $i_\alpha \geq 1$:

- any horizontal hyperface of $[\delta^k;\boldsymbol{\alpha}]$ is contained in X_2;
- $[\delta^k;\boldsymbol{\alpha}] \cdot \delta_v^{\ell;j}$ is contained in X_3 for any $\ell \neq k$ and for any $j \in [p_\ell]$;
- $[\delta^k;\boldsymbol{\alpha}] \cdot \delta_v^{k;j}$, where $j_\alpha \neq j \neq j_\alpha + 1$, is:
  - contained in X_3 if $\alpha_{k+1} \cdot \delta^j$ is not surjective; and
  - a (possibly trivial) degeneracy of some cell that satisfies (4a-e) otherwise; and
- $[\delta^k;\boldsymbol{\alpha}] \cdot \delta_v^{k;j_\alpha+1}$ is:
  - contained in $\Theta_2[n;\mathbf{q}]$ if $\alpha_k(j) = \lozenge$ for all $j \neq j_\alpha + 1$; and
  - a (possibly trivial) degeneracy of some $(n-1;\mathbf{p}')$-cell $[\delta^k;\boldsymbol{\beta}]$ that satisfies (4a-d) otherwise.

Note in the last clause, the cell $[\delta^k;\boldsymbol{\beta}]$ may not satisfy (4e). However, at least we know $\dim[n-1;\mathbf{p}'] < \dim[n-1;\boldsymbol{p}]$ and $|\beta_k^{-1}(\blacklozenge)| < |\alpha_k^{-1}(\blacklozenge)|$. Hence if $[\delta^k;\boldsymbol{\gamma}]$ is an $(n-1;\mathbf{p}'')$-cell satisfying (4a-e) such that $[\delta^k;\boldsymbol{\beta}] = [\delta^k;\boldsymbol{\gamma}] \cdot \delta_v^{k;j_\gamma}$ then

$$\dim[n-1;\mathbf{p}''] = \dim[n-1;\mathbf{p}'] + 1 \leq \dim[n-1;\boldsymbol{p}]$$

and

$$|\gamma_k^{-1}(\blacklozenge)| = |\beta_k^{-1}(\blacklozenge)| < |\alpha_k^{-1}(\blacklozenge)|.$$

A similar analysis can be done for the case $i_\alpha = 0$ too, and this completes the proof. □

3.2 Alternative horizontal horns

We now consider a slightly different set of horn inclusions.

Definition 3.2.1. Given $[n;\mathbf{q}] \in \Theta_2$, $1 \leq k \leq n-1$ and a (q_k, q_{k+1})-shuffle $\langle \alpha, \alpha' \rangle$, we write $\Lambda_h^{k;\langle \alpha,\alpha' \rangle}[n;\mathbf{q}] \subset \Theta_2[n;\mathbf{q}]$ for the cellular subset generated by all hyperfaces of $\Theta_2[n;\mathbf{q}]$ except for $\delta_h^{k;\langle \alpha,\alpha' \rangle}$. We denote by $\mathcal{H}_h'$ the set of all such *alternative inner horizontal horn inclusions* $\Lambda_h^{k;\langle \alpha,\alpha' \rangle}[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$.

We prove that $\mathcal{H}_h \subset \mathrm{cell}(\mathcal{H}_h' \cup \mathcal{H}_v)$ holds and that $\mathcal{H}_h'$ is contained in the class of trivial cofibrations.

3.2.1 Oury's horn inclusions can be obtained from the alternative ones

The purpose of this subsection is to prove the following lemma.

Lemma 3.2.2. *Every map in $\mathcal{H}_h$ is contained in* $\mathrm{cell}(\mathcal{H}_h' \cup \mathcal{H}_v)$.

Similarly to the proof of Lemma 3.1.4, we must consider a wider class of horn inclusions.

Definition 3.2.3. Given a set S of hyperfaces of $\Theta_2[n;\mathbf{q}]$, let $\Lambda^S[n;\mathbf{q}] \subset \Theta_2[n;\mathbf{q}]$ denote the cellular subset generated by all hyperfaces except for those in S.

Proposition 3.2.4. *For any set S of inner hyperfaces of $\Theta_2[n;\mathbf{q}]$, the cellular subset $\Lambda^S[n;\mathbf{q}] \subset \Theta_2[n;\mathbf{q}]$ is equal to $\Upsilon^T[n;\mathbf{q}]$ where T is the set of all inner hyperfaces of $\Theta_2[n;\mathbf{q}]$ that are not in S.*

Proof. Compare Definitions 3.1.6 and 3.2.3. □

Recall that if S is a set of faces of $\Theta_2[n;\mathbf{q}]$ and $1 \leq k \leq n-1$ then we write

$$\mathbf{Shfl}_S(q_k, q_{k+1}) = \left\{ \langle \alpha, \alpha' \rangle \in (q_k, q_{k+1}) : \delta_h^{k;\langle \alpha,\alpha' \rangle} \in S \right\}.$$

Lemma 3.2.5. *The inclusion $\Lambda^S[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ is contained in:*

(i) $\mathrm{cell}(\mathcal{H}_v)$ *if S is a non-empty set of k-th vertical hyperfaces for some $1 \leq k \leq n$; and*

(ii) $\mathrm{cell}(\mathcal{H}_h' \cup \mathcal{H}_v)$ *if S is a non-empty set of k-th horizontal hyperfaces for some $1 \leq k \leq n-1$ and $\mathbf{Shfl}_S(q_k, q_{k+1})$ is upward closed.*

Note that Lemma 3.2.2 follows from Lemma 3.2.5(ii) by setting S to be the set of all k-th horizontal hyperfaces of $\Theta_2[n;\mathbf{q}]$.

Proof. We will prove (i) by induction on $|S|$. By assumption, we can write S as

$$S = \left\{ \delta_v^{k;i} : i \in I_S \right\}$$

for some $1 \leq k \leq n$ and $\varnothing \neq I_S \subset \{1, \ldots, q_k - 1\}$. If $I_S = \{i\}$ is a singleton, then $\Lambda^S[n;\mathbf{q}] = \Lambda_v^{k;i}[n;\mathbf{q}]$ and hence the result follows trivially. So assume $|S| \geq 2$. Choose $i \in I_S$

and let $S' = \{\delta_v^{k;j} : j \in I_S \setminus \{i\}\}$. Then $\Lambda^{S'}[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ is in cell($\mathcal{H}_v$) by the inductive hypothesis. Therefore it suffices to prove that the upper horizontal map in the gluing square

$$\begin{tikzcd} X \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near end] & \Theta_2[n;\boldsymbol{p}] \arrow[d, "\delta_v^{k;i}"] \\ \Lambda^S[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Lambda^{S'}[n;\mathbf{q}] \end{tikzcd}$$

belongs to cell($\mathcal{H}_v$), where $\boldsymbol{p} = (q_1, \ldots, q_k - 1, \ldots, q_n)$. Indeed, one can check that $X = \Lambda^T[n;\boldsymbol{p}]$ where

$$T = \{\delta_v^{k;j} : j \in I_S,\ j < i\} \cup \{\delta_v^{k;j-1} : j \in I_S,\ j > i\}.$$

Since $|T| = |S| - 1$, the desired inclusion is in cell($\mathcal{H}_v$) by the inductive hypothesis.

Now we prove (ii) by induction on $|S|$. If $S = \{\delta_h^{k;\langle\alpha,\alpha'\rangle}\}$ is a singleton then $\Lambda^S[n;\mathbf{q}] = \Lambda_h^{k;\langle\alpha,\alpha'\rangle}[n;\mathbf{q}]$ and hence the result follows trivially. So assume $|S| \geq 2$. Choose a minimal element $\langle\alpha,\alpha'\rangle \in \mathbf{Shfl}_S(q_k, q_{k+1})$ and let $S' = S \setminus \{\delta_h^{k;\langle\alpha,\alpha'\rangle}\}$. Then by the inductive hypothesis, $\Lambda^{S'}[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ is in cell($\mathcal{H}_h' \cup \mathcal{H}_v$). Thus it suffices to prove that $\Lambda^S[n;\mathbf{q}] \hookrightarrow \Lambda^{S'}[n;\mathbf{q}]$ too is in cell($\mathcal{H}_h' \cup \mathcal{H}_v$). Indeed, it follows from Claims 0 to 4 in the proof of Lemma 3.1.11 that we have a gluing square

$$\begin{tikzcd} \Lambda^T[n-1;\boldsymbol{p}] \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\ulcorner", very near end] & \Theta_2[n-1;\boldsymbol{p}] \arrow[d, "\delta_h^{k;\langle\alpha,\alpha'\rangle}"] \\ \Lambda^S[n;\mathbf{q}] \arrow[r, hook, "\subset"] & \Lambda^{S'}[n;\mathbf{q}] \end{tikzcd}$$

where $\boldsymbol{p} = (q_1, \ldots, q_k + q_{k+1}, \ldots, q_n)$, and $T = \{\delta_v^{k;i} : i \in {}^\ulcorner\langle\alpha,\alpha'\rangle\}$. (In fact, this square is essentially the first square that appears in the proof of Lemma 3.1.11.) Note that ${}^\ulcorner\langle\alpha,\alpha'\rangle = \varnothing$ if and only if $\langle\alpha,\alpha'\rangle$ is the maximum (q_k, q_{k+1})-shuffle, but the latter is impossible since $|S| \geq 2$ and $\langle\alpha,\alpha'\rangle$ is minimal in S. Hence $\Lambda^T[n-1;\boldsymbol{p}] \hookrightarrow \Theta_2[n-1;\boldsymbol{p}]$ is in cell($\mathcal{H}_v$) by (i). □

3.2.2 Alternative horn inclusions are trivial cofibrations

The purpose of this subsection is to prove the following lemma.

Lemma 3.2.6. *Every map in $\mathcal{H}_h'$ is a trivial cofibration.*

Once again, we consider a wider class of horn inclusions. Suppose we have fixed $[n;\mathbf{q}] \in \Theta_2$, $1 \leq k \leq n-1$ and a (q_k, q_{k+1})-shuffle $\langle\zeta,\zeta'\rangle$. (Note that the inequality $1 \leq n-1$ in particular implies that $[n;\mathbf{q}]$ is poly-vertebral.) Let

$$I = \{\langle\alpha,\alpha'\rangle \in \mathbf{Shfl}(q_k, q_{k+1}) : \langle\alpha,\alpha'\rangle \leq \langle\zeta,\zeta'\rangle\}.$$

Lemma 3.2.7. *If U is a set of the form*

$$U = \left\{\delta_h^{k;\langle\alpha,\alpha'\rangle} : \langle\alpha,\alpha'\rangle \in I_U\right\}$$

for some non-empty, upward closed subset $I_U \subset I$, then the inclusion $\Lambda^U[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]$ is a trivial cofibration.

Here I_U is upward closed in I but not necessarily in $\mathbf{Shfl}(q_k, q_{k+1})$. Thus in general $\mathbf{Shfl}(q_k, q_{k+1}) \setminus I_U$ is not downward closed, and this is why Lemma 3.2.7 does not follow directly from Proposition 3.2.4 and Lemma 3.1.11.

Note that since $\langle \zeta, \zeta' \rangle$ is the maximum element in I, any non-empty, upward closed $I_U \subset I$ will always have $\langle \zeta, \zeta' \rangle \in I_U$. Also observe that Lemma 3.2.6 follows from Lemma 3.2.7 by setting $U = \{\delta_h^{k;\langle \zeta, \zeta' \rangle}\}$.

Proof. We prove Lemma 3.2.7 by induction on $|I \setminus I_U|$ (so we start with the case $I_U = I$ and progressively make I_U smaller). For the base case, observe that

$$\{\langle \alpha, \alpha' \rangle \in \mathbf{Shfl}(q_k, q_{k+1}) : \langle \alpha, \alpha' \rangle \nleq \langle \zeta, \zeta' \rangle\}$$

is an upward closed, proper subset of $\mathbf{Shfl}(q_k, q_{k+1})$. Thus, when $I_U = I$ the inclusion $\Lambda^U[n; \mathbf{q}] \hookrightarrow \Theta_2[n; \mathbf{q}]$ is a trivial cofibration by Proposition 3.2.4 and the dual of Lemma 3.1.11.

For the inductive step, assume $I_U \neq I$. Choose a maximal element $\langle \alpha, \alpha' \rangle \in I \setminus I_U$ and let $I_{U'} = I_U \cup \{\langle \alpha, \alpha' \rangle\}$. Then $\Lambda^{U'}[n; \mathbf{q}] \hookrightarrow \Theta_2[n; \mathbf{q}]$ is a trivial cofibration by the inductive hypothesis, and hence it suffices to show the upper horizontal map in

$$\begin{tikzcd} X \arrow[r, hook, "\subset"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Theta_2[n-1; \boldsymbol{p}] \arrow[d, "\delta_h^{k;\langle \alpha, \alpha' \rangle}"] \\ \Lambda^{U'}[n; \mathbf{q}] \arrow[r, hook, "\subset"] \arrow[ur, phantom, "\ulcorner", very near end] & \Lambda^U[n; \mathbf{q}] \end{tikzcd}$$

(where $\boldsymbol{p} = (q_1, \ldots, q_k + q_{k+1}, \ldots, q_n)$) is a trivial cofibration. We again use Lemma 3.1.11. More precisely, we claim that X has the form $X = \Upsilon^T[n-1; \boldsymbol{p}]$ for

$$T = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_4' \cup T_5$$

where

$$\begin{aligned} T_1 &= \left\{\delta_h^{\ell;\langle \gamma, \gamma' \rangle}[n-1; \boldsymbol{p}] : 1 \leq \ell \leq n-1,\ \langle \gamma, \gamma' \rangle \in \mathbf{Shfl}(p_\ell, p_{\ell+1})\right\}, \\ T_2 &= \left\{\delta_v^{k;j}[n-1; \boldsymbol{p}] : j \in \lrcorner\langle \alpha, \alpha' \rangle\right\}, \\ T_3 &= \left\{\delta_v^{\ell;j}[n-1; \boldsymbol{p}] : \ell \neq k,\ 1 \leq j \leq q_\ell - 1\right\}, \\ T_4 &= \left\{\delta_v^{k;j}[n-1; \boldsymbol{p}] : (\exists i \in [q_k]) \left[1 \leq i \leq q_k - 1,\ \alpha^{-1}(i) = \{j\}\right]\right\}, \\ T_4' &= \left\{\delta_v^{k;j}[n-1; \boldsymbol{p}] : (\exists i \in [q_{k+1}]) \left[1 \leq i \leq q_{k+1} - 1,\ (\alpha')^{-1}(i) = \{j\}\right]\right\}, \\ T_5 &= \left\{\delta_v^{k;j}[n-1; \boldsymbol{p}] : j \in \ulcorner\langle \alpha, \alpha' \rangle,\ \alpha(j) = \zeta(j)\right\}. \end{aligned}$$

Aside from T_5, these sets are essentially special cases of the sets with the same names in the proof of Lemma 3.1.11. More precisely, we have set S' to be the set of inner hyperfaces of $\Theta_2[n; \mathbf{q}]$ that are not in U', then merged T_1 and T_1' into a single set and similarly for T_3, and unwound the conditions involving S'. Thus for much of the proof that $X = \Upsilon^T[n-1; \boldsymbol{p}]$ holds, we can reuse Claims 0 to 4 from the proof of Lemma 3.1.11.

The following claim relates the hyperfaces $\delta_h^{k;\langle \beta, \beta' \rangle}$ of $\Theta_2[n; \mathbf{q}]$ with $\langle \beta, \beta' \rangle \notin I$ to the elements of T_5. Note that if $1 \leq j \leq p_k - 1$ and $j \notin \ulcorner\langle \alpha, \alpha' \rangle$ then Proposition 2.1.5 implies that $\delta_v^{k;j}[n-1; \boldsymbol{p}]$ is contained in T_2, T_4 or T_4'.

Claim 5.

(i) For any $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_k,q_{k+1})$ with $\langle\beta,\beta'\rangle \not\leq \langle\zeta,\zeta'\rangle$, the pullback of $\delta_h^{k;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is contained in $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ for some $1 \leq j \leq p_k-1$ such that either $j \notin {}^\ulcorner\langle\alpha,\alpha'\rangle$ or $\alpha(j) = \zeta(j)$.

(ii) For any $j \in {}^\ulcorner\langle\alpha,\alpha'\rangle$ with $\alpha(j) = \zeta(j)$, the hyperface $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ is the pullback of $\delta_h^{k;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ for some $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_k,q_{k+1})$ with $\langle\beta,\beta'\rangle \not\leq \langle\zeta,\zeta'\rangle$.

Proof. For (i), fix $\langle\beta,\beta'\rangle \in \mathbf{Shfl}(q_k,q_{k+1})$ with $\langle\beta,\beta'\rangle \not\leq \langle\zeta,\zeta'\rangle$. Note that if $\beta(j) \neq \alpha(j)$ for some $j \in [q_k+q_{k+1}]$, then the pullback of $\delta_h^{k;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$ is contained in $\delta_v^{k;j}[n-1;\boldsymbol{p}]$. Thus it suffices to prove that there exists some $j \in [q_k+q_{k+1}]$ such that $\beta(j) \neq \alpha(j)$ and either $j \notin {}^\ulcorner\langle\alpha,\alpha'\rangle$ or $\alpha(j) = \zeta(j)$.

Suppose otherwise. Then in particular $\beta(j) = \alpha(j)$ for all $j \in [q_k+q_{k+1}] \setminus {}^\ulcorner\langle\alpha,\alpha'\rangle$. Since ${}^\ulcorner\langle\alpha,\alpha'\rangle$ contains no two consecutive integers, it follows that $\beta(j) \neq \alpha(j)$ implies $\beta(j) = \alpha(j)+1$ for any $j \in [q_k+q_{k+1}]$. Now for each $j \in [q_k+q_{k+1}]$:

- if $\beta(j) = \alpha(j)$ then $\beta(j) \leq \zeta(j)$ because $\langle\alpha,\alpha'\rangle < \langle\zeta,\zeta'\rangle$; and
- if $\beta(j) \neq \alpha(j)$, then $\alpha(j) < \zeta(j)$ by assumption and hence $\beta(j) = \alpha(j)+1 \leq \zeta(j)$.

Therefore we have $\langle\beta,\beta'\rangle \leq \langle\zeta,\zeta'\rangle$, which is the desired contradiction.

To prove (ii), let $j \in {}^\ulcorner\langle\alpha,\alpha'\rangle$ and suppose $\alpha(j) = \zeta(j)$. Then $\langle\beta,\beta'\rangle \not\leq \langle\zeta,\zeta'\rangle$ for the immediate successor $\langle\beta,\beta'\rangle$ of $\langle\alpha,\alpha'\rangle$ corresponding to j, and $\delta_v^{k;j}[n-1;\boldsymbol{p}]$ is the pullback of $\delta_h^{k;\langle\beta,\beta'\rangle}$ along $\delta_h^{k;\langle\alpha,\alpha'\rangle}$. □

We can deduce $X = \Upsilon^T[n-1;\boldsymbol{p}]$ from Claims 0 to 5, and it remains to check that T is admissible, *i.e.* T satisfies Definition 3.1.10(i-iii). Since it contains all of the inner horizontal hyperfaces of $\Theta_2[n-1;\boldsymbol{p}]$, T clearly satisfies (ii) and (iii). For (i), observe that $\langle\alpha,\alpha'\rangle < \langle\zeta,\zeta'\rangle$ implies there exists an immediate successor $\langle\beta,\beta'\rangle$ of $\langle\alpha,\alpha'\rangle$ such that $\langle\beta,\beta'\rangle \leq \langle\zeta,\zeta'\rangle$. If $j \in {}^\ulcorner\langle\alpha,\alpha'\rangle$ is the element corresponding to $\langle\beta,\beta'\rangle$, then $\alpha(j) = \beta(j)-1 < \zeta(j)$ and so T does not contain $\delta_v^{k;j}[n-1;\boldsymbol{p}]$. Therefore T is not the set of all inner hyperfaces of $\Theta_2[n-1;\boldsymbol{p}]$. □

3.3 Most horizontal equivalence extensions are redundant

The aim of this very short section is to prove the following lemma.

Lemma 3.3.1. *For any* $[0] \neq [n;\mathbf{q}] \in \Theta_2$*, the horizontal equivalence extension*

$$(\Theta_2[0] \overset{e}{\hookrightarrow} J)\hat{\times}(\partial\Theta_2[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}])$$

is contained in $\mathrm{cell}(\mathcal{H}_h)$*.*

Proof. Fix $[0] \neq [n;\mathbf{q}] \in \Theta_2$ and consider $e\hat{\times}(\partial\Theta_2[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}])$, whose domain we denote by X. Let $Y \subset J \times \Theta_2[n;\mathbf{q}]$ be the cellular subset generated by X and all cells that do not contain the vertex $(\blacklozenge,n)$. Then for any non-degenerate $\phi : \Theta_2[m;\mathbf{p}] \to Y$ that does not factor through X, there is unique $1 \leq k_\phi \leq m$ such that $\pi_1 \circ \phi(k_\phi - 1) = \blacklozenge$ and $\pi_1 \circ \phi(k) = \lozenge$ for all $k_\phi \leq k \leq m$, where $\pi_1 : J \times \Theta_2[n;\mathbf{q}] \to J$ is the projection. Observe that for any non-degenerate ϕ in $Y \setminus X$, either ϕ satisfies

(†) $\pi_2 \circ \phi(k_\phi) = \pi_2 \circ \phi(k_\phi - 1)$

or there is a unique non-degenerate cell ψ in $Y \setminus X$ satisfying (†) such that ϕ is a (unique) k_ψ-th horizontal hyperface of ψ. Therefore the non-degenerate cells in $Y \setminus X$ can be partitioned into pairs of the form

$$\{\phi, \phi \cdot \delta_h^{k_\alpha;\langle !,\mathrm{id}\rangle}\}$$

where ϕ is an $(m;\boldsymbol{p})$-cell satisfying (†) (which necessarily has $p_{k_\phi} = 0$). We prove that Y may be obtained from X by gluing such ϕ along the horn $\Lambda_h^{k_\phi}[m;\mathbf{p}]$ in lexicographically increasing order of $\dim[m;\mathbf{p}]$ and $\left|(\pi_1 \circ \phi)^{-1}(\blacklozenge)\right|$. (Here $|-|$ counts the number of objects.)

Fix an $(m;\boldsymbol{p})$-cell ϕ in $Y \setminus X$ satisfying (†). We must check that all hyperfaces of ϕ except for the (unique) k_α-th horizontal one are contained either in X or in some $(m';\mathbf{p}')$-cell ψ satisfying (†) such that either:

- $\dim[m';\mathbf{p}'] < \dim[m;\mathbf{p}]$; or
- $\dim[m';\mathbf{p}'] = \dim[m;\mathbf{p}]$ and $\left|(\pi_1 \circ \psi)^{-1}(\blacklozenge)\right| < \left|(\pi_1 \circ \phi)^{-1}(\blacklozenge)\right|$.

Indeed:

- $\phi \cdot \delta_h^{k_\phi - 1;\langle \mathrm{id},!\rangle}$ may or may not satisfy (†), but we know
$$\left|\left(\pi_1 \circ (\phi \cdot \delta^{k_\phi - 1})\right)^{-1}(\blacklozenge)\right| = \left|(\pi_1 \circ \phi)^{-1}(\blacklozenge)\right| - 1;$$
- for any $0 \le k < k_\phi - 1$ with
$$\left|(\pi_2 \circ \phi)^{-1}(\pi_2 \circ \phi(k))\right| \ge 2,$$
any k-th horizontal hyperface of ϕ is a (possibly trivial) degeneracy of some cell satisfying (†) with dimension strictly lower than $\dim[m;\mathbf{p}]$; and
- any other hyperface of ϕ (excluding the k_α-th horizontal one) is contained in X.

Moreover $\phi(k_\phi - 1) = (\blacklozenge, \pi_2 \circ \phi(k_\phi))$ and hence $\pi_2 \circ \phi(k_\phi) \neq n$. This implies $k_\phi \neq m$ and it follows that the horn $\Lambda_h^{k_\phi}[m;\mathbf{p}]$ is inner. Thus the inclusion $X \hookrightarrow Y$ is in $\mathrm{cell}(\mathcal{H}_h)$.

Now consider the remaining non-degenerate cells $\phi : \Theta_2[m;\mathbf{p}] \to J \times \Theta_2[n;\mathbf{q}]$ that are not in Y. Let $k_\phi \in [m]$ be the smallest such that $\pi_2 \circ \phi(k_\phi) = n$. Note that $[n;\mathbf{q}] \neq [0]$ implies $k_\phi \neq 0$. Observe that for any non-degenerate cell ϕ in $(J \times \Theta_2[n;\mathbf{q}]) \setminus Y$, either ϕ satisfies

(‡) $\pi_1 \circ \phi(k_\phi) = \lozenge$

or there is a unique non-degenerate cell ψ in $(J \times \Theta_2[n;\mathbf{q}]) \setminus Y$ satisfying (‡) such that ϕ is a (unique) k_ψ-th horizontal hyperface of ψ. Therefore the non-degenerate cells in $(J \times \Theta_2[n;\mathbf{q}]) \setminus Y$ can be partitioned into pairs of the form

$$\{\phi, \phi \cdot \delta_h^{k_\phi;\langle \mathrm{id},!\rangle}\}$$

where ϕ is an $(m;\boldsymbol{p})$-cell satisfying (‡) (which necessarily has $p_{k_\phi+1} = 0$). We prove that $J \times \Theta_2[n;\mathbf{q}]$ may be obtained from Y by gluing such ϕ along the horn $\Lambda_h^{k_\phi}[m;\mathbf{p}]$ in increasing order of $\dim[m;\mathbf{p}]$.

Fix an $(m;\boldsymbol{p})$-cell ϕ in $(J \times \Theta_2[n;\mathbf{q}]) \setminus Y$ satisfying (‡). We must check that all hyperfaces of ϕ except for the (unique) k_α-th one are contained either in Y or in some cell that satisfies (‡) and has dimension strictly smaller than $\dim[m;\mathbf{p}]$. Indeed:

- the unique $(k_\phi + 1)$-th horizontal hyperface of ϕ (which may be inner or outer depending on whether $k_\phi + 1 = m$) is:
  - a degeneracy of some non-degenerate cell in $(J \times \Theta_2[n;\mathbf{q}]) \setminus Y$ satisfying (‡) of dimension $\dim[m;\mathbf{p}] - 2$ if $k_\phi + 3 \leq m$ (in which case we necessarily have $\phi(k_\phi + 3) = (\blacklozenge, n)$); and
  - contained in Y otherwise;
- for any $k_\phi + 1 < k \leq m$, the unique k-th horizontal hyperface of ϕ is a (possibly trivial) degeneracy of some cell satisfying (‡) of dimension strictly lower than $\dim[m;\mathbf{p}]$;
- for any $0 \leq k < k_\phi$ with
$$\left|(\pi_2 \circ \phi)^{-1}(\pi_2 \circ \phi(k))\right| \geq 2,$$
any k-th horizontal hyperface of ϕ is a (possibly trivial) degeneracy of some cell satisfying (‡) of dimension strictly lower than $\dim[m;\mathbf{p}]$; and
- any other hyperface of ϕ (excluding the k_α-th horizontal one) is contained in Y.

Moreover, the horn $\Lambda_h^{k_\alpha}[m;\mathbf{p}]$ is inner since (‡) implies $k_\phi \neq m$. This completes the proof. □

3.4 Characterisation of fibrations into 2-quasi-categories

Recall the sets $\mathcal{J}_A$, $\mathcal{H}_h$, $\mathcal{H}_v$, $\mathcal{E}_v$ and $\mathcal{H}'_h$ as defined in Definitions 2.2.23, 2.3.8 and 3.2.1. By combining Theorem 2.3.9 and all of the results we have proved, we obtain the following theorem.

Theorem 3.4.1. *Let $f : X \to Y$ be a map in $\widehat{\Theta_2}$ and suppose that Y is a 2-quasi-category. Then the following are equivalent:*

(i) f is a fibration with respect to Ara's model structure;

(ii) f has the right lifting property with respect to all maps in $\mathcal{J}_A$;

(iii) f has the right lifting property with respect to all maps in $\mathcal{H}_h \cup \mathcal{H}_v \cup \mathcal{E}_v \cup \{e\}$; and

(iv) f has the right lifting property with respect to all maps in $\mathcal{H}'_h \cup \mathcal{H}_v \cup \mathcal{E}_v \cup \{e\}$.

Proof. (i) ⇔ (ii): This equivalence is part of Theorem 2.3.9.

(i) ⇒ (iv): The elements of $\mathcal{H}'_h$ are trivial cofibrations by Lemma 3.2.6. Similarly for $\mathcal{H}_v$ and $\mathcal{E}_v$ by Lemma 3.1.4 and Lemma 3.1.12 respectively. The horizontal equivalence extension e is also a trivial cofibration since $e \in \mathcal{E}_h \subset \mathcal{J}_A$.

(iv) ⇒ (iii): This follows from the containment $\mathcal{H}_h \subset \mathrm{cell}(\mathcal{H}'_h \cup \mathcal{H}_v)$, which is precisely the statement of Lemma 3.2.2.

(iii) ⇒ (ii): We have the containment $\mathcal{J}_A \subset \mathrm{cell}(\mathcal{J}_O) = \mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v \cup \mathcal{E}_h \cup \mathcal{E}_v)$ by Lemma 3.1.1. But $\mathcal{E}_h \subset \{e\} \cup \mathrm{cell}(\mathcal{H}_h)$ holds by Lemma 3.3.1, which implies that $\mathrm{cell}(\mathcal{J}_O) = \mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v \cup \mathcal{E}_v \cup \{e\})$. □

Since e admits a retraction, we obtain the following corollary by setting Y to be the terminal cellular set $\Theta_2[0]$.

Corollary 3.4.2. *Let $X \in \widehat{\Theta_2}$ be a cellular set. Then the following are equivalent:*

(i) X is a 2-quasi-category;

(ii) X has the right lifting property with respect to all maps in $\mathcal{J}_A$*;*

(iii) X has the right lifting property with respect to all maps in $\mathcal{H}_h \cup \mathcal{H}_v \cup \mathcal{E}_v$*; and*

(iv) X has the right lifting property with respect to all maps in $\mathcal{H}'_h \cup \mathcal{H}_v \cup \mathcal{E}_v$*.*

The following corollary says that, when detecting left Quillen functors out of $\widehat{\Theta_2}$, we may replace the infinite family $\mathcal{E}_v$ by a single map $[\mathrm{id}; e] : \Theta_2[1;0] \to \Theta_2[1;J]$. Recall that $\mathcal{I}$ denotes the set of boundary inclusions.

Definition 3.4.3. Let $\mathcal{J} \overset{\text{def}}{=} \mathcal{H}_h \cup \mathcal{H}_v \cup \{e, [\mathrm{id}; e]\}$ and $\mathcal{J}' \overset{\text{def}}{=} \mathcal{H}'_h \cup \mathcal{H}_v \cup \{e, [\mathrm{id}; e]\}$.

Corollary 3.4.4. *Let*

$$F : \widehat{\Theta_2} \times \cdots \times \widehat{\Theta_2} \to \mathscr{M}$$

be an n-ary functor into a model category $\mathscr{M}$*. Suppose that F satisfies Definition 2.3.4(1). Then the following are equivalent:*

(i) F is left Quillen;

(ii) each map in $\hat{F}(\mathcal{I}, \ldots, \mathcal{I})$ *is a cofibration and each map in* $\hat{F}(\mathcal{I}, \ldots, \mathcal{I}, \mathcal{J}, \mathcal{I}, \ldots, \mathcal{I})$ *is a trivial cofibration regardless of the position of* $\mathcal{J}$*; and*

(iii) each map in $\hat{F}(\mathcal{I}, \ldots, \mathcal{I})$ *is a cofibration and each map in* $\hat{F}(\mathcal{I}, \ldots, \mathcal{I}, \mathcal{J}', \mathcal{I}, \ldots, \mathcal{I})$ *is a trivial cofibration regardless of the position of* $\mathcal{J}'$*.*

Proof. (i) $\Rightarrow$ (iii) follows from Lemmas 3.1.4 and 3.2.6, and (iii) $\Rightarrow$ (ii) follows from Lemmas 2.2.11 and 3.2.2.

For (ii) $\Rightarrow$ (i), suppose that F satisfies (ii). It follows from Lemma 2.2.11 and Proposition 2.2.16 that $\hat{F}(f_1, \ldots, f_n)$ is a cofibration for any monomorphisms $f_1, \ldots, f_n$. This proves that F satisfies the first part of Definition 2.3.4(2).

Recall that for any $[1;0] \neq [n;\mathbf{q}] \in \Theta_2$ and any $1 \leq k \leq n$ satisfying $q_k = 0$, both of $\Theta_2[n;\mathbf{q}] \hookrightarrow \Psi^k[n;\mathbf{q}]$ and $\Theta_2[n;\mathbf{q}] \hookrightarrow \Phi^k[n;\mathbf{q}]$ are in

$$\mathrm{cell}\Big(\mathcal{H}_h \cup \mathcal{H}_v \cup \big\{\Psi^\ell[m;\mathbf{p}] \hookrightarrow \Phi^\ell[m;\mathbf{p}] : \dim\,[m;\mathbf{p}] < \dim\,[n;\mathbf{q}]\big\}\Big)$$

by Lemmas 3.1.13 to 3.1.17. Since the n-ary version of [RV14, Observation 5.1] shows that a map of the form $\hat{F}(f_1, \ldots, f_{k-1}, hg, f_{k+1}, \ldots, f_n)$ may be obtained as a composite of $\hat{F}(f_1, \ldots, f_{k-1}, h, f_{k+1}, \ldots, f_n)$ and a pushout of $\hat{F}(f_1, \ldots, f_{k-1}, g, f_{k+1}, \ldots, f_n)$, it now follows by the 2-out-of-3 property and induction on $\dim\,[n;\mathbf{q}]$ that $\hat{F}(\mathcal{I}, \ldots, \mathcal{I}, \mathcal{E}_v, \mathcal{I}, \ldots, \mathcal{I})$ is contained in the class of trivial cofibrations. Thus we have deduced from (ii) that:

(★) each map in $\hat{F}(\mathcal{I}, \ldots, \mathcal{I}, \mathcal{H}_h \cup \mathcal{H}_v \cup \mathcal{E}_v \cup \{e\}, \mathcal{I}, \ldots, \mathcal{I})$ is a trivial cofibration.

Now let $f_1, \ldots, f_n$ be monomorphisms in $\widehat{\Theta_2}$ and suppose that f_k is a trivial cofibration for some k. We wish to show that $\hat{F}(f_1, \ldots, f_n)$ is a trivial cofibration. We already know that it is at least a cofibration. Hence by [JT07, Lemma 7.14], $\hat{F}(f_1, \ldots, f_n)$ is trivial if and only if it has the left lifting property with respect to all fibrations between fibrant objects. By Proposition 2.3.5, the latter is equivalent to the statement that f_k has the left lifting property with respect to $\check{R}_k(f_1,\dots,f_{k-1},f_{k+1},\dots,f_n,g)$ for any fibration g between fibrant objects, where

$$R_k : \widehat{\Theta_2}^{\mathrm{op}} \times \cdots \times \widehat{\Theta_2}^{\mathrm{op}} \times \mathscr{M} \to \widehat{\Theta_2}$$

is defined as in Section 2.3.1. Thus it suffices to show that $\check{R}_k(f_1,\dots,f_{k-1},f_{k+1},\dots,f_n,g)$ is a fibration between fibrant objects whenever g is so. By Proposition 2.3.5, Theorem 3.4.1 and (★), this reduces to showing that the codomain of $\check{R}_k(f_1,\dots,f_{k-1},f_{k+1},\dots,f_n,g)$ is fibrant whenever g is a fibration between fibrant objects.

Write X_i^0 and X_i^1 for the domain and the codomain of f_i respectively, and fix a fibration $g : Y^0 \to Y^1$ between fibrant objects in $\mathscr{M}$. We proceed by induction on the cardinality of

$$\{i : i \neq k,\ X_i^0 \neq 0\} \cup \{* : Y^1 \neq 1\}$$

where 0 and 1 denote the initial and terminal objects in appropriate categories. (The second set simply contributes 1 to the cardinality if $Y^1 \neq 1$ and contributes 0 if $Y^1 = 1$.) The base case is trivial since $X_i^0 = 0$ for all $i \neq k$ and $Y^1 = 1$ would imply that the codomain of $\check{R}_k(f_1,\dots,f_{k-1},f_{k+1},\dots,f_n,g)$ is the terminal cellular set.

For the inductive step, let $G : \mathbb{2}^n \to \widehat{\Theta_2}$ be the functor given by

$$G(\epsilon_1,\dots,\epsilon_{k-1},\epsilon_{k+1},\dots,\epsilon_n,\epsilon) = R_k(X_1^{1-\epsilon_1},\dots,X_{k-1}^{1-\epsilon_{k-1}},X_{k+1}^{1-\epsilon_{k+1}},\dots,X_n^{1-\epsilon_n},Y^\epsilon)$$

and let $I : \mathscr{C} \hookrightarrow \mathbb{2}^n$ denote the inclusion of the full subcategory spanned by all non-initial objects. Then the codomain of $\check{R}_k(f_1,\dots,f_{k-1},f_{k+1},\dots,f_n,g)$ is the limit of GI. Observe that $\mathscr{C}$ admits a Reedy structure with $\deg(\boldsymbol{\epsilon}) = n - \sum \boldsymbol{\epsilon}$ such that all maps are degree-lowering. Since there is no degree-raising map in $\mathscr{C}$, the diagonal functor $\widehat{\Theta_2} \to [\mathscr{C},\widehat{\Theta_2}]$ is left Quillen. Thus it remains to show that GI is Reedy fibrant.

Fix an object $\boldsymbol{\epsilon} \in \mathscr{C}$. We wish to show that the $\boldsymbol{\epsilon}$-th matching map for GI is a fibration. Observe that this matching map is precisely $\check{R}_k(f'_1,\dots,f'_{k-1},f'_{k+1},\dots,f'_n,g')$ where

$$f'_i = \begin{cases} f_i & \text{if } \epsilon_i = 0, \\ 0 \hookrightarrow X_i^0 & \text{if } \epsilon_i = 1 \end{cases}$$

for each $i \neq k$ and

$$g' = \begin{cases} g & \text{if } \epsilon = 0, \\ Y^1 \to 1 & \text{if } \epsilon = 1. \end{cases}$$

Since $\boldsymbol{\epsilon} \in \mathscr{C}$ (and hence $\boldsymbol{\epsilon} \neq (0,\dots,0)$), it follows by the inductive hypothesis that the codomain of $\check{R}_k(f'_1,\dots,f'_{k-1},f'_{k+1},\dots,f'_n,g')$ is fibrant. Moreover, this map has the right lifting property with respect to all maps in $\mathcal{H}_h \cup \mathcal{H}_v \cup \mathcal{E}_v \cup \{e\}$ by Proposition 2.3.5 and (★). Therefore Theorem 3.4.1 implies that $\check{R}_k(f'_1,\dots,f'_{k-1},f'_{k+1},\dots,f'_n,g')$ is a fibration. This completes the proof. □

3.5 Special outer horns

The aim of this section is to prove that certain *special outer horn inclusions* are trivial cofibrations. We will use the vertical case of this result in Chapter 4.

We first consider the horizontal case. Let $[n;\mathbf{q}] \in \Theta_2$ with $n \geq 2$ and $q_1 = 0$.

Definition 3.5.1. We will denote by $\widetilde{\Lambda}^0_h[n;\mathbf{q}]$ and $\widetilde{\Theta}^0_2[n;\mathbf{q}]$ the cellular sets defined by the following pushout squares

$$\begin{tikzcd}
\Theta_2[1;0] \arrow[r] \arrow[d, hook] & \Lambda^0_h[n;\mathbf{q}] \arrow[r, hook] \arrow[d, hook] & \Theta_2[n;\mathbf{q}] \arrow[d, hook] \\
J \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \widetilde{\Lambda}^0_h[n;\mathbf{q}] \arrow[r, hook] \arrow[ur, phantom, "\ulcorner" very near end] & \widetilde{\Theta}^0_2[n;\mathbf{q}]
\end{tikzcd}$$

where the composite of the upper row is (isomorphic to) $\big[\{0,1\};\mathrm{id}\big] : [1;0] \to [n;\mathbf{q}]$ and the left vertical maps pick out the $(1;0)$-cell $\{\lozenge \to \blacklozenge\}$.

Lemma 3.5.2. *The map $\widetilde{\Lambda}^0_h[n;\mathbf{q}] \hookrightarrow \widetilde{\Theta}^0_2[n;\mathbf{q}]$ is a trivial cofibration.*

Proof. Let $\mathscr{J}_h = \{\lozenge \cong \blacklozenge\}$ be the chaotic category on two objects so that $N\mathscr{J}_h \cong J$. Let $\mathscr{H}$ denote the 2-category defined by the pushout

$$\begin{tikzcd}
{[1;0]} \arrow[r] \arrow[d] & {[n;\mathbf{q}]} \arrow[d] \\
\mathscr{J}_h \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathscr{H}
\end{tikzcd}$$

where the upper horizontal map is $\big[\{0,1\};\mathrm{id}\big]$ and the left vertical map picks out the 1-cell $\lozenge \to \blacklozenge$. Define a preorder $\preceq$ on the set $[n] = \{0,\dots,n\}$ so that $i \preceq j$ if and only if:

- $i \le j$ (with respect to the usual order); or
- $i = 1$ and $j = 0$.

Then an $(m;\boldsymbol{p})$-cell $[\alpha;\boldsymbol{\alpha}]$ in the nerve $N\mathscr{H}$ consists of an order preserving map

$$\alpha : \big([m],\le\big) \to \big([n],\preceq\big)$$

together with a simplicial operator $\alpha_k : [p_\ell] \to [q_k]$ for each $\ell \in [m]$ and $\alpha(\ell-1) < k \le \alpha(\ell)$.

We will regard $\widetilde{\Lambda}^0_h[n;\mathbf{q}]$ and $\widetilde{\Theta}^0_2[n;\mathbf{q}]$ as cellular subsets of $N\mathscr{H}$ via the obvious monomorphisms $\widetilde{\Lambda}^0_h[n;\mathbf{q}] \hookrightarrow \widetilde{\Theta}^0_2[n;\mathbf{q}] \hookrightarrow N\mathscr{H}$. The desired result follows once we prove that both of the inclusions $\widetilde{\Theta}^0_2[n;\mathbf{q}] \hookrightarrow N\mathscr{H}$ and $\widetilde{\Lambda}^0_h[n;\mathbf{q}] \hookrightarrow N\mathscr{H}$ are trivial cofibrations. These facts are proved in Lemmas 3.5.3 and 3.5.4 below. □

Observe that an $(m;\boldsymbol{p})$-cell $[\alpha;\boldsymbol{\alpha}]$ in $N\mathscr{H}$ is contained in $N\mathscr{H} \setminus \widetilde{\Theta}^0_2[n;\mathbf{q}]$ if and only if:

(a) there is $0 \le \ell < m$ such that $\alpha(\ell) = 1$ and $\alpha(\ell+1) = 0$; and

(b) $\alpha(m) \ge 2$.

The only non-degenerate cells in $\widetilde{\Theta}^0_2[n;\mathbf{q}] \setminus \widetilde{\Lambda}^0_h[n;\mathbf{q}]$ are $[\mathrm{id};\mathbf{id}]$ and δ^0_h.

Lemma 3.5.3. *The inclusion $\widetilde{\Theta}^0_2[n;\mathbf{q}] \hookrightarrow N\mathscr{H}$ is in* $\mathrm{cell}(\mathcal{H}_h)$.

Proof. We say an order-preserving map $\alpha : ([m], \leq) \to ([n], \leq)$ is *dull* if $\alpha(0) \geq 2$. An $(m; \boldsymbol{p})$-cell $[\alpha; \boldsymbol{\alpha}]$ in $N\mathscr{H}$ is called *dull* if α is dull. For any non-dull $\alpha : ([m], \leq) \to ([n], \leq)$, we define

$$\ell_\alpha \stackrel{\text{def}}{=} \max\bigl(\alpha^{-1}(\{0, 1\})\bigr).$$

We say a non-degenerate, non-dull cell $[\alpha; \boldsymbol{\alpha}]$ in $N\mathscr{H}$ is of:

- *type 0* if $\alpha(\ell_\alpha) = 0$; and
- *type 1* if $\alpha(\ell_\alpha) = 1$.

Then it is easy to check (using the conditions (a) and (b) above) that the set of non-degenerate cells in $N\mathscr{H} \setminus \widetilde{\Theta}_2^0[n; \mathbf{q}]$ can be partitioned into pairs of the form

$$\bigl\{[\alpha; \boldsymbol{\alpha}], [\alpha; \boldsymbol{\alpha}] \cdot \delta_h^{\ell_\alpha; \langle !, \mathrm{id} \rangle}\bigr\}$$

where $[\alpha; \boldsymbol{\alpha}]$ is of type 1. Moreover, for any $[\alpha; \boldsymbol{\alpha}]$ of type 1 in $N\mathscr{H} \setminus \widetilde{\Theta}_2^0[n; \mathbf{q}]$, any of its hyperfaces other than the (unique) ℓ_α-th horizontal one is:

- degenerate;
- contained in $\widetilde{\Theta}_2^0[n; \mathbf{q}]$; or
- of type 1.

It follows that $N\mathscr{H}$ may be obtained from $\widetilde{\Theta}_2^0[n; \mathbf{q}]$ by gluing those $(m; \boldsymbol{p})$-cells $[\alpha; \boldsymbol{\alpha}]$ of type 1 in $N\mathscr{H} \setminus \widetilde{\Theta}_2^0[n; \mathbf{q}]$ along the horn $\Lambda_h^{\ell_\alpha}[m; \mathbf{p}]$ in increasing order of $\dim [m; \mathbf{p}]$. This horn is inner since (a) implies $\ell_\alpha \neq 0$ and (b) implies $\ell_\alpha \neq m$. This completes the proof. □

Lemma 3.5.4. *The inclusion and* $\widetilde{\Lambda}_h^0[n; \mathbf{q}] \hookrightarrow N\mathscr{H}$ *is in* $\mathrm{cell}(\mathcal{H}_h)$*.*

Proof. Let $X \subset N\mathscr{H}$ denote the cellular subset consisting of those cells that do not contain $[\delta^0; \mathbf{id}]$. Then this inclusion can be factorised as

$$\widetilde{\Lambda}_h^0[n; \mathbf{q}] \hookrightarrow X \hookrightarrow N\mathscr{H}.$$

Moreover:

- the non-degenerate cells in $X \setminus \widetilde{\Lambda}_h^0[n; \mathbf{q}]$ can be partitioned into pairs of the form
  $$\bigl\{[\alpha; \boldsymbol{\alpha}], [\alpha; \boldsymbol{\alpha}] \cdot \delta_h^{\ell_\alpha; \langle !, \mathrm{id} \rangle}\bigr\}$$
  where $[\alpha; \boldsymbol{\alpha}]$ is of type 1; and
- the non-degenerate cells in $N\mathscr{H} \setminus X$ can be partitioned into pairs of the form
  $$\bigl\{[\alpha; \boldsymbol{\alpha}], [\alpha; \boldsymbol{\alpha}] \cdot \delta_h^{\ell_\alpha; \langle !, \mathrm{id} \rangle}\bigr\}$$
  where $[\alpha; \boldsymbol{\alpha}]$ is of type 0.

The rest of the proof is similar to that of Lemma 3.5.3 and is left to the reader. □

Taking the "suspension" of the above argument yields the following vertical case. Fix $[1; q] \in \Theta_2$ with $q \geq 2$.

Definition 3.5.5. We denote by $\widetilde{\Lambda}^{1;0}_v[1;q]$ and $\widetilde{\Theta}^{1;0}_2[1;q]$ the cellular sets defined by the following pushout squares

$$\begin{tikzcd}
\Theta_2[1;1] \arrow[r] \arrow[d, hook] & \Lambda^{1;0}_v[1;q] \arrow[r, hook] \arrow[d, hook] & \Theta_2[1;q] \arrow[d, hook] \\
\Theta_2[1;J] \arrow[r] & \widetilde{\Lambda}^{1;0}_v[1;q] \arrow[ul, phantom, "\ulcorner" very near start] \arrow[r, hook] & \widetilde{\Theta}^{1;0}_2[1;q] \arrow[ul, phantom, "\ulcorner" very near start]
\end{tikzcd}$$

where the left vertical map picks out the $(1;1)$-cell $\lozenge \Rightarrow \blacklozenge$ and the composite of the upper row is (isomorphic to) $\big[\mathrm{id};\{0,1\}\big] : [1;1] \to [1;q]$.

Lemma 3.5.6. *The map $\widetilde{\Lambda}^{1;0}_v[1;q] \hookrightarrow \widetilde{\Theta}^{1;0}_2[1;q]$ is a trivial cofibration.*

4

The Gray tensor product for 2-quasi-categories

In this chapter, we make precise and prove the following statement:

the 2-quasi-categorical Gray tensor product is
part of an up-to-homotopy monoidal closed structure.

4.1 The Gray tensor product

4.1.1 Classical version

The (*lax*) *Gray tensor product* [Gra74, Theorem I.4.9] of two (small) 2-categories $\mathscr{A}$ and $\mathscr{B}$ is the 2-category $\mathscr{A} \boxtimes \mathscr{B}$ given by the following generators-and-relations presentation. Its object set is $\mathrm{ob}(\mathscr{A} \boxtimes \mathscr{B}) = \mathrm{ob}\,\mathscr{A} \times \mathrm{ob}\,\mathscr{B}$. Its underlying 1-category is generated by the maps of the form

$$\begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] & (x',y) \end{tikzcd}\ , \qquad \begin{tikzcd} (x,y) \arrow[d, "{(x,g)}"] \\ (x,y') \end{tikzcd} \tag{4.1}$$

where $f : x \to x'$ in $\mathscr{A}$ and $g : y \to y'$ in $\mathscr{B}$, subject to the relations $(f',y)(f,y) = (f'f,y)$ and $(x,g')(x,g) = (x,g'g)$ whenever these composites make sense, and $\mathrm{id}_{(x,y)} = (\mathrm{id}_x, y) = (x, \mathrm{id}_y)$. Similarly, we have generating 2-cells

$$\begin{tikzcd} (x,y) \arrow[r, bend left=50, "{(f',y)}"{name=U}] \arrow[r, bend right=50, "{(f,y)}"'{name=D}] & (x',y) \arrow[from=D, to=U, phantom, "{\Uparrow (\alpha,y)}"] \end{tikzcd}\ , \qquad \begin{tikzcd} (x,y) \arrow[d, bend right=50, "{(x,g)}"'{name=L}] \arrow[d, bend left=50, "{(x,g')}"{name=R}] \\ (x,y') \arrow[from=L, to=R, phantom, "{\xRightarrow{(x,\beta)}}"] \end{tikzcd} \tag{4.2}$$

for any 2-cells $\alpha : f \Rightarrow f' : x \to x'$ in $\mathscr{A}$ and $\beta : g \Rightarrow g' : y \to y'$ in $\mathscr{B}$, subject to the obvious relations involving the horizontal and vertical compositions in $\mathscr{A}$ and $\mathscr{B}$. There are additional generating 2-cells of the form

$$\begin{tikzcd}
(x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] & (x',y) \arrow[d, "{(x',g)}"] \\
(x,y') \arrow[r, "{(f,y')}"'] \arrow[ur, phantom, "\gamma_{f,g}"] & (x',y')
\end{tikzcd} \tag{4.3}$$

for 1-cells f in $\mathscr{A}$ and g in $\mathscr{B}$. The relations we impose on these 2-cells are:

$$\begin{tikzcd}
(x,y) \arrow[r, bend left, "{(f',y)}"] \arrow[r, bend right, "{(f,y)}"'] \arrow[r, phantom, "\Uparrow (\alpha,y)"] \arrow[d, "{(x,g)}"'] & (x',y) \arrow[d, "{(x',g)}"] \\
(x,y') \arrow[r, "{(f,y')}"'] \arrow[ur, phantom, "\gamma_{f,g}"] & (x',y')
\end{tikzcd}
=
\begin{tikzcd}
(x,y) \arrow[r, "{(f',y)}"] \arrow[d, "{(x,g)}"'] & (x',y) \arrow[d, "{(x',g)}"] \\
(x,y') \arrow[r, bend left, "{(f',y')}"] \arrow[r, bend right, "{(f,y')}"'] \arrow[r, phantom, "\Uparrow (\alpha,y')"] \arrow[ur, phantom, "\gamma_{f',g}"] & (x',y')
\end{tikzcd} \tag{4.4}$$

$$\begin{tikzcd}
(x,y) \arrow[r, "{(\mathrm{id}_x,y)}"] \arrow[d, "{(x,g)}"'] & (x,y) \arrow[d, "{(x,g)}"] \\
(x,y') \arrow[r, "{(\mathrm{id}_x,y')}"'] \arrow[ur, phantom, "\gamma_{\mathrm{id}_x,g}"] & (x,y')
\end{tikzcd}
=
\begin{tikzcd}
(x,y) \arrow[r, "\mathrm{id}_{(x,y)}"] \arrow[d, "{(x,g)}"'] & (x,y) \arrow[d, "{(x,g)}"] \\
(x,y') \arrow[r, "\mathrm{id}_{(x,y')}"'] \arrow[ur, phantom, "\mathrm{id}"] & (x,y')
\end{tikzcd} \tag{4.5}$$

$$\begin{tikzcd}
(x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] & (x',y) \arrow[r, "{(f',y)}"] \arrow[d, "{(x',g)}"] & (x'',y) \arrow[d, "{(x'',g)}"] \\
(x,y') \arrow[r, "{(f,y')}"'] \arrow[ur, phantom, "\gamma_{f,g}"] & (x',y') \arrow[r, "{(f',y')}"'] \arrow[ur, phantom, "\gamma_{f',g}"] & (x'',y')
\end{tikzcd}
=
\begin{tikzcd}
(x,y) \arrow[r, "{(f'f,y)}"] \arrow[d, "{(x,g)}"'] & (x'',y) \arrow[d, "{(x'',g)}"] \\
(x,y') \arrow[r, "{(f'f,y')}"'] \arrow[ur, phantom, "\gamma_{f'f,g}"] & (x'',y')
\end{tikzcd} \tag{4.6}$$

and their "vertical" counterparts, involving the 2-category structure of $\mathscr{B}$. A description of the 2-cells in $\mathscr{A} \boxtimes \mathscr{B}$ as equivalence classes of (vertically composable) strings of equivalence classes of (horizontally composable) strings of generating 2-cells, making this presentation more explicit, can be found in [Gra74, Theorem I.4.9].

This tensor product extends to a functor $2\text{-}\underline{\mathrm{Cat}} \times 2\text{-}\underline{\mathrm{Cat}} \to 2\text{-}\underline{\mathrm{Cat}}$, and forms part of a biclosed monoidal structure on $2\text{-}\underline{\mathrm{Cat}}$. In particular, there are natural bijections

$$2\text{-}\underline{\mathrm{Cat}}(\mathscr{B}, [\mathscr{A},\mathscr{C}]_{\mathrm{lax}}) \cong 2\text{-}\underline{\mathrm{Cat}}(\mathscr{A} \boxtimes \mathscr{B}, \mathscr{C}) \cong 2\text{-}\underline{\mathrm{Cat}}(\mathscr{A}, [\mathscr{B},\mathscr{C}]_{\mathrm{oplax}})$$

where $[\mathscr{A},\mathscr{C}]_{\mathrm{lax}}$ is the 2-category of 2-functors $\mathscr{A} \to \mathscr{C}$, lax natural transformations and modifications, and $[\mathscr{B},\mathscr{C}]_{\mathrm{oplax}}$ is similar but has oplax natural transformations as 1-cells. This monoidal structure is not braided, but we have natural isomorphisms $(\mathscr{A} \boxtimes \mathscr{B})^{\mathrm{op}} \cong \mathscr{B}^{\mathrm{op}} \boxtimes \mathscr{A}^{\mathrm{op}}$ and $(\mathscr{A} \boxtimes \mathscr{B})^{\mathrm{co}} \cong \mathscr{B}^{\mathrm{co}} \boxtimes \mathscr{A}^{\mathrm{co}}$.

Remark. The functor $\boxtimes : 2\text{-}\underline{\mathrm{Cat}} \times 2\text{-}\underline{\mathrm{Cat}} \to 2\text{-}\underline{\mathrm{Cat}}$ does not extend to a 2-functor. For instance, regard the unique non-identity 1-cell in $[1;0]$ as a 2-natural transformation between two 2-functors $[0] \to [1;0]$, and consider the "tensor" of this 2-natural transformation with another copy of $[1;0]$.

4.1.2 Θ_2-version

Definition 4.1.1. For each $a \geq 2$, the *a-ary Gray tensor product* functor

$$\otimes_a : \underbrace{\widehat{\Theta_2} \times \cdots \times \widehat{\Theta_2}}_{a \text{ times}} \to \widehat{\Theta_2}$$

is obtained by extending the composite

$$\Theta_2 \times \cdots \times \Theta_2 \hookrightarrow 2\text{-}\underline{\mathrm{Cat}} \times \cdots \times 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{\boxtimes_a} 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{N} \widehat{\Theta_2}$$

cocontinuously in each variable, where the second map $\boxtimes_a$ is the a-ary Gray tensor product of 2-categories. We define $\otimes_0 \overset{\text{def}}{=} \Theta_2[0]$ and $\otimes_1(X) \overset{\text{def}}{=} X$.

Therefore the tensor product $\otimes_a(X^1, \ldots, X^a)$ admits a coend description

$$\otimes_a(X^1, \ldots, X^a) \cong \int^{\theta_1, \ldots, \theta_a \in \Theta_2} \big(X^1_{\theta_1} \times \cdots \times X^a_{\theta_a}\big) * N\big(\boxtimes_a(\theta_1, \ldots, \theta_a)\big)$$

where $*$ denotes the copower. More explicitly, a ζ-cell in $\otimes_a(X^1, \ldots, X^a)$ is represented by $(\theta_1, \ldots, \theta_a, \phi, x_1, \ldots, x_a)$ where $\phi : \zeta \to \boxtimes_a(\theta_1, \ldots, \theta_a)$ is a 2-functor and $x_i \in X^i_{\theta_i}$ for each i. Two such $(2a+1)$-tuples represent the same ζ-cell if and only if they are related by the equivalence relation generated by $\sim$ defined as follows: for any cellular operators $\alpha : \theta_i \to \theta'_i$, any 2-functor $\phi : \zeta \to \boxtimes_a(\theta_1, \ldots, \theta_a)$, and any $x_i \in X^i_{\theta'_i}$,

$$(\boldsymbol{\theta}, \phi, \boldsymbol{x} \cdot \boldsymbol{\alpha}) \sim \big(\boldsymbol{\theta}', \boxtimes_a(\boldsymbol{\alpha}) \circ \phi, \boldsymbol{x}\big).$$

(See the remark after Lemma 4.2.2.)

Note that the obvious 2-functors $\pi_i : \boxtimes_a(\mathscr{A}_1, \ldots, \mathscr{A}_a) \to \mathscr{A}_i$ induce cellular maps $\pi_i : \otimes_a(X^1, \ldots, X^a) \to X^i$ for $1 \leq i \leq a$.

Remark. One might (rightly) object that, although Definition 4.1.1 involves the ordinary Gray tensor product $\boxtimes_a$, this does not fully justify calling the functor $\otimes_a$ the Gray tensor product. Ideally we would respond to such an objection by exhibiting that "everything" we ever do with $\boxtimes_a$ admits an analogue for $\otimes_a$. Our main results (showing that $\otimes_a$ is part of an up-to-homotopy monoidal closed structure) may be seen as a partial justification along this line, and we hope to strengthen this justification by reconstructing the formal theory of monads in future.

The following argument provides another justification for calling $\otimes_a$ the Gray tensor product. For mono-vertebral objects $\theta, \theta' \in \big\{[0], [1;0], [1;1]\big\}$ in Θ_2, it is easy to compute and even draw the binary tensor $\Theta_2^\theta \otimes \Theta_2^{\theta'} = \otimes_2\big(\Theta_2^\theta, \Theta_2^{\theta'}\big)$. If one is convinced that these low-dimensional examples "look correct" (in the sense that they match what one expects the Gray tensor products of these simple $(\infty,2)$-categories to be), then:

1. $\Theta_2^\theta \otimes \Theta_2^{\theta'}$ must be "correct" for any $\theta, \theta' \in \Theta_2$ since $\otimes$ is left Quillen (Theorem 4.4.1) and any Θ_2^θ is a homotopy colimit of $\Theta_2[0]$, $\Theta_2[1;0]$ and $\Theta_2[1;1]$ (via the spine);

2. thus $\otimes = \otimes_2$ must be "correct" on arbitrary inputs since it is left Quillen; and
3. it follows that $\otimes_a$ must be "correct" for arbitrary a since $\otimes$ is associative up to homotopy (Corollary 4.5.11).

4.1.3 Tensoring cells

Since the objects of Θ_2 are very simple 2-categories, we can describe $\boxtimes_a(\boldsymbol{\theta}) = \boxtimes_a(\theta_1, \ldots, \theta_a)$ explicitly for any $\theta_i \in \Theta_2$. First, consider the case where $\theta_i = [n_i; \mathbf{0}]$ for each i. We will describe a 2-category $\mathscr{T}$, and prove that $\mathscr{T} \cong \boxtimes_a(\boldsymbol{\theta})$. The underlying 1-category of $\mathscr{T}$ is the free one on the directed graph determined by the following conditions:

- the vertex set is $\{0, \ldots, n_1\} \times \cdots \times \{0, \ldots, n_a\}$; and
- there is a unique edge
$$(x_1, \ldots, x_i - 1, \ldots, x_a) \longrightarrow (x_1, \ldots, x_i, \ldots, x_a) \tag{4.7}$$
whenever $0 < x_i \leq n_i$ and $0 \leq x_j \leq n_j$ for $j \neq i$.

Before describing the 2-cells in $\mathscr{T}$, let us analyse the 1-cells.

Definition 4.1.2. Given $0 \leq s_i \leq t_i \leq n_i$ for each i, let
$$S(\boldsymbol{s}, \boldsymbol{t}) \stackrel{\text{def}}{=} \big\{(i|k) : 1 \leq i \leq n,\ s_i < k \leq t_i\big\}.$$

We have adopted the notation $(i|k)$ in order to distinguish the elements of $S(\boldsymbol{s}, \boldsymbol{t})$ from other kinds of pairs, *e.g.* objects in $\theta_1 \boxtimes \theta$.

Observe that for any 1-cell f in $\mathscr{T}$ from $\boldsymbol{s} = (s_1, \ldots, s_n)$ to $\boldsymbol{t} = (t_1, \ldots, t_n)$, assigning the pair $(i|x_i)$ to the atomic factor of the form (4.7) yields a bijection between $S(\boldsymbol{s}, \boldsymbol{t})$ and the set of atomic factors of f. Moreover, the obvious total order on the latter set induces a total order $\leq$ on $S(\boldsymbol{s}, \boldsymbol{t})$ satisfying

(※) $(i|k) \leq (i|\ell)$ for any $1 \leq i \leq a$ and $s_i < k \leq \ell \leq t_i$.

Informally speaking, $\leq$ orders the set $S(\boldsymbol{s}, \boldsymbol{t})$ of "instructions" where $(i|k)$ is to be interpreted as "move in the i-th direction by one step so that the new i-th coordinate is k". Conversely, any total order $\leq$ satisfying (※) uniquely determines a 1-cell from $\boldsymbol{s}$ to $\boldsymbol{t}$. Hence we may identify the objects in the hom-category $\mathscr{T}(\boldsymbol{s}, \boldsymbol{t})$ with the set of such total orders on $S(\boldsymbol{s}, \boldsymbol{t})$.

Definition 4.1.3. A *shuffle* on $S(\boldsymbol{s}, \boldsymbol{t})$ is a total order $\leq$ on $S(\boldsymbol{s}, \boldsymbol{t})$ satisfying (※).

Remark. This definition is consistent with Definition 2.1.1 in the sense that there is an obvious bijection between $\mathbf{Shfl}(m, n)$ and the set of shuffles on $S\big((0,0), (m, n)\big)$.

Finally we define the hom-category $\mathscr{T}(\boldsymbol{s}, \boldsymbol{t})$ to be the poset given by the partial order $\blacktriangleleft$ defined below. It is straightforward to check that $\mathscr{T}$ is a poset-enriched category and hence a 2-category.

Definition 4.1.4. Let $\leq$ and $\leq'$ be shuffles on $S(\boldsymbol{s}, \boldsymbol{t})$. Then $\leq \blacktriangleleft \leq'$ if and only if $(i|k) \leq (j|\ell)$ and $i < j$ imply $(i|k) \leq' (j|\ell)$ for any $(i|k), (j|\ell) \in S(\boldsymbol{s}, \boldsymbol{t})$.

For instance, when $a = 2$ and $n_1 = n_2 = 1$, the 2-category $\mathscr{T}$ looks like

$$\begin{tikzcd}
(0,0) \arrow[r, "(1|1)"] \arrow[d, "(2|1)"'] & (1,0) \arrow[d, "(2|1)"] \\
(0,1) \arrow[r, "(1|1)"'] \arrow[ur, Rightarrow, "\blacktriangleleft"] & (1,1)
\end{tikzcd}$$

where the 2-cell corresponds to the relation

$$\big((2|1) \preceq (1|1)\big) \blacktriangleleft \big((1|1) \preceq' (2|1)\big).$$

Lemma 4.1.5. $\boxtimes_a(\theta_1, \ldots, \theta_a) \cong \mathscr{T}$.

Proof of Lemma 4.1.5. It is easy to see from the generators-and-relations presentation of the Gray tensor product that $\boxtimes_a(\boldsymbol{\theta})$ and $\mathscr{T}$ have isomorphic underlying 1-categories. Moreover, the 2-cells in the former 2-category are generated (under vertical and horizontal compositions) by those of the form

$$\begin{tikzcd}
& (x_1, \ldots, x_i - 1, \ldots, x_j - 1, \ldots, x_n) \arrow[dl] \arrow[dr] & \\
(x_1, \ldots, x_i - 1, \ldots, x_j, \ldots, x_n) \arrow[dr] \arrow[rr, Rightarrow] & & (x_1, \ldots, x_i, \ldots, x_j - 1, \ldots, x_n) \arrow[dl] \\
& (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) &
\end{tikzcd}$$

(see (4.3)) and this 2-cell has the same domain and codomain as the 2-cell

$$\begin{tikzcd}
(x_1, \ldots, x_i - 1, \ldots, x_j - 1, \ldots, x_n) \arrow[r, bend left, "(j|x_j) \preceq (i|x_i)"] \arrow[r, bend right, "(i|x_i) \preceq (j|x_j)"'] \arrow[r, phantom, "\Downarrow"] & (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)
\end{tikzcd}$$

in $\mathscr{T}$. It follows that we have a 2-functor $F : \boxtimes_a(\boldsymbol{\theta}) \to \mathscr{T}$ that is bijective on objects and 1-cells.

We next prove that this 2-functor F is locally full. Consider a morphism in the hom-category $\mathscr{T}(\boldsymbol{s}, \boldsymbol{t})$, or equivalently, a pair of shuffles $\preceq_0$ and $\preceq_1$ on $S(\boldsymbol{s}, \boldsymbol{t})$ such that $\preceq_0 \blacktriangleleft \preceq_1$. By the *variance* of this morphism, we mean the cardinality of the set

$$\Big\{\big((i|k), (j|\ell)\big) \in S(\boldsymbol{s}, \boldsymbol{t})^2 : (i|k) \succeq_0 (j|\ell),\ (i|k) \preceq_1 (j|\ell)\Big\}.$$

We prove by induction on the variance that this morphism is in the image of F. The base case is easy since the variance of a morphism is 0 if and only if it is the identity. For the inductive step, assume that the variance is positive. Then there is a pair $(i_\dagger|k_\dagger), (j_\dagger|\ell_\dagger) \in S(\boldsymbol{s}, \boldsymbol{t})$ such that:

- $(i_\dagger|k_\dagger)$ is the immediate $\preceq_0$-successor of $(j_\dagger|\ell_\dagger)$; and
- $(i_\dagger|k_\dagger) \preceq_1 (j_\dagger|\ell'_\dagger)$

(if such a pair does not exist then $\preceq_0$ and $\preceq_1$ coincide). Now define a total order $\preceq$ on $S(\boldsymbol{s},\boldsymbol{t})$ so that it agrees with $\preceq_0$ on all pairs of elements in $S(\boldsymbol{s},\boldsymbol{t})$ except that $(i_\dagger|k_\dagger) \preceq (j_\dagger|\ell_\dagger)$. Then clearly $\preceq$ is a shuffle and moreover we have $\preceq_0\ \blacktriangleleft\ \preceq\ \blacktriangleleft\ \preceq_1$, giving a factorisation of the original morphism. The first factor is in the image of F by the first paragraph of this proof, and the second factor is in the image too by the inductive hypothesis. This proves that F is locally full.

The proof that F is locally faithful is deferred to Appendix A.1. □

Definition 4.1.6. Given any $\theta = [n;\mathbf{q}] \in \Theta_2$, we will write $\bar{\theta}$ for $[n;\mathbf{0}] \in \Theta_2$.

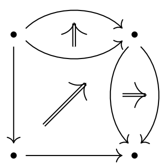
In the rest of this paper, any unlabelled cellular operator of the form $\theta \to \bar{\theta}$ is assumed to be the unique one $[n;\mathbf{q}] \to [n;\mathbf{0}]$ whose horizontal component is the identity.

Lemma 4.1.7. *For any $\theta_1,\dots,\theta_a \in \Theta_2$, the square*

$$\begin{tikzcd}
\boxtimes_a(\theta_1,\dots,\theta_a) \arrow[r] \arrow[d, "\langle\pi_1,\dots,\pi_a\rangle"'] & \boxtimes_a(\bar{\theta}_1,\dots,\bar{\theta}_a) \arrow[d, "\langle\pi_1,\dots,\pi_a\rangle"] \\
\theta_1\times\cdots\times\theta_a \arrow[r] & \bar{\theta}_1\times\cdots\times\bar{\theta}_a
\end{tikzcd}$$

is a pullback in 2-<u>Cat</u>*.*

Proof. We will sketch the proof and leave the details to the reader. Clearly the above square is at least commutative, and thus there is an induced 2-functor from $\boxtimes_a(\boldsymbol{\theta})$ to the pullback of the cospan. It is straightforward to see that this 2-functor is bijective on objects and 1-cells. Moreover one can check that it is locally full, similarly to the proof of the previous lemma. It then suffices to prove that $\boxtimes_a(\boldsymbol{\theta})$ is poset-enriched. It follows from Eqs. (4.4) and (4.6) that any 2-cell in $\boxtimes_a(\boldsymbol{\theta})$ can be (vertically) factorised as a composite of γ's (4.3) followed by a composite of 2-cells "coming from θ_i's" (4.2); *e.g.* for $a = 2$ such a factorisation typically looks like



where the second factor is the horizontal composite of the two globe-shaped 2-cells. Observe that for any parallel pair of 2-cells, this factorisation yields the same middle 1-cell. Therefore the desired result follows from Lemma 4.1.5 and the observation that each θ_i is poset-enriched. □

The lemmas below are straightforward to prove using the explicit description of $\boxtimes_a(\boldsymbol{\theta})$ provided by Lemmas 4.1.5 and 4.1.7. Note that the underlying 1-category of $\boxtimes_a(\boldsymbol{\theta})$ is free on the obvious graph and hence each 1-cell in $\boxtimes_a(\boldsymbol{\theta})$ admits a unique atomic decomposition.

Definition 4.1.8. By the *endpoints* of an $(n;\boldsymbol{q})$-cell $\phi : [n;\mathbf{q}] \to \boxtimes_a(\boldsymbol{\theta})$, we mean the objects $\phi(0)$ and $\phi(n)$.

Definition 4.1.9. Let $\phi : [1;q] \to \boxtimes_a(\theta_1, \ldots, \theta_a)$ be a $(1;q)$-cell with endpoints $\boldsymbol{s}, \boldsymbol{t}$. By the *underlying shuffles* of ϕ, we mean the $q+1$ shuffles on $S(\boldsymbol{s}, \boldsymbol{t})$ corresponding to the composite

$$[1;q] \xrightarrow{\phi} \boxtimes_a(\theta_1, \ldots, \theta_a) \longrightarrow \boxtimes_a(\bar{\theta}_1, \ldots, \bar{\theta}_a).$$

Definition 4.1.10. Given a 1-cell $f : \boldsymbol{s} \to \boldsymbol{t}$ and an object $\boldsymbol{x}$ in $\boxtimes_a(\boldsymbol{\theta})$, we say f *visits* $\boldsymbol{x}$ to mean that the atomic decomposition of f involves $\boldsymbol{x}$. Equivalently, f visits x if and only if either $\boldsymbol{x} = \boldsymbol{s}$ or there is (necessarily unique) $(j|\ell) \in S(\boldsymbol{s}, \boldsymbol{t})$ such that

$$x_i = \min\bigl(\{k : (i|k) \preceq (j|\ell)\} \cup \{s_i\}\bigr)$$

for each i where $\preceq$ is the underlying shuffle of f.

Lemma 4.1.11. *Let $\delta_i : \zeta_i \to \theta_i$ be a face operator in Θ_2 for $1 \le i \le a$. Then*

$$\boxtimes_a(\delta_1, \ldots, \delta_n) : \boxtimes_a(\zeta_1, \ldots, \zeta_a) \to \boxtimes_a(\theta_1, \ldots, \theta_a)$$

is a monomorphism in 2-$\underline{\mathrm{Cat}}$. *Consequently, its nerve*

$$\otimes_a(\delta_1, \ldots, \delta_n) : \otimes_a\bigl(\Theta_2^{\zeta_1}, \ldots, \Theta_2^{\zeta_a}\bigr) \to \otimes_a\bigl(\Theta_2^{\theta_1}, \ldots, \Theta_2^{\theta_a}\bigr)$$

is a monomorphism in $\widehat{\Theta_2}$. Moreover, a κ-cell $\phi : \kappa \to \boxtimes_a(\boldsymbol{\theta})$ in $\otimes_a\bigl(\Theta_2^{\theta_1}, \ldots, \Theta_2^{\theta_a}\bigr)$ is in the image of this map if and only if:

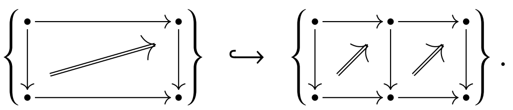
(i) $\kappa \xrightarrow{\phi} \boxtimes_a(\boldsymbol{\theta}) \xrightarrow{\pi_i} \theta_i$ *factors through δ_i for each i; and*

(ii) if some 1-cell f in the image of ϕ visits two distinct objects $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $x_i = y_i$, then the object $x_i = y_i \in \theta_i$ is in the image of δ_i.

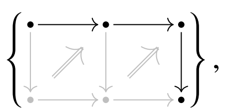
For example, consider the map

$$\delta_h^1 \otimes \mathrm{id} : \Theta_2[1;0] \otimes \Theta_2[1;0] \to \Theta_2[2;0] \otimes \Theta_2[1;0]$$

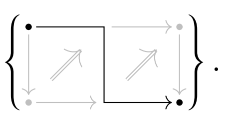
where $\otimes = \otimes_2$. This map is the nerve of the inclusion 2-functor



A cell in the codomain violates (i) if and only if it contains an object in the middle column, *e.g.*



and it violates (ii) if and only if it involves moving down in the middle column, *e.g.*



4.2 $\hat{\otimes}_a$ preserves monomorphisms

Let $a \geq 2$ and let $\theta_1, \ldots, \theta_a \in \Theta_2$. The aim of this section is to prove the following lemma.

Lemma 4.2.1. *The Leibniz Gray tensor product*

$$\hat{\otimes}_a\big(\partial\Theta_2^{\theta_1} \hookrightarrow \Theta_2^{\theta_1}, \ldots, \partial\Theta_2^{\theta_a} \hookrightarrow \Theta_2^{\theta_n}\big) \tag{4.8}$$

is a monomorphism.

Proof. By Lemma 2.2.9, it suffices to prove that the functor

$$G : \mathbb{2}^a \to \widehat{\Theta_2}$$

(defined as in Section 2.2.3 with $F = \otimes_a$) sends each square of the form (2.1) to a pullback square of monomorphisms. We will prove in Lemma 4.2.2 below that G sends each map in $\mathbb{2}^a$ to a monomorphism. Assuming this fact, it is straightforward to deduce using Lemma 4.1.11 that the desired square is indeed a pullback. □

Observe that $\Theta_2^{\theta_i}$ and $\partial\Theta_2^{\theta_i}$ satisfy the hypothesis in the following lemma.

Lemma 4.2.2. *Fix $1 \leq b \leq a$ and let $X^i \in \widehat{\Theta_2}$ for $1 \leq i \leq a$ with $i \neq b$. Suppose that in each X^i, any face of a non-degenerate cell is itself non-degenerate. Then*

$$\otimes_a(X^1, \ldots, X^{b-1}, \partial\Theta_2^{\theta} \hookrightarrow \Theta_2^{\theta}, X^{b+1}, \ldots, X^a) \tag{4.9}$$

is a monomorphism for any $\theta \in \Theta_2$.

Remark. Given a small category $\mathscr{X}$, a functor $G : \mathscr{X} \to \underline{\mathrm{Set}}$ and a weight $W : \mathscr{X}^{\mathrm{op}} \to \underline{\mathrm{Set}}$, the colimit of G weighted by W is isomorphic to the (conical) colimit of the composite

$$H : \int W \longrightarrow \mathscr{X} \xrightarrow{G} \underline{\mathrm{Set}}$$

where $\int W$ is the Grothendieck construction of W and the first factor is the canonical projection. But the colimit of any $\underline{\mathrm{Set}}$-valued functor H may be computed as the set of connected components of $\int H$, thus we can describe the original weighted colimit as the set of connected components of this iterated Grothendieck construction $\int H$.

Since the coend formula expresses the cellular set $\otimes_a(X^1, \ldots, X^a)$ as the weighted colimit of the composite

$$\Theta_2 \times \cdots \times \Theta_2 \hookrightarrow 2\text{-}\underline{\mathrm{Cat}} \times \cdots \times 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{\boxtimes_a} 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{N} \widehat{\Theta_2}$$

with weight given by

$$(\theta_1, \ldots, \theta_a) \mapsto X^1_{\theta_1} \times \cdots \times X^a_{\theta_a},$$

it follows that the value of $\otimes_a(X^1, \ldots, X^a)$ at any $\zeta \in \Theta_2$ may be described as the set of connected components of an appropriate iterated Grothendieck construction. This is how we obtain the categories $\mathscr{B}$ and $\mathscr{C}$ in the proof below.

Proof. Fix $\zeta \in \Theta_2$. We will give a more explicit description of the ζ-component of the natural transformation (4.9).

Let $\mathscr{A}$ be the category whose objects are $(2a+1)$-tuples

$$(\boldsymbol{\kappa}, \phi, \boldsymbol{x}) = (\kappa_1, \ldots, \kappa_a, \phi, x_1, \ldots, x_a)$$

where:

- $\kappa_i \in \Theta_2$ for each $1 \le i \le a$;
- $\phi : \zeta \to \boxtimes_a(\kappa_1, \ldots, \kappa_a)$ is a 2-functor;
- $x_i \in X^i_{\kappa_i}$ for $i \ne b$; and
- $x_b : \kappa_b \to \theta$ is a 2-functor

and whose morphisms $\boldsymbol{\alpha} : (\boldsymbol{\kappa}, \phi, \boldsymbol{x}) \to (\boldsymbol{\lambda}, \chi, \boldsymbol{y})$ consist of cellular operators $\alpha_i : \kappa_i \to \lambda_i$ for $1 \le i \le a$ such that:

- $\chi = \boxtimes_a(\alpha_1, \ldots, \alpha_a) \circ \phi$; and
- $x_i = y_i \cdot \alpha_i$ for $1 \le i \le a$.

Notation. If $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_a)$ is an a-tuple of “things” and ψ is another “thing” then we will denote by $\boldsymbol{\omega}\{\psi\}$ the a-tuple

$$\boldsymbol{\omega}\{\psi\} \overset{\text{def}}{=} (\omega_1, \ldots, \omega_{b-1}, \psi, \omega_{b+1}, \ldots, \omega_a).$$

Let $\mathscr{B}$ be the full subcategory of $\mathscr{A}$ spanned by those $(\boldsymbol{\kappa}, \phi, \boldsymbol{x})$ with $x_b \in \partial\Theta_2^\theta$, and let $\mathscr{C}$ be the full subcategory of $\mathscr{A}$ spanned by those $(\boldsymbol{\kappa}, \phi, \boldsymbol{x})$ with $\kappa_b = \theta$ and $x_b = \mathrm{id}_\theta$. Then there is a functor $F : \mathscr{B} \to \mathscr{C}$ given by

$$F(\boldsymbol{\kappa}, \phi, \boldsymbol{x}) = \big(\boldsymbol{\kappa}\{\theta\}, \boxtimes_a(\mathbf{id}\{x_b\}) \circ \phi, \boldsymbol{x}\{\mathrm{id}_\theta\}\big)$$

and

$$F(\boldsymbol{\alpha}) = \boldsymbol{\alpha}\{\mathrm{id}_\theta\}.$$

Then the ζ-component of the natural transformation (4.9) can be identified with the function $\pi_0(F) : \pi_0(\mathscr{B}) \to \pi_0(\mathscr{C})$ where $\pi_0 : \underline{\mathrm{Cat}} \to \underline{\mathrm{Set}}$ is the connected components functor.

Thus, to prove that (4.9) is a monomorphism, it suffices to show that if $(\boldsymbol{\kappa}, \phi, \boldsymbol{x})$ and $(\boldsymbol{\kappa}', \phi', \boldsymbol{x}')$ are objects in $\mathscr{B}$ and there is a zigzag of (possibly identity) arrows

$$\begin{gathered} F(\boldsymbol{\kappa}, \phi, \boldsymbol{x}) \xrightarrow{\ \boldsymbol{\alpha}^1\ } (\boldsymbol{\lambda}^1, \chi^1, \boldsymbol{y}^1) \xleftarrow{\quad \boldsymbol{\alpha}^2 \quad} \cdots \\ \cdots \xrightarrow{\quad \boldsymbol{\alpha}^m \quad} (\boldsymbol{\lambda}^m, \chi^m, \boldsymbol{y}^m) \xleftarrow{\boldsymbol{\alpha}^{m+1}} F(\boldsymbol{\kappa}', \phi', \boldsymbol{x}') \end{gathered} \tag{4.10}$$

in $\mathscr{C}$ then $(\boldsymbol{\kappa}, \phi, \boldsymbol{x})$ and $(\boldsymbol{\kappa}', \phi', \boldsymbol{x}')$ lie in the same connected component of $\mathscr{B}$. (Here we are assuming m to be odd so that $\boldsymbol{\alpha}^{m+1}$ does really point away from the endpoint; we do not lose generality by doing so since $\boldsymbol{\alpha}^{m+1}$ is allowed to be the identity.)

First, we prove that we may assume each object $(\boldsymbol{\lambda}^k, \chi^k, \boldsymbol{y}^k)$ to be in the image of F.

Temporary definition. We call a zigzag of the form (4.10) *k-admissible* if $(\boldsymbol{\lambda}^\ell, \chi^\ell, \boldsymbol{y}^\ell)$ is in the image of F for all $1 \le \ell \le k$.

Claim. For any $(k-1)$-admissible zigzag of the form (4.10), there exists a k-admissible zigzag that has the same length and the same endpoints.

Proof of the claim. The easier case is when k is odd. In this case, by the inductive hypothesis we have a map

$$\boldsymbol{\alpha^k} : F(\boldsymbol{\lambda}, \chi, \boldsymbol{y}) \to (\boldsymbol{\lambda^k}, \chi^k, \boldsymbol{y^k})$$

for some $(\boldsymbol{\lambda}, \chi, \boldsymbol{y}) \in \mathscr{B}$. Then it is easy to check that

$$(\boldsymbol{\lambda^k}, \chi^k, \boldsymbol{y^k}) = F\big(\boldsymbol{\lambda^k}\{\lambda_b\}, \boxtimes_a(\boldsymbol{\alpha^k}\{\mathrm{id}_{\lambda_b}\}) \circ \chi, \boldsymbol{y^k}\{y_b\}\big).$$

Next suppose that k is even so that we have

$$F(\boldsymbol{\lambda}, \chi, \boldsymbol{y}) \xleftarrow{\boldsymbol{\alpha^k}} (\boldsymbol{\lambda^k}, \chi^k, \boldsymbol{y^k})$$

for some $(\boldsymbol{\lambda}, \chi, \boldsymbol{y}) \in \mathscr{B}$. We first treat the special case where each α^k_i is a face operator. By the definition of $\mathscr{B}$, we have $y_b \in \partial\Theta_2^\theta$. Thus in the Reedy factorisation

$$y_b : \ \lambda_b \xrightarrow{\sigma} \lambda'_b \xrightarrow{y'_b} \theta$$

the second factor y'_b is a non-identity face map. But then we have

$$F(\boldsymbol{\lambda}, \chi, \boldsymbol{y}) = F\big(\boldsymbol{\lambda}\{\lambda'_b\}, \boxtimes_a(\mathbf{id}\{\sigma\}) \circ \chi, \boldsymbol{y}\{y'_b\}\big).$$

Thus we may assume that y_b is itself a non-identity face map. Then the inner square in

$$\begin{tikzcd}
\zeta \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "\chi^k"] \arrow[ddr, bend right, "\chi"'] & & \\
& \boxtimes_a(\boldsymbol{\lambda^k}\{\lambda_b\}) \arrow[r, "\boxtimes_a(\mathbf{id}\{y_b\})"] \arrow[d, "\boxtimes_a(\boldsymbol{\alpha^k}\{\mathrm{id}_{\lambda_b}\})"] & \boxtimes_a(\boldsymbol{\lambda^k}\{\theta\}) \arrow[d, "\boxtimes_a(\boldsymbol{\alpha^k})"] \\
& \boxtimes_a(\boldsymbol{\lambda}) \arrow[r, "\boxtimes_a(\mathbf{id}\{y_b\})"'] & \boxtimes_a(\boldsymbol{\lambda}\{\theta\})
\end{tikzcd}$$

is a pullback square (which can be checked using Lemma 4.1.11), and the outer square commutes since $\boldsymbol{\alpha^k}$ is a morphism in $\mathscr{C}$. Hence we obtain the induced map ψ which then satisfies

$$(\boldsymbol{\lambda^k}, \chi^k, \boldsymbol{y^k}) = F(\boldsymbol{\lambda^k}\{\lambda_b\}, \psi, \boldsymbol{y^k}\{y_b\}).$$

This completes the proof of the special case where each α^k_i is a face operator.

Now consider the general case. Note that $(\boldsymbol{\lambda^{k+1}}, \chi^{k+1}, \boldsymbol{y^{k+1}})$ is well-defined since k is even and m is odd. If $y^{k+1}_i = z_i \cdot \iota_i$ for some $z_i \in X^i_{\mu_i}$ and $\iota_i : \lambda_i \to \mu_i$, then we can replace $\boldsymbol{\alpha^{k+1}}$ and $\boldsymbol{\alpha^{k+2}}$ by their respective composites with $\boldsymbol{\iota}$:

$$\begin{tikzcd}
& (\boldsymbol{\mu}, \boxtimes_a(\boldsymbol{\iota}) \circ \chi^{k+1}, \boldsymbol{z}) & \\
(\boldsymbol{\lambda^k}, \chi^k, \boldsymbol{y^k}) \arrow[r, "\boldsymbol{\alpha^{k+1}}"] & (\boldsymbol{\lambda^{k+1}}, \chi^{k+1}, \boldsymbol{y^{k+1}}) \arrow[u, "\boldsymbol{\iota}"] & ? \arrow[l, "\boldsymbol{\alpha^{k+2}}"']
\end{tikzcd}$$

(in which "?" is either $(\boldsymbol{\lambda}^{\boldsymbol{k+2}}, \chi^{k+2}, \boldsymbol{y}^{\boldsymbol{k+2}})$ or $F(\boldsymbol{\kappa}', \phi', \boldsymbol{x}'))$ to obtain a new zigzag. Thus we may assume that each y_i^{k+1} is non-degenerate. Similarly we may assume that each y_i is non-degenerate.

Let $\alpha_i^k = \delta_i^k \circ \sigma_i^k$ and $\alpha_i^{k+1} = \delta_i^{k+1} \circ \sigma_i^{k+1}$ be the Reedy factorisations of α_i^k and α_i^{k+1} respectively. Then we have the solid part of the following commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd}
& \big(\boxtimes_a(\boldsymbol{\sigma}^{\boldsymbol{k}}) \circ \chi^k, \boldsymbol{y} \cdot \boldsymbol{\delta}^{\boldsymbol{k}}\big) \arrow[dl, "\boldsymbol{\delta}^{\boldsymbol{k}}"'] \arrow[rr, dashed, equal] & & \big(\boxtimes_a(\boldsymbol{\sigma}^{\boldsymbol{k+1}}) \circ \chi^k, \boldsymbol{y}^{\boldsymbol{k+1}} \cdot \boldsymbol{\delta}^{\boldsymbol{k+1}}\big) \arrow[dr, "\boldsymbol{\delta}^{\boldsymbol{k+1}}"] & \\
F(\chi, \boldsymbol{y}) & & (\chi^k, \boldsymbol{y}^{\boldsymbol{k}}) \arrow[ll, "\boldsymbol{\alpha}^{\boldsymbol{k}}"] \arrow[rr, "\boldsymbol{\alpha}^{\boldsymbol{k+1}}"'] \arrow[ul, "\boldsymbol{\sigma}^{\boldsymbol{k}}"'] \arrow[ur, "\boldsymbol{\sigma}^{\boldsymbol{k+1}}"] & & (\chi^{k+1}, \boldsymbol{y}^{\boldsymbol{k+1}})
\end{tikzcd}$$

where we are omitting the first a coordinates of each object. For each $i \neq b$, the cells $y_i \cdot \delta_i^k$ and $y_i^{k+1} \cdot \delta_i^{k+1}$ are non-degenerate since y_i and y_i^{k+1} are non-degenerate and the non-degenerate cells in X^i are assumed to be closed under taking faces. Thus both

$$y_i^k = y_i^{k+1} \cdot \alpha_i^{k+1} = (y_i^{k+1} \cdot \delta_i^{k+1}) \cdot \sigma_i^{k+1}$$

and

$$y_i^k = y_i \cdot \alpha_i^k = (y_i \cdot \delta_i^k) \cdot \sigma_i^k$$

express y_i^k as a degeneracy of a non-degenerate cell. By the uniqueness of such a presentation, we must have $y_i^{k+1} \cdot \delta_i^{k+1} = y_i \cdot \delta_i^k$ and $\sigma_i^{k+1} = \sigma_i^k$, and so we have an equality as indicated above. Therefore we can replace the segment

$$F(\chi, \boldsymbol{y}) \xleftarrow{\boldsymbol{\alpha}^{\boldsymbol{k}}} (\chi^k, \boldsymbol{y}^{\boldsymbol{k}}) \xrightarrow{\boldsymbol{\alpha}^{\boldsymbol{k+1}}} (\chi^{k+1}, \boldsymbol{\alpha}^{\boldsymbol{k+1}})$$

of the zigzag by

$$F(\chi, \boldsymbol{y}) \xleftarrow{\boldsymbol{\delta}^{\boldsymbol{k}}} \big(\boxtimes_a(\boldsymbol{\sigma}^{\boldsymbol{k}}) \circ \chi^k, \boldsymbol{y} \cdot \boldsymbol{\delta}^{\boldsymbol{k}}\big) \xrightarrow{\boldsymbol{\delta}^{\boldsymbol{k+1}}} (\chi^{k+1}, \boldsymbol{\alpha}^{\boldsymbol{k+1}})$$

which reduces the problem to the special case treated above. This completes the proof of the claim. □

Thus by induction, we can turn any zigzag of the form (4.10) into a k-admissible one for any k. In particular, we may assume that the zigzag is m-admissible so that each object $(\boldsymbol{\lambda}^{\boldsymbol{k}}, \chi^k, \boldsymbol{y}^{\boldsymbol{k}})$ is in the image of F. Therefore it suffices to prove that, if $(\boldsymbol{\kappa}, \phi, \boldsymbol{x}), (\boldsymbol{\kappa}', \phi', \boldsymbol{x}') \in \mathscr{B}$ and there is a morphism

$$\boldsymbol{\alpha} : F(\boldsymbol{\kappa}, \phi, \boldsymbol{x}) \to F(\boldsymbol{\kappa}', \phi', \boldsymbol{x}')$$

in $\mathscr{C}$ then $(\boldsymbol{\kappa}, \phi, \boldsymbol{x})$ and $(\boldsymbol{\kappa}', \phi', \boldsymbol{x}')$ lie in the same connected component of $\mathscr{B}$. Note that if $x_b : \ \kappa_b \xrightarrow{\sigma} \lambda \xrightarrow{\delta} \theta$ is the Reedy factorisation of x_b then

$$\mathbf{id}\{\sigma\} : (\boldsymbol{\kappa}, \phi, \boldsymbol{x}) \to (\boldsymbol{\kappa}\{\lambda\}, \boxtimes_a(\mathbf{id}\{\sigma\}) \circ \phi, \boldsymbol{x}\{\delta\})$$

is a map in $\mathscr{B}$ and F sends it to the identity at $F(\boldsymbol{\kappa}, \phi, \boldsymbol{x})$. Thus we may assume that x_b is a non-identity face map into θ, and similarly for x_b'. We construct the dashed part of the following diagram as follows:

$$\begin{tikzcd}
\zeta \arrow[rr, "\phi"] \arrow[dr, dashed] \arrow[dd, "\phi'"'] & & \boxtimes_a(\boldsymbol{\kappa}) \arrow[d, "\pi_b"] \\
& \lambda \arrow[r, dashed, "\iota"] \arrow[d, dashed, "\iota'"'] & \kappa_b \arrow[d, "x_b"] \\
\boxtimes_a(\boldsymbol{\kappa}') \arrow[r, "\pi'_b"'] & \kappa'_b \arrow[r, "x'_b"'] & \theta
\end{tikzcd}$$

(Note that the solid part commutes since α is a morphism in $\mathscr{C}$.) Let $s \in \theta$ and $t \in \theta$ be the images of the first and last objects under the (unique) composite $\zeta \to \theta$ respectively. Let $m_0, m_1, \ldots, m_n \in \theta$ be the increasingly ordered list of objects m such that $s \le m \le t$ and m is in the images of both x_b and x'_b. For each $1 \le k \le n$, let $f^k_0, f^k_1, \ldots, f^k_{q_k} \in \theta(m_{k-1}, m_k)$ be those 1-cells through which some 1-cell in the image of $\zeta \to \theta$ factors (again increasingly ordered). Then we set $\lambda = [n; \mathbf{q}] \in \Theta_2$, and the obvious maps

$$\zeta \to \lambda, \quad \iota : \lambda \to \kappa_b, \quad \iota' : \lambda \to x'_b$$

fit into the above commutative diagram. Now consider the following diagram:

$$\begin{tikzcd}
\zeta \arrow[r, "\phi"] \arrow[d, dashed, "\chi"] \arrow[dd, bend right, "\phi'"'] & \boxtimes_a(\boldsymbol{\kappa}) \arrow[r, "\boxtimes_a(\mathbf{id}\{x_b\})"] \arrow[d, "\boxtimes_a(\alpha\{\mathrm{id}_{\kappa_b}\})"] & \boxtimes_a(\boldsymbol{\kappa}\{\theta\}) \arrow[dd, "\boxtimes_a(\alpha)"] \\
\boxtimes_a(\boldsymbol{\kappa}'\{\lambda\}) \arrow[r, "\boxtimes_a(\mathbf{id}\{\iota\})"] \arrow[d, "\boxtimes_a(\mathbf{id}\{\iota'\})"] & \boxtimes_a(\boldsymbol{\kappa}'\{\kappa_b\}) \arrow[dr, "\boxtimes_a(\mathbf{id}\{x_b\})"] & \\
\boxtimes_a(\boldsymbol{\kappa}') \arrow[rr, "\boxtimes_a(\mathbf{id}\{x'_b\})"'] & & \boxtimes_a(\boldsymbol{\kappa}'\{\theta\})
\end{tikzcd}$$

The perimeter commutes because $\boldsymbol{\alpha}$ is a morphism in $\mathscr{C}$, whereas the bottom quadrangle commutes because it is the image of the inner square in the previous diagram under $\boxtimes_a(\boldsymbol{\kappa}'\{-\})$. That the right quadrangle commutes is just functoriality of $\boxtimes_a$. It can be seen from our construction of $\kappa'_b \xleftarrow{\iota'} \lambda \xrightarrow{\iota} \kappa_b$ and Lemma 4.1.11 that there is a map χ that renders the whole diagram commutative. Thus the following zigzag in $\mathscr{B}$ connects $(\boldsymbol{\kappa}, \phi, \boldsymbol{x})$ and $(\boldsymbol{\kappa}', \phi', \boldsymbol{x}')$:

$$\begin{tikzcd}
(\boldsymbol{\kappa}, \phi, \boldsymbol{x}) \arrow[r, "\alpha\{\mathrm{id}_{\kappa_b}\}"] & \big(\boldsymbol{\kappa}'\{\kappa_b\}, \boxtimes_a(\alpha\{\mathrm{id}_{\kappa_b}\}) \circ \phi, \boldsymbol{x}'\{x_b\}\big) \arrow[d, equal] \\
\big(\boldsymbol{\kappa}'\{\lambda\}, \chi, \boldsymbol{x}'\{x_b \cdot \iota\}\big) \arrow[r, "\mathbf{id}\{\iota\}"] \arrow[d, equal] & \big(\boldsymbol{\kappa}'\{\kappa_b\}, \boxtimes_a(\mathbf{id}\{\iota\}) \circ \chi, \boldsymbol{x}'\{x_b\}\big) \\
\big(\boldsymbol{\kappa}'\{\lambda\}, \chi, \boldsymbol{x}'\{x'_b \cdot \iota'\}\big) \arrow[r, "\mathbf{id}\{\iota'\}"] & (\boldsymbol{\kappa}', \phi', \boldsymbol{x}')
\end{tikzcd}$$

This completes the proof. $\square$

4.3 Some visual concepts

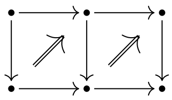
In the next section, we prove that the functor $\otimes = \otimes_2$ is left Quillen. Note that $\mathcal{I} \hat{\otimes} \mathcal{I} \subseteq \mathrm{cell}(\mathcal{I})$ is an instance of Lemma 4.2.1. Thus by Corollary 3.4.4, it suffices to show that $\mathcal{I} \hat{\otimes} \mathcal{J} \subseteq \mathrm{cell}(\mathcal{J})$ and $\mathcal{J} \hat{\otimes} \mathcal{I} \subseteq \mathrm{cell}(\mathcal{J})$.

4.3.1 A low dimensional example

The aim of this subsection is to illustrate our general strategy by considering the special instance

$$\begin{pmatrix} \Lambda^1_h[2;0,0] \\ \big\downarrow \\ \Theta_2[2;0,0] \end{pmatrix} \hat{\otimes} \begin{pmatrix} \partial\Theta_2[1;0] \\ \big\downarrow \\ \Theta_2[1;0] \end{pmatrix}.$$

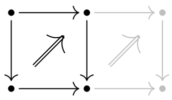
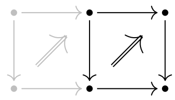
By definition, the codomain $\Theta_2[2;0,0] \otimes \Theta_2[1;0]$ of this map is the nerve of the 2-category $[2;0,0] \boxtimes [1;0]$ which looks like



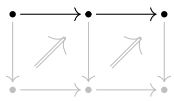
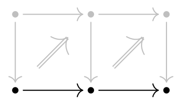
and its domain is the cellular subset

$$X = \left(\Lambda^1_h[2;0,0] \otimes \Theta_2[1;0]\right) \cup \left(\Theta_2[2;0,0] \otimes \partial\Theta_2[1;0]\right)$$

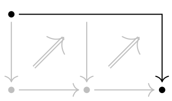
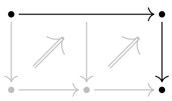
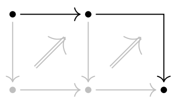
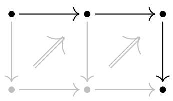
of $\Theta_2[2;0,0] \otimes \Theta_2[1;0]$. The first part $\Lambda^1_h[2;0,0] \otimes \Theta_2[1;0]$ is generated by the nerves of the sub-2-categories

 and 

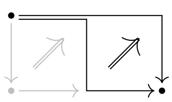
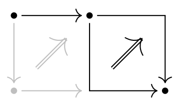
and $\Theta_2[2;0,0] \otimes \partial\Theta_2[1;0]$ is generated by the nerves of

 and  .

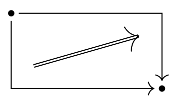
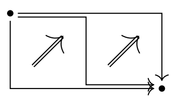
We wish to show that $X \hookrightarrow \Theta_2[2;0,0] \otimes \Theta_2[1;0]$ is a trivial cofibration. We categorise the non-degenerate cells in $(\Theta_2[2;0,0] \otimes \Theta_2[1;0]) \setminus X$ into six kinds according to their "silhouette". The cells

 ,  ,  and 

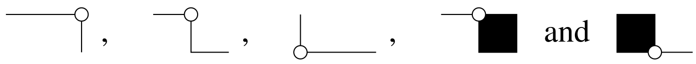
have the same silhouette "──┐". Similarly there are four cells of silhouette "─┐└─" and four of silhouette "└──". There are two cells

 and 

of silhouette "─■", and similarly for "■─". Finally, the cells

 and 

have silhouette "■". We can associate a cut-point (= a point that disconnects the shape if removed) to each silhouette except for the last one as follows:



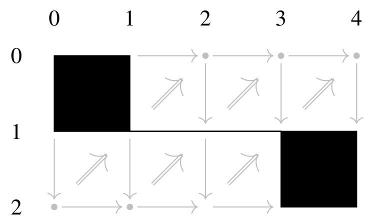
Observe that the set of non-degenerate cells of these "cuttable" silhouettes can then be partitioned into pairs of the form $\{\phi, \phi \cdot \delta_h^{k_\phi}\}$ where the k_ϕ-th vertex of ϕ is the cut-point associated to the silhouette of ϕ. We can glue such ϕ to X along $\Lambda_h^{k_\phi}$ in increasing order of $\dim \phi$, and then glue the above $(1;2)$-cell of silhouette "■" along $\Lambda_v^{1;1}[1;2]$. This exhibits the inclusion $X \hookrightarrow \Theta_2[2;0,0] \otimes \Theta_2[1;0]$ as a member of $\mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v)$.

4.3.2 Silhouettes and cut-points

We will formalise the notions of *silhouette* and *cut-point* which were vaguely defined in the previous subsection. Fix $\theta_1, \ldots, \theta_a \in \Theta_2$.

Definition 4.3.1. A *silhouette* σ in $\otimes_a\left(\Theta_2^{\theta_1}, \ldots, \Theta_2^{\theta_a}\right)$ is a $(1;1)$-cell regarded as a pair of $(1;0)$-cells $\sigma = (\sigma_0, \sigma_1)$ where $\sigma_0 = \sigma \cdot \delta_v^{1;1}$ is the source and $\sigma_1 = \sigma \cdot \delta_v^{1;0}$ is the target.[1] We write $\leq_0^\sigma$ and $\leq_1^\sigma$ for the underlying shuffles of σ_0 and σ_1 respectively.

For example, the following picture depicts a silhouette in $\Theta_2^{4;\mathbf{0}} \otimes \Theta_2^{2;\mathbf{0}}$:



(4.11)

For each $\boldsymbol{s}, \boldsymbol{t} \in \boxtimes_a(\boldsymbol{\theta})$, we put a partial order on the set of silhouettes with endpoints $\boldsymbol{s}, \boldsymbol{t}$ so that $\sigma \leq \tau$ if and only if

$$\tau_0 \leq \sigma_0 \leq \sigma_1 \leq \tau_1$$

holds in the poset $\boxtimes_a(\boldsymbol{\theta})(\boldsymbol{s}, \boldsymbol{t})$. This should be thought of as the containment relation between the silhouettes.

Definition 4.3.2. Let σ be a silhouette with endpoints $\boldsymbol{s}, \boldsymbol{t}$. Then a *cut-point* in σ is an object $\boldsymbol{x}$ with $\boldsymbol{s} \neq \boldsymbol{x} \neq \boldsymbol{t}$ such that both σ_0 and σ_1 visit $\boldsymbol{x}$. We call a silhouette *cuttable* if it admits a cut-point.

For example, the silhouette (4.11) has cut-points $(1,1)$, $(2,1)$ and $(3,1)$. The following proposition follows from Definition 4.1.10.

Proposition 4.3.3. *Let σ be a silhouette in $\boxtimes_a(\boldsymbol{\theta})$ with endpoints $\boldsymbol{s}, \boldsymbol{t}$ and let $\boldsymbol{x} \in \boxtimes_a(\boldsymbol{\theta})$. Then $\boldsymbol{x}$ is a cut-point in σ if and only if:*

- *$s_i \leq x_i \leq t_i$ for each i (which implies $S(\boldsymbol{s}, \boldsymbol{t}) = S(\boldsymbol{s}, \boldsymbol{x}) \cup S(\boldsymbol{x}, \boldsymbol{t})$);*
- *$\boldsymbol{s} \neq \boldsymbol{x} \neq \boldsymbol{t}$; and*

[1] We are making this distinction between a silhouette and a $(1;1)$-cell mainly so that Definitions 4.3.2, 4.3.7 and 4.3.10 do not cause ambiguity.

- *both* $(i|k) \leq_0^\sigma (j|\ell)$ *and* $(i|k) \leq_1^\sigma (j|\ell)$ *hold for any* $(i|k) \in S(\boldsymbol{s}, \boldsymbol{x})$ *and* $(j|\ell) \in S(\boldsymbol{x}, \boldsymbol{t})$.

Definition 4.3.4. A cut-point $\boldsymbol{x}$ in a silhouette σ is *right-angled* if for any i with $s_i < x_i < t_i$, either:

- $(i|x_i + 1)$ is not the immediate $\leq_0^\sigma$-successor of $(i|x_i)$; or
- $(i|x_i + 1)$ is not the immediate $\leq_1^\sigma$-successor of $(i|x_i)$.

To continue our example (4.11), the cut-point $(2, 1)$ is not right-angled since $(1|3)$ is the immediate successor of $(1|2)$ with respect to both $\leq_0^\sigma$ and $\leq_1^\sigma$. The other two cut-points $(1, 1)$ and $(3, 1)$ are right-angled.

Definition 4.3.5. A silhouette in $\otimes_a\big(\Theta_2^{\theta_1}, \dots, \Theta_2^{\theta_a}\big)$ is said to be *non-linear* if it has endpoints $\boldsymbol{s}$ and $\boldsymbol{t}$ such that $s_i < t_i$ for at least two i's.

Lemma 4.3.6. *Let σ be a non-linear, cuttable silhouette. Then σ admits a right-angled cut-point.*

Proof. Let $\boldsymbol{s}$ and $\boldsymbol{t}$ denote the endpoints of σ. We will first treat the case where σ_0 and σ_1 visit exactly the same set of objects. Note that in this case any object that σ_0 visits is a cut-point in σ. By non-linearity, we must have $(j|\ell), (j'|\ell') \in S(\boldsymbol{s}, \boldsymbol{t})$ with $j \neq j'$ such that $(j'|\ell')$ is the immediate $\leq_0^\sigma$-successor of $(j|\ell)$. Then the object $\boldsymbol{x}$ defined by

$$x_i = \min\Big(\big\{(i|k) \in S(\boldsymbol{s}, \boldsymbol{t}) : (i|k) \leq_0^\sigma (j|\ell)\big\} \cup \{s_i\}\Big)$$

is a right-angled cut-point.

In the other case, there must be a cut-point $\boldsymbol{x}$ such that σ_0 visits a non-cut-point object $\boldsymbol{y}$ with $\boldsymbol{s} \neq \boldsymbol{y} \neq \boldsymbol{t}$ immediately before or immediately after $\boldsymbol{x}$. Such $\boldsymbol{x}$ then is necessarily right-angled. □

Note that for any silhouette σ, the set of cut-points in σ admits a total order given by $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if $x_i \leq y_i$ for each i.

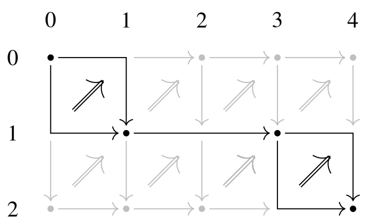
Definition 4.3.7. If σ is a non-linear, cuttable silhouette, then we write $\mathrm{cut}(\sigma)$ for the first right-angled cut-point in σ (whose existence is guaranteed by Lemma 4.3.6).

4.3.3 Silhouettes of cells

Definition 4.3.8. For any $(n; \boldsymbol{q})$-cell ϕ in $\otimes_a\big(\Theta_2^{\theta_1}, \dots, \Theta_2^{\theta_a}\big)$, the *silhouette of* ϕ is

$$\mathrm{sil}(\phi) \overset{\text{def}}{=} \Big(\phi \cdot \big[\{0, n\}; \{0\}, \dots, \{0\}\big],\ \phi \cdot \big[\{0, n\}; \{q_1\}, \dots, \{q_n\}\big]\Big).$$

For example, if ϕ is the $(3; 1, 0, 1)$-cell



(4.12)

in $\Theta_2^{4;\boldsymbol{0}} \otimes \Theta_2^{2;\boldsymbol{0}}$ then $\mathrm{sil}(\phi)$ is the silhouette (4.11).

Proposition 4.3.9. *Let ϕ be a non-degenerate cell in $\otimes_a(\Theta_2^{\theta_1}, \dots, \Theta_2^{\theta_a})$. Then a face of ϕ has the same silhouette as that of ϕ if and only if it is an inner face.*

Definition 4.3.10. A non-degenerate, non-linear cell $\phi : [n; \mathbf{q}] \to \boxtimes_a(\boldsymbol{\theta})$ is said to be:

- *potentially cuttable* if $\mathrm{sil}(\phi)$ is cuttable;
- *cuttable* if it is potentially cuttable and moreover there is $k \in [n; \mathbf{q}]$ such that $\phi(k) = \mathrm{cut}\big(\mathrm{sil}(\phi)\big)$; and
- *absolutely uncuttable* if $\mathrm{sil}(\phi)$ is not cuttable.

If ϕ is a cuttable cell, we write $\mathrm{cut}(\phi)$ for the necessarily unique $0 < k < n$ satisfying $\phi(k) = \mathrm{cut}\big(\mathrm{sil}(\phi)\big)$.

Proposition 4.3.11. *Let χ be a potentially cuttable cell in $\otimes_a(\Theta_2^{\theta_1}, \dots, \Theta_2^{\theta_a})$ that is not cuttable. Then there exists a unique cuttable cell ϕ such that χ is a $\mathrm{cut}(\phi)$-th horizontal face of ϕ.*

Conversely, if ϕ is a cuttable $(n; \boldsymbol{q})$-cell and $\delta : [n-1; \boldsymbol{p}] \to [n; \mathbf{q}]$ is a $\mathrm{cut}(\phi)$-th horizontal face operator, then $\phi \cdot \delta$ is potentially cuttable but not cuttable.

Proof. The second part follows from Proposition 4.3.9. We will prove the first part in the special case where $\theta_i = [n_i; \mathbf{0}]$ for each i and χ is a $(1; q)$-cell. The general case can be treated similarly and is left to the reader.

In this special case, χ is solely determined by its underlying shuffles $\preceq_p$ on $S(\boldsymbol{s}, \boldsymbol{t})$ where $\boldsymbol{s}, \boldsymbol{t} \in \boxtimes_a(\boldsymbol{\theta})$ are the endpoints of χ. Let $\boldsymbol{x} = \mathrm{cut}(\mathrm{sil}(\chi))$ and suppose we are given $(i|k) \in S(\boldsymbol{s}, \boldsymbol{x})$ and $(j|\ell) \in S(\boldsymbol{x}, \boldsymbol{t})$. Then $\preceq_0 = \preceq_0^{\mathrm{sil}(\chi)}$ and $\preceq_q = \preceq_1^{\mathrm{sil}(\chi)}$ by the definition of $\mathrm{sil}(\chi)$, hence we have $(i|k) \preceq_0 (j|\ell)$ and $(i|k) \preceq_q (j|\ell)$ by Proposition 4.3.3. Thus for any $0 \le p \le q$:

- if $i < j$ then we must have $(i|k) \preceq_p (j|\ell)$ since $\preceq_0 \blacktriangleleft \preceq_p$;
- if $i > j$ then we must have $(i|k) \preceq_p (j|\ell)$ for otherwise it contradicts our assumption that $\preceq_p \blacktriangleleft \preceq_q$; and
- if $i = j$ then $(i|k) \preceq_0 (j|\ell)$ implies $k < \ell$ since $\preceq_0$ is a shuffle, which in turn implies $(i|k) \preceq_p (i|\ell)$ since $\preceq_p$ is a shuffle.

This shows that $(i|k) \preceq_p (j|\ell)$ holds for any $(i|k) \in S(\boldsymbol{s}, \boldsymbol{x})$, $(j|\ell) \in S(\boldsymbol{x}, \boldsymbol{t})$ and $0 \le p \le q$.

Define two equivalence relations $\sim_1$, $\sim_2$ on the set $[q]$ so that:

- $p \sim_1 p'$ if and only if $\preceq_p$ and $\preceq_{p'}$ restrict to the same shuffle on $S(\boldsymbol{s}, \boldsymbol{x})$; and
- $p \sim_2 p'$ if and only if $\preceq_p$ and $\preceq_{p'}$ restrict to the same shuffle on $S(\boldsymbol{x}, \boldsymbol{t})$.

Then the desired cuttable cell ϕ is the obvious $(2; q_1, q_2)$-cell where $[q_1] \cong [q]/\sim_1$ and $[q_2] \cong [q]/\sim_2$. □

Definition 4.3.12. In the situation of Proposition 4.3.11, we say ϕ is the *cuttable parent* of χ.

Lemma 4.3.13. *Let $f_i : X^i \to \Theta_2^{\theta_i}$ be a monomorphism in $\widehat{\Theta_2}$ for each i, and let χ be a potentially cuttable cell in $\otimes_a\big(\Theta_2^{\theta_1}, \dots, \Theta_2^{\theta_a}\big)$ that is not cuttable. Then χ is in the image of the monomorphism*

$$\otimes_a(f_1, \dots, f_a) : \otimes_a(X^1, \dots, X^a) \to \otimes_a\big(\Theta_2^{\theta_1}, \dots, \Theta_2^{\theta_a}\big)$$

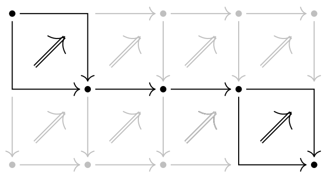
if and only if the cuttable parent of χ is in the image.

Proof. This follows from Lemma 4.1.11. □

Remark. Lemma 4.3.13 relies crucially on the fact that $\mathrm{cut}(\sigma)$ is right-angled. For example, if we had defined $\mathrm{cut}(\mathrm{sil}(\phi)) = (2, 1)$ for the $(3; 1, 0, 1)$-cell ϕ from (4.12) then ϕ is in the image of

$$\delta_h^2 \otimes \mathrm{id} : \Theta_2^{3;\mathbf{0}} \otimes \Theta_2^{2;\mathbf{0}} \to \Theta_2^{4;\mathbf{0}} \otimes \Theta_2^{2;\mathbf{0}}$$

whereas its parent



is not. On the other hand, the fact that $\mathrm{cut}(\sigma)$ is the first one among all right-angled cut-points is not really necessary. Any right-angled cut-point would suffice for our purposes, and we are choosing the first one purely for the sake of definiteness.

4.4 $\otimes_2$ is left Quillen

This section is devoted to proving the following theorem.

Theorem 4.4.1. *The binary Gray tensor product functor $\otimes = \otimes_2$ is left Quillen.*

Proof. By Corollary 3.4.4, it suffices to prove that the map $f \hat{\otimes} g$ is a cofibration if $f, g \in \mathcal{I}$ and it is a trivial cofibration if one of f and g is in $\mathcal{I}$ and the other is in $\mathcal{J}$. The first part is an instance of Lemma 4.2.1, and the second part follows from Lemmas 4.4.2 to 4.4.4 and 4.4.7 proved below (and their duals). □

4.4.1 Inner horizontal horn inclusion $\hat{\otimes}$ boundary inclusion

Let $[m; \mathbf{p}], [n; \mathbf{q}] \in \Theta_2$ and let $1 \le k \le m - 1$. The aim of this subsection is to prove the following lemma.

Lemma 4.4.2. *The map*

$$\big(\Lambda_h^k[m; \mathbf{p}] \hookrightarrow \Theta_2[m; \mathbf{p}]\big) \hat{\otimes} \big(\partial\Theta_2[n; \mathbf{q}] \hookrightarrow \Theta_2[n; \mathbf{q}]\big)$$

is in $\mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v)$.

Proof. We will denote this map by $A \hookrightarrow B$. It is a monomorphism by Lemmas 2.2.11 and 4.2.1 so we may regard A as a cellular subset of $B = N\big([m; \mathbf{p}] \boxtimes [n; \mathbf{q}]\big)$. Since the case $[n; \mathbf{q}] = [0]$ is trivial, we will assume $n \ge 1$.

Let $A' \subset B$ be the cellular subset generated by A and the (potentially) cuttable cells. Note that any cell in $B \setminus A$ is non-linear. Moreover, it follows from Proposition 4.3.11

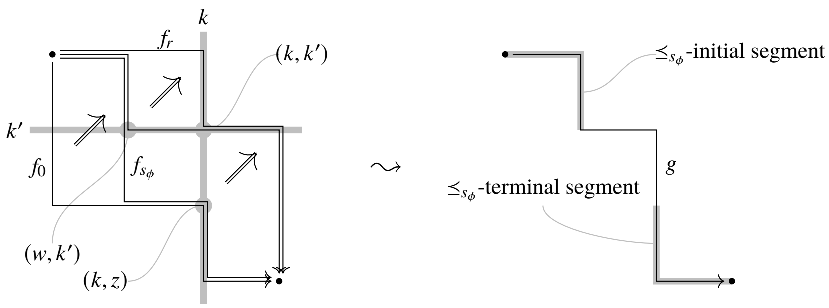


Figure 4.1: Example: $[m;\mathbf{p}] = [n;\mathbf{q}] = [3;\mathbf{0}]$ and $k = 2$.

and Lemma 4.3.13 that the set of non-degenerate cells in $A' \setminus A$ can be partitioned into subsets of the form

$$\{\phi \text{ and all of its cut}(\phi)\text{-th horizontal faces}\}$$

where ϕ is a cuttable cell. We prove that A' may be obtained from A by gluing cuttable ϕ along the inner horn $\Lambda_h^{\mathrm{cut}(\phi)}$ in lexicographically increasing order of $\mathrm{sil}(\phi)$ and $\dim(\phi)$. That is, given two cuttable cells χ and ϕ, we glue χ before ϕ if:

- $\mathrm{sil}(\chi) < \mathrm{sil}(\phi)$; or
- $\mathrm{sil}(\chi) = \mathrm{sil}(\phi)$ and $\dim(\chi) < \dim(\phi)$.

Fix a cuttable cell ϕ in $A' \setminus A$. We must check that all hyperfaces of ϕ except for the cut(ϕ)-th horizontal ones are contained either in A or in some cuttable χ satisfying one of the two conditions described above. Indeed, all outer hyperfaces of ϕ have smaller silhouettes than ϕ, and all inner hyperfaces χ of ϕ except for the cut(ϕ)-th horizontal ones are cuttable and satisfy $\mathrm{sil}(\chi) = \mathrm{sil}(\phi)$ and $\dim(\chi) < \dim(\phi)$. Thus the inclusion $A \hookrightarrow A'$ is in $\mathrm{cell}(\mathcal{H}_h)$.

Now we consider an absolutely uncuttable cell ϕ in B with endpoints $(0,0)$ and (m,n) (which may or may not be contained in A). Such ϕ is necessarily a $(1;r)$-cell for some $r \geq 1$. Thus ϕ can be identified with a chain $f_0 < \cdots < f_r$ in the poset $\big([m;\mathbf{p}] \boxtimes [n;\mathbf{q}]\big)\big((0,0),(m,n)\big)$. For each $0 \leq s \leq r$, we write $\leq_s$ for the underlying shuffle of f_s.

Let $k' \in [n]$ be the largest element such that f_r visits (k,k'). Note that we must have $k' < n$ for otherwise $(k,k') = (k,n)$ would be a cut-point in $\mathrm{sil}(\phi)$. Define s_ϕ to be the largest element $s \in [r]$ such that

$$(2|k'+1) \leq_s (1|k)$$

holds; equivalently, s_ϕ is the largest s such that f_s does not visit (k,k') (see Fig. 4.1). Such s_ϕ indeed exists for otherwise (k,k') is a cut-point in $\mathrm{sil}(\phi)$.

We will construct the “best approximation” g to f_{s_ϕ} that visits (k,k'). Let $w \in [m]$ be the maximum such that f_{s_ϕ} visits (w,k') and let $z \in [n]$ be the minimum such that f_{s_ϕ} visits (k,z). Then we must have $0 \leq w < k$ and $k' < z \leq n$. Now let $g : (0,0) \to (m,n)$ be the 1-cell determined by the following conditions:

- both of the projections $[m;\mathbf{p}] \leftarrow [m;\mathbf{p}] \boxtimes [n;\mathbf{q}] \to [n;\mathbf{q}]$ send g and f_{s_ϕ} to the same 1-cell; and
- the underlying shuffle $\leq$ of g is obtained by patching together the following (see Fig. 4.1):
  - the $\leq_{s_\phi}$-initial segment up to just before $(2|k'+1)$;

– the $\leq_{s_\phi}$-terminal segment starting just after $(1|k)$; and
– the interval

$$(1|w+1) \leq (1|w+2) \leq \ldots \leq (1|k) \leq (2|k'+1) \leq (2|k'+2) \leq \ldots \leq (2|z).$$

These data indeed specify a unique 1-cell by Lemma 4.1.7, and moreover it is easy to see that $f_{s_\phi} < g \leq f_{s_\phi+1}$ holds in the hom-poset. Consider the following condition on ϕ:

(hh) $f_{s_\phi+1} = g$.

(Here “hh” stands for “horizontal horn”.)

It is obvious that the set of absolutely uncuttable cells in B with endpoints $(0,0)$ and (m,n) can be partitioned into pairs of the form $\{\phi, \phi \cdot \delta_v^{1;s_\phi+1}\}$ where ϕ satisfies (hh). Note that if ϕ satisfies (hh) then $s_\phi + 1 \neq r$ since $f_{s_\phi+1}$ visits $(k, k'+1)$ while f_r does not. Also we have $s_\phi + 1 \neq 0$ since $s_\phi \geq 0$.

Claim. A cell ϕ satisfying (hh) is contained in A' (or equivalently in A) if and only if $\phi \cdot \delta_v^{1;s_\phi+1}$ is contained in A' (or equivalently in A).

Proof of the claim. The “only if” part is obvious. For the “if” part, we first treat the case where $\phi \cdot \delta_v^{1;s_\phi+1}$ is contained in the cellular subset $\Theta_2[m;\mathbf{p}] \otimes \partial\Theta_2[n;\mathbf{q}]$. For most hyperface maps δ into $[n;\mathbf{q}]$, if $\phi \cdot \delta_v^{1;s_\phi+1}$ is contained in the image of some $\mathrm{id} \otimes \delta$ then we can apply Lemma 4.1.11 twice to deduce that ϕ is in the image of same map, using the fact that the 1-cell g constructed above is “almost” f_{s_ϕ}. The only non-trivial sub-case is when $\phi \cdot \delta_v^{1;s_\phi+1}$ is in the image of

$$\mathrm{id} \otimes \delta_h^{k';\langle\alpha,\alpha'\rangle} : \Theta_2[m;\mathbf{p}] \otimes \Theta_2[n-1;\boldsymbol{q}'] \to \Theta_2[m;\mathbf{p}] \otimes \Theta_2[n;\mathbf{q}]$$

for some (q_k, q_{k+1})-shuffle $\langle\alpha,\alpha'\rangle$. Here the same argument does not apply since it may be possible that $(2|k'+1)$ is the immediate successor of $(2|k')$ with respect to $\leq_{s_\phi}$ but not with respect to $\leq$. However, we have

$$(2|k') \leq_r (1|k+1) \leq_r (2|k'+1)$$

by our definition of k' which implies that $\phi \cdot \delta_v^{1;s_\phi+1}$ is never contained in the image of $\mathrm{id} \otimes \delta_h^{k';\langle\alpha,\alpha'\rangle}$.

Next, suppose that $\phi \cdot \delta_v^{1;s_\phi+1}$ is contained in the cellular subset $\Lambda_h^k[m;\mathbf{p}] \otimes \Theta_2[n;\mathbf{q}]$. Note that, by construction of g, if g visits two distinct objects (ℓ,ℓ') and (ℓ,ℓ'') for some ℓ but f_{s_ϕ} does not then we must have $\ell = k$. Since all of the generating hyperfaces in $\Lambda_h^k[m;\mathbf{p}] \otimes \Theta_2[n;\mathbf{q}]$ contain the object k, it follows from Lemma 4.1.11 that ϕ is contained in $\Lambda_h^k[m;\mathbf{p}] \otimes \Theta_2[n;\mathbf{q}]$. □

We prove that B may be obtained from A' by gluing those $(1;r)$-cells ϕ in $B \setminus A'$ satisfying (hh) along the inner horn $\Lambda_v^{1;s_\phi+1}[1;r]$ in lexicographically increasing order of $\mathrm{sil}(\phi)$, $\dim(\phi)$ and s_ϕ. We must check that, for any such ϕ, all of its hyperfaces except for the $(1;s_\phi+1)$-th vertical one are contained either in A' or in some cell χ satisfying (hh) such that:

- $\mathrm{sil}(\chi) < \mathrm{sil}(\phi)$;
- $\mathrm{sil}(\chi) = \mathrm{sil}(\phi)$ and $\dim(\chi) < \dim(\phi)$; or

- $\mathrm{sil}(\chi) = \mathrm{sil}(\phi)$, $\dim(\chi) = \dim(\phi)$ and $s_\chi < s_\phi$.

Indeed:

- $\phi \cdot \delta_v^{1;0}$ and $\phi \cdot \delta_v^{1;r}$ have smaller silhouettes than $\mathrm{sil}(\phi)$;
- if $s_\phi \neq 0$ then $\phi \cdot \delta_v^{1;s_\phi}$:
  - is contained in A';
  - satisfies (hh); or
  - is of the form $\phi \cdot \delta_v^{1;s_\phi} = \chi \cdot \delta_v^{1;s_\chi+1}$ for some cell χ satisfying (hh) which necessarily has $\dim \chi = \dim \phi$ and $s_\chi = s_\phi - 1$; and
- for any other value of s, the hyperface $\phi \cdot \delta_v^{1;s}$:
  - is contained in A'; or
  - satisfies (hh) and has dimension strictly smaller than $\dim(\phi)$.

This completes the proof. $\square$

4.4.2 Inner vertical horn inclusion $\hat{\otimes}$ boundary inclusion

Let $[m;\mathbf{p}], [n;\mathbf{q}] \in \Theta_2$, $1 \leq k \leq m$ and $1 \leq i \leq p_k - 1$. In this subsection, we will prove the following lemma.

Lemma 4.4.3. *The map*

$$\left(\Lambda_v^{k;i}[m;\mathbf{p}] \hookrightarrow \Theta_2[m;\mathbf{p}]\right)\hat{\otimes}\left(\partial\Theta_2[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]\right)$$

is in $\mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v)$.

Proof. We will regard this map as a cellular subset inclusion and denote it as $A \hookrightarrow B$. Since the case $[n;\mathbf{q}] = [0]$ is trivial, we will assume $n \geq 1$.

Similarly to the proof of Lemma 4.4.2, we can show that gluing the cuttable cells ϕ to A along the inner horn $\Lambda^{\mathrm{cut}(\phi)}$ in lexicographically increasing order of $\mathrm{sil}(\phi)$ and $\dim(\phi)$ yields the cellular subset $A' \subset \Theta_2[m;\mathbf{p}] \otimes \Theta_2[n;\mathbf{q}]$ generated by A and the (potentially) cuttable cells.

Temporary definition. For any 1-cell $f : (0,0) \to (m,n)$ in $[m;\mathbf{p}] \boxtimes [n;\mathbf{q}]$ and for any $1 \leq \ell \leq m$, the composite

$$[1;0] \xrightarrow{f} [m;\mathbf{p}] \boxtimes [n;\mathbf{q}] \xrightarrow{\pi_1} [m;\mathbf{p}]$$

corresponds to a cellular operator $[\{0,m\};\boldsymbol{\alpha}] : [1;0] \to [m;\mathbf{p}]$. We will write $f \upharpoonright \ell$ for $\alpha_\ell(0) \in [p_\ell]$.

Let ϕ be a non-degenerate $(1;r)$-cell in $B \setminus A'$ (which necessarily has endpoints $(0,0)$ and (m,n)) corresponding to 1-cells $f_0, \ldots, f_r : (0,0) \to (m,n)$ with underlying shuffles $\leq_0, \ldots, \leq_r$ respectively. Let

$$s_\phi \overset{\text{def}}{=} \max\left\{s : f_s \upharpoonright k = i - 1\right\}.$$

To see that this is well-defined, observe that if $f_s \upharpoonright k \neq i - 1$ for all $0 \leq s \leq r$ then ϕ is contained in the image of $\delta_v^{k;i-1} \otimes \mathrm{id}$ which contradicts our assumption that ϕ is not in A'.

We construct the "best approximation" g to f_{s_ϕ} with $g \upharpoonright k = i$. Let $g : (0,0) \to (m,n)$ be the 1-cell determined by the following conditions:

- the second projection $[m;\mathbf{p}] \boxtimes [n;\mathbf{q}] \to [n;\mathbf{q}]$ sends f_{s_ϕ} and g to the same 1-cell;
- f_{s_ϕ} and g have the same underlying shuffle; and
- $g \upharpoonright \ell = \begin{cases} i & \text{if } \ell = k, \\ f_{s_\phi} \upharpoonright \ell & \text{otherwise.} \end{cases}$

Then clearly we have $f_{s_\phi} < g \leq f_{s_\phi+1}$. Consider the following condition on ϕ:

(vh) $f_{s_\phi+1} = g$.

Note that if ϕ satisfies (vh) then $s_\phi + 1 \neq r$ since $f_{s_\phi+1} \upharpoonright k = i$ while $f_r \upharpoonright k = p_k$. Also we have $s_\phi + 1 \neq 0$ since $s_\phi \geq 0$.

It can be easily checked using Lemma 4.1.11 that the set of non-degenerate cells in $B \setminus A'$ can be partitioned into pairs of the form

$$\{\phi, \phi \cdot \delta_v^{1;s_\phi+1}\}$$

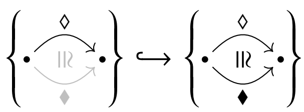
where ϕ is a $(1;r)$-cell satisfying (vh). We claim that B may be obtained from A' by gluing such ϕ along the inner horn $\Lambda_v^{1;s_\phi+1}[1;r]$ in lexicographically increasing order of $\mathrm{sil}(\phi)$, $\dim(\phi)$ and s_ϕ. Indeed, for any such ϕ:

- $\phi \cdot \delta_v^{1;0}$ and $\phi \cdot \delta_v^{1;r}$ have smaller silhouettes than $\mathrm{sil}(\phi)$;
- if $s_\phi \neq 0$ then $\phi \cdot \delta_v^{1;s_\phi}$ is:
  - contained in A'; or
  - of the form $\phi \cdot \delta_v^{1;s_\phi} = \chi \cdot \delta_v^{1;s_\chi+1}$ for some cell χ satisfying (vh) which necessarily has $\mathrm{sil}(\chi) = \mathrm{sil}(\phi)$, $\dim(\chi) = \dim(\phi)$ and $s_\chi = s_\phi - 1$; and
- for any other value of s, the hyperface $\phi \cdot \delta_v^{1;s}$:
  - is contained in A'; or
  - satisfies (vh) and has dimension strictly smaller than $\dim(\phi)$.

This completes the proof. □

4.4.3 Vertical equivalence extension $\hat{\otimes}$ boundary inclusion

Any unlabelled map of the form $\Theta_2[1;0] \hookrightarrow \Theta_2[1;J]$ in this subsection is assumed to be $[\mathrm{id}; e]$, which looks like:



Fix $[n;\mathbf{q}] \in \Theta_2$. We will prove the following lemma in this subsection.

Lemma 4.4.4. *The map*

$$(\Theta_2[1;0] \hookrightarrow \Theta_2[1;J]) \hat{\otimes} (\partial\Theta_2[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}])$$

is a trivial cofibration.

We will first analyse the Gray tensor product $\Theta_2[1;J] \otimes \Theta_2[n;\mathbf{q}]$. Let $\mathscr{J}_v$ be the 2-category whose object set is $\{0,1\}$ and whose hom-categories are

$$\begin{aligned}
\mathscr{J}_v(0,0) &= [0],\\
\mathscr{J}_v(1,1) &= [0],\\
\mathscr{J}_v(0,1) &= \{\lozenge \cong \blacklozenge\},\\
\mathscr{J}_v(1,0) &= \varnothing
\end{aligned}$$

so that we have $N\mathscr{J}_v \cong \Theta_2[1;J]$. The following lemma can be proved in essentially the same way as Lemma 4.1.7.

Lemma 4.4.5. *The square*

$$\begin{tikzcd}
\mathscr{J}_v \boxtimes [n;\mathbf{q}] \arrow[r] \arrow[d, "\langle \pi_1,\pi_2 \rangle"'] & {[1;0]} \boxtimes [n;\mathbf{q}] \arrow[d, "\langle \pi_1,\pi_2 \rangle"] \\
\mathscr{J}_v \times [n;\mathbf{q}] \arrow[r] & {[1;0]} \times [n;\mathbf{q}]
\end{tikzcd}$$

is a pullback in 2-Cat*, where the horizontal maps are induced by the unique identity-on-objects 2-functor* $\mathscr{J}_v \to [1;0]$.

For any 2-categories $\mathscr{A}$ and $\mathscr{B}$, a ζ-cell in the Gray tensor product $N\mathscr{A} \otimes N\mathscr{B}$ is represented (non-uniquely) by $\theta_1, \theta_2 \in \Theta_2$ together with three 2-functors

$$\begin{aligned}
\phi &: \zeta \to \theta_1 \boxtimes \theta_2,\\
\chi_1 &: \theta_1 \to \mathscr{A},\\
\chi_2 &: \theta_2 \to \mathscr{B}.
\end{aligned}$$

Such 2-functors may be combined into a single 2-functor

$$\zeta \xrightarrow{\phi} \theta_1 \boxtimes \theta_2 \xrightarrow{\chi_1 \boxtimes \chi_2} \mathscr{A} \boxtimes \mathscr{B}$$

which corresponds to a ζ-cell in $N(\mathscr{A} \boxtimes \mathscr{B})$. This defines a comparison map

$$N\mathscr{A} \otimes N\mathscr{B} \to N(\mathscr{A} \boxtimes \mathscr{B}).$$

Lemma 4.4.6. *The comparison map*

$$\Theta_2[1;J] \otimes \Theta_2[n;\mathbf{q}] \to N\big(\mathscr{J}_v \boxtimes [n;\mathbf{q}]\big)$$

is invertible.

Proof. Observe that $\Theta_2[1;J]$ may be obtained from $\Theta_2[0] \amalg \Theta_2[0]$ by gluing two copies of $\Theta_2[1;r]$ along the boundary for each $r \geq 0$ in increasing order of r. Since the functor $\otimes$ preserves colimits in each variable, it follows that $\Theta_2[1;J] \otimes \Theta_2[n;\mathbf{q}]$ may be obtained from $\Theta_2[n;\mathbf{q}] \amalg \Theta_2[n;\mathbf{q}]$ by gluing two copies of $\Theta_2[1;r] \otimes \Theta_2[n;\mathbf{q}]$ along $\partial\Theta_2[1;r] \otimes \Theta_2[n;\mathbf{q}]$ for each $r \geq 0$. This presentation of $\Theta_2[1;J] \otimes \Theta_2[n;\mathbf{q}]$ can be made more explicit using Lemma 4.1.11, and comparing it to Lemma 4.4.5 yields the desired result. $\square$

Proof of Lemma 4.4.4. We will regard the map

$$\big(\Theta_2[1;0] \hookrightarrow \Theta_2[1;J]\big)\hat{\otimes}\big(\partial\Theta_2[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]\big)$$

as a cellular subset inclusion and denote it by $A \hookrightarrow B$. Let

$$P : \Theta_2[1;J] \otimes \Theta_2[n;\mathbf{q}] \to \Theta_2[1;0] \otimes \Theta_2[n;\mathbf{q}]$$

be the map induced by the unique map $\Theta_2[1;J] \to \Theta_2[1;0]$ that is bijective on 0-cells. Given any cell ϕ in B, we will write $\mathrm{sil}(\phi)$ for $\mathrm{sil}(P(\phi))$ and say ϕ is *non-linear*, (*potentially*) *cuttable* or *absolutely uncuttable* if $P(\phi)$ is so. If ϕ is a non-linear cuttable cell, we write $\mathrm{cut}(\phi)$ for $\mathrm{cut}(P(\phi))$. Let $A' \subset B$ be the cellular subset generated by A and the (potentially) cuttable cells. Then one can prove, using the obvious analogues of Lemma 4.3.6 and Proposition 4.3.11, that the inclusion $A \hookrightarrow A'$ is in $\mathrm{cell}(\mathcal{H}_h)$.

Now we consider the absolutely uncuttable cells ϕ in $B \setminus A'$. By Lemmas 4.4.5 and 4.4.6, any $(1;r)$-cell ϕ with endpoints $(0,0)$ and $(1,n)$ is uniquely determined by:

- a chain $\leq_0 \blacktriangleleft \cdots \blacktriangleleft \leq_r$ in the poset $S\big((0,0),(1,n)\big)$;
- a chain $f_0 \leq \cdots \leq f_r$ in the poset $[q_1] \times \cdots \times [q_n]$; and
- a sequence $(\epsilon_0, \dots, \epsilon_r)$ in $\{\lozenge, \blacklozenge\}$.

Since ϕ is not contained in A', we must have $\epsilon_s = \blacklozenge$ for at least one s. Thus

$$s_\phi \overset{\text{def}}{=} \max\{s : \epsilon_s = \blacklozenge\}$$

is well-defined. Consider the following condition on ϕ:

(ve) $s_\phi < r$, $\leq_{s_\phi+1} = \leq_{s_\phi}$ and $f_{s_\phi+1} = f_{s_\phi}$.

Note that, since we are assuming ϕ to be (absolutely uncuttable and hence) non-degenerate, (ve) implies $\epsilon_{s_\phi+1} = \lozenge$. It also implies $r \geq 2$ for otherwise $\leq_{s_\phi+1} = \leq_{s_\phi}$ is the only underlying shuffle of ϕ which contradicts our assumption that ϕ is absolutely uncuttable.

Clearly the set of non-degenerate cells in $B \setminus A'$ can be partitioned into pairs of the form

$$\big\{\phi, \phi \cdot \delta_v^{1;s_\phi+1}\big\}$$

where ϕ is a $(1;r)$-cell satisfying (ve). We claim that B may be obtained from A' by gluing such ϕ along the horn $\Lambda_v^{1;s_\phi+1}[1;r]$ in lexicographically increasing order of $\dim(\phi)$ and s_ϕ. Indeed, for any $(1;r)$-cell ϕ satisfying (ve):

- $\phi \cdot \delta_v^{1;s_\phi}$:
  - is contained in A';
  - is degenerate;
  - satisfies (ve); or
  - is of the form $\phi \cdot \delta_v^{1;s_\phi} = \chi \cdot \delta_v^{1;s_\chi+1}$ for some cell χ satisfying (ve) which necessarily has $\dim(\chi) = \dim(\phi)$ and $s_\chi < s_\phi$; and
- $\phi \cdot \delta_v^{1;s}$ where $s_\phi \neq s \neq s_\phi + 1$ is:

- contained in A'; or
- a (possibly trivial) degeneracy of some cell χ satisfying (ve) which necessarily has $\dim(\chi) < \dim(\phi)$.

The horn $\Lambda_v^{1;s_\phi+1}[1;r]$ is not necessarily inner since $s_\phi + 1$ may be equal to r. Nevertheless, in that case the outer horn is a *special* one in the sense that the composite map

$$\Lambda_v^{1;s_\phi+1}[1;r] = \Lambda_v^{1;r}[1;r] \hookrightarrow \Theta_2[1;r] \xrightarrow{\phi} \Theta_2[1;J] \otimes \Theta_2[n;\mathbf{q}]$$

can be extended to one from $\widetilde{\Lambda}_v^{1;r}[1;r]$ as defined in Section 3.5. Moreover, the images of the cells in $\widetilde{\Lambda}_v^{1;r}[1;r] \setminus \Lambda_v^{1;r}[1;r]$ are cuttable and hence contained in A'. Since the special outer horn inclusions $\widetilde{\Lambda}_v^{1;r}[1;r] \hookrightarrow \widetilde{\Theta}_2^{1;r}[1;r]$ are trivial cofibrations by the dual of Lemma 3.5.6, we can deduce that the inclusion $A' \hookrightarrow B$ is a trivial cofibration. This completes the proof. □

4.4.4 Horizontal equivalence extension $\hat{\otimes}$ boundary inclusion

Recall that the monomorphism $e : \Theta_2[0] \hookrightarrow J$ is (isomorphic to) the nerve of the inclusion

$$\{\lozenge\} \hookrightarrow \{\lozenge \cong \blacklozenge\} = \mathscr{I}_h.$$

We will prove the following lemma in this subsection.

Lemma 4.4.7. *The map*

$$\big(\Theta_2[0] \overset{e}{\hookrightarrow} J\big)\hat{\otimes}\big(\partial\Theta_2[n;\mathbf{q}] \hookrightarrow \Theta_2[n;\mathbf{q}]\big)$$

is a trivial cofibration for any $[n;\mathbf{q}] \in \Theta_2$.

First we analyse the Gray tensor product $J \otimes \Theta_2[1;q]$ for $q \geq 0$. Consider the (2-categorical) Gray tensor product $\mathscr{I}_h \boxtimes [1;q]$. Its object set is obviously $\{\lozenge, \blacklozenge\} \times \{0,1\}$.

Lemma 4.4.8. *For any* $\star, \star' \in \{\lozenge, \blacklozenge\}$ *and for any* $k, \ell \in \{0,1\}$*, the hom-category of* $\mathscr{I}_h \boxtimes [1;q]$ *is given by*

$$\big(\mathscr{I}_h \boxtimes [1;q]\big)\big((\star,k),(\star',\ell)\big) \cong \begin{cases} [0] & \text{if } k = \ell, \\ \{\cdot \cong \cdot\} \times [q] & \text{if } k = 0 \text{ and } \ell = 1, \\ \varnothing & \text{if } k = 1 \text{ and } \ell = 0. \end{cases}$$

Proof. The proof is similar to that of Lemma 4.1.7. The inverse to a generating 2-cell of the form

$$\begin{tikzcd} (\lozenge,0) \arrow[r] \arrow[d, "(\lozenge,p)"'] & (\blacklozenge,0) \arrow[d, "(\blacklozenge,p)"] \\ (\lozenge,1) \arrow[r] \arrow[ur, phantom, "\Nearrow\gamma"] & (\blacklozenge,1) \end{tikzcd}$$

is obtained by whiskering the 2-cell

$$\begin{tikzcd} (\lozenge,0) \arrow[d, "(\lozenge,p)"'] & (\blacklozenge,0) \arrow[l] \arrow[d, "(\blacklozenge,p)"] \\ (\lozenge,1) & (\blacklozenge,1) \arrow[l] \arrow[ul, phantom, "\Nwarrow\gamma"] \end{tikzcd}$$

with the obvious 1-cells. □

Lemma 4.4.9. *The comparison map*

$$J \otimes \Theta_2[1;q] \to N(\mathscr{J}_h \boxtimes [1;q])$$

is invertible for any $q \geq 0$.

Proof. For the sake of simplicity, we will only prove that the comparison map acts bijectively on the $(1;r)$-cells with endpoints $(\lozenge, 0)$ and $(\blacklozenge, 1)$; the general case can be treated similarly. By Lemma 4.4.8, such $(1;r)$-cells correspond to those sequences in $\{L, R\} \times [q]$ of length $r+1$ that are increasing in the second coordinate; here L and R correspond to 1-cells of the form

$$\begin{tikzcd} (\lozenge,0) \arrow[d] & \\ (\lozenge,1) \arrow[r] & (\blacklozenge,1) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} (\lozenge,0) \arrow[r] & (\blacklozenge,0) \arrow[d] \\ & (\blacklozenge,1) \end{tikzcd}$$

respectively.

Observe that J has precisely two non-degenerate $(d;\mathbf{0})$-cells $e^d_\lozenge, e^d_\blacklozenge$ for each $d \geq 0$ where $e^d_\star \cdot [\{0\}] = \star$ for $\star \in \{\lozenge, \blacklozenge\}$. Thus $J \otimes \Theta_2[1;q]$ may be obtained from $\varnothing$ by gluing two copies of $\Theta_2[d;\mathbf{0}] \otimes \Theta_2[1;q]$ along $\partial\Theta_2[d;\mathbf{0}] \otimes \Theta_2[1;q]$ in increasing order of d. By Lemma 4.1.11, a sequence of 2-cells

$$f_0 \Rightarrow \cdots \Rightarrow f_r$$

in $[d;\mathbf{0}] \boxtimes [1;q]$ corresponds to a $(1;r)$-cell in $\big(\Theta_2[d;\mathbf{0}] \otimes \Theta_2[1;q]\big) \setminus \big(\partial\Theta_2[d;\mathbf{0}] \otimes \Theta_2[1;q]\big)$ if and only if:

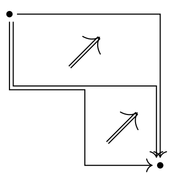
- $f_s : (0,0) \to (d,1)$ for each s; and
- for each $1 \leq c \leq d-1$, there exists $0 \leq s \leq r$ such that $\leq_s$ visits both $(c,0)$ and $(c,1)$.

There are four kinds of such cells, depending on whether f_0 visits $(0,1)$ and whether f_r visits $(d,0)$ (see Fig. 4.2). The images of these cells under

$$e^d_\star \otimes \mathrm{id} : \Theta_2[d;\mathbf{0}] \otimes \Theta_2[1;q] \to J \otimes \Theta_2[1;q]$$

have endpoints $(\lozenge,0)$ and $(\blacklozenge,1)$ if and only if $\star = \lozenge$ and $d = 2d'+1$ is odd. Moreover, in such case $e^{2d'+1}_\lozenge \otimes \mathrm{id}$ sends these cells bijectively to those sequences in $\{L,R\} \times [q]$ for which "RL" appears exactly d' times in their first projections (which are sequences in $\{L,R\}$). This completes the proof. $\square$

Remark. Lemma 4.4.8 can be generalised in the obvious way to general $[n;\mathbf{q}] \in \Theta_2$ (in place of $[1;q]$), but Lemma 4.4.9 is no longer true if we replace $[1;q]$ by $[n;\mathbf{q}]$ with $n \geq 2$. For example, consider the $(1;2)$-cell



in $\Theta_2[2;\mathbf{0}] \otimes \Theta_2[2;\mathbf{0}]$. For each $\star \in \{\lozenge, \blacklozenge\}$, the image of this cell under $e^2_\star \otimes \mathrm{id}$ is an example of a non-degenerate cell in $J \otimes \Theta_2[2;\mathbf{0}]$ that is sent to a degenerate one by the comparison map

$$J \otimes \Theta_2[2;\mathbf{0}] \to N\big(\mathscr{J}_h \boxtimes [2;\mathbf{0}]\big).$$

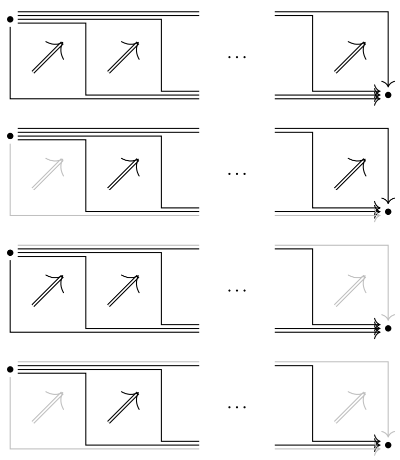


Figure 4.2: Cells in $\Theta_2[d;\mathbf{0}] \otimes \Theta_2[1;q]$

In general, the cellular set $J \otimes \Theta_2[n;\mathbf{q}]$ does not seem to admit a simple description. Therefore the rest of our proof of Lemma 4.4.7 will be less combinatorial compared to those of Lemmas 4.4.2 to 4.4.4.

We will make use of the following result of Campbell.

Theorem 4.4.10 ([Cam, Theorem 10.11])**.** *A 2-functor $\mathscr{B} \to \mathscr{C}$ is a biequivalence if and only if its nerve $N\mathscr{B} \to N\mathscr{C}$ is a weak equivalence of cellular sets.*

Proof of Lemma 4.4.7. Fix $[n;\mathbf{q}] \in \Theta_2$, and let $\mathscr{D}$ be the full subcategory of the category of elements of $\Xi[n;\mathbf{q}]$ spanned by the non-degenerate cells. Then $\mathscr{D}$ has an obvious Reedy category structure in which every map is degree-raising. Since $\mathscr{D}$ has no degree-lowering maps, the diagonal functor $\widehat{\Theta_2} \to \left[\mathscr{D}, \widehat{\Theta_2}\right]$ is trivially right Quillen. Now both composites

$$\mathscr{D} \longrightarrow \Theta_2 \xrightarrow{\text{Yoneda}} \widehat{\Theta_2} \xrightarrow{J\otimes(-)} \widehat{\Theta_2}$$

$$\mathscr{D} \longrightarrow \Theta_2 \xrightarrow{\text{Yoneda}} \widehat{\Theta_2} \xrightarrow{J\times(-)} \widehat{\Theta_2}$$

(where $\mathscr{D} \to \Theta_2$ is the canonical projection) can be easily checked to be Reedy cofibrant by direct calculation. Moreover, there is a natural transformation between them whose components are given by

$$J \otimes \Theta_2[0] \cong J \cong J \times \Theta_2[0]$$

for objects of degree 0, and

$$J \otimes \Theta_2[1;q] \cong N\big(\mathscr{J}_h \boxtimes [1;q]\big) \to N\big(\mathscr{J}_h \times [1;q]\big) \cong J \times \Theta_2[1;q]$$

for objects of degree $q+1$ (with $q \in \{0,1\}$), where the middle map is the nerve of the obvious 2-functor. It is easy to check that $\mathscr{J}_h \boxtimes [1;q] \to \mathscr{J}_h \times [1;q]$ is a biequivalence, and hence its nerve is a weak equivalence by Theorem 4.4.10. Thus by taking the colimit, we can conclude that $J \otimes \Xi[n;\mathbf{q}] \to J \times \Xi[n;\mathbf{q}]$ is a weak equivalence. This map fits into the following commutative square:

$$\begin{tikzcd}
J \otimes \Xi[n;\mathbf{q}] \arrow[r, "\langle\pi_1,\pi_2\rangle"] \arrow[d, hook] & J \times \Xi[n;\mathbf{q}] \arrow[d, hook] \\
J \otimes \Theta_2[n;\mathbf{q}] \arrow[r, "\langle\pi_1,\pi_2\rangle"] & J \times \Theta_2[n;\mathbf{q}]
\end{tikzcd}$$

Moreover, the right vertical map is a trivial cofibration because Ara's model structure is cartesian [Ara14, Corollary 8.5], and the left vertical map is so by Lemmas 3.1.2, 4.4.2 and 4.4.3. Therefore $\langle \pi_1, \pi_2 \rangle : J \otimes \Theta_2[n; \mathbf{q}] \to J \times \Theta_2[n; \mathbf{q}]$ is a weak equivalence.

Finally, we prove the statement of the lemma by induction on $\dim [n; \mathbf{q}]$. The base case is trivial. For the inductive step, consider the following commutative diagram:

$$\begin{tikzcd}
\Theta_2[0] \otimes \Theta_2[n; \mathbf{q}] \arrow[r, "\langle \pi_1, \pi_2 \rangle"] \arrow[d, hook] & \Theta_2[0] \times \Theta_2[n; \mathbf{q}] \arrow[dd, hook, "e \times \mathrm{id}"] \\
\big(\Theta_2[0] \otimes \Theta_2[n; \mathbf{q}]\big) \cup \big(J \otimes \partial\Theta_2[n; \mathbf{q}]\big) \arrow[d, hook] & \\
J \otimes \Theta_2[n; \mathbf{q}] \arrow[r, "\langle \pi_1, \pi_2 \rangle"] & J \times \Theta_2[n; \mathbf{q}]
\end{tikzcd}$$

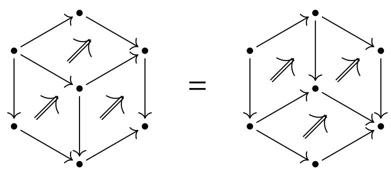
Here the upper horizontal map is an isomorphism, the lower horizontal map is a weak equivalence as we have just proved, and the right vertical map is a trivial cofibration since Ara's model structure is cartesian. Moreover, the upper left vertical map can be obtained by composing pushouts of maps of the form

$$\big(\Theta_2[0] \overset{e}{\hookrightarrow} J\big) \hat{\otimes} \big(\partial\Theta_2[m; \mathbf{p}] \hookrightarrow \Theta_2[m; \mathbf{p}]\big)$$

with $\dim [m; \mathbf{p}] < \dim [n; \mathbf{q}]$, hence it is a trivial cofibration by the inductive hypothesis. Thus the desired result follows by the 2-out-of-3 property. □

4.5 Monoidal structure up to homotopy

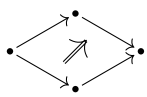
In this section, we will prove that the Gray tensor product forms an up-to-homotopy monoidal structure on $\widehat{\Theta_2}$ in an appropriate sense. Let us first illustrate why it is not a genuine monoidal structure, or more specifically, how it fails to be associative up to isomorphism. One would expect the Gray tensor product of three copies of $\Theta_2[1; 0]$ to "be" the commutative cube:



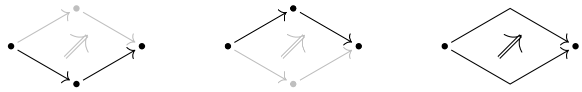
Indeed, the "total" tensor product $B = \otimes_3\big(\Theta_2[1; 0], \Theta_2[1; 0], \Theta_2[1; 0]\big)$ is by definition the nerve of this 2-category. Now consider the nested tensor product

$$A = \otimes_2\Big(\otimes_2\big(\Theta_2[1; 0], \Theta_2[1; 0]\big), \Theta_2[1; 0]\Big).$$

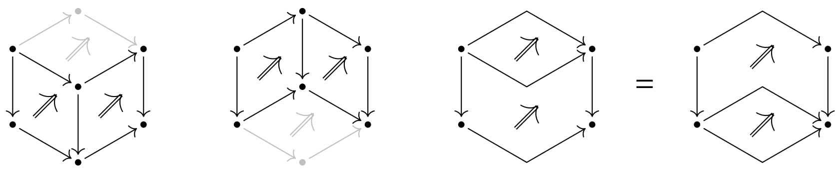
The binary tensor product $\otimes_2\big(\Theta_2[1; 0], \Theta_2[1; 0]\big)$ is the nerve of the 2-category



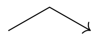
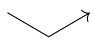
and hence it has the following maximal non-degenerate cells:



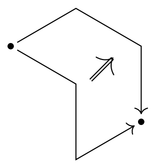
Thus A is obtained by pasting together the nerves of two copies of $[2;\mathbf{0}] \boxtimes [1;0]$ and one copy of $[1;1] \boxtimes [1;0]$ appropriately. In fact, A turns out to be (isomorphic to) the cellular subset of B generated by the nerves of the following sub-2-categories:



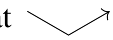
Informally speaking, a cell in B is contained in $B \setminus A$ if and only if it remembers both of:

- the decomposability of a 1-cell of shape  or ; and
- the existence of the top or bottom face of the cube.

For example, the following $(1;1)$-cell is not contained in A:



(4.13)

Here we can "see" the top face of the cube, and moreover the vertical segment in the lower path "remembers" that  is decomposable. This geometric intuition is formalised in Theorem 4.5.4. There is a similar description of the other nested tensor product

$$A' = \otimes_2\Big(\Theta_2[1;0], \otimes_2\big(\Theta_2[1;0], \Theta_2[1;0]\big)\Big)$$

as a cellular subset of B, and it is easy to see that $A \ncong A'$.

Now, although A and A' are not isomorphic to each other, they both admit an inclusion into B. In general, we always have a comparison map from a nested tensor product to the corresponding total tensor product. The functors $\otimes_a$ together with these comparison maps form a *normal lax monoidal structure* on the category $\widehat{\Theta_2}$ (Proposition 4.5.2). Moreover the (relative version of the) comparison maps are trivial cofibrations, hence the Gray tensor product is associative up to homotopy. As a bonus, this associativity can be used to upgrade Theorem 4.4.1 (which states that the binary Gray tensor product is left Quillen) to the general a-ary version.

4.5.1 Lax monoidal structure

Definition 4.5.1. A *lax monoidal structure* on a category $\mathscr{C}$ consists of:

- a functor $\odot_a : \mathscr{C}^a \to \mathscr{C}$ for each $a \in \mathbb{N}$;
- a natural transformation $\iota : \mathrm{id}_{\mathscr{C}} \to \odot_1$; and

- a natural transformation

$$\mu_{b_1,\dots,b_a} : \odot_a\left(\odot_{b_1},\dots,\odot_{b_a}\right) \to \odot_{b_1+\cdots+b_a}$$

  for each $a, b_1, \dots, b_a \in \mathbb{N}$

such that the following diagrams commute:

$$\begin{tikzcd}
\odot_a \arrow[r, "\iota\odot_a"] \arrow[dr, "\mathrm{id}"'] & \odot_1(\odot_a) \arrow[d, "\mu_a"] \\
& \odot_a
\end{tikzcd}
\qquad
\begin{tikzcd}
\odot_a(\odot_1,\dots,\odot_1) \arrow[d, "\mu_{1,\dots,1}"'] & \odot_a \arrow[l, "\odot_a(\iota,\dots,\iota)"'] \arrow[dl, "\mathrm{id}"] \\
\odot_a
\end{tikzcd}$$

$$\begin{tikzcd}
& \odot_a\big(\odot_{b_1}(\odot_{c_{11}},\dots,\odot_{c_{1b_1}}),\dots,\odot_{b_a}(\odot_{c_{a1}},\dots,\odot_{c_{ab_a}})\big) \arrow[dl, "\mu_{b_1,\dots,b_a}"'] \arrow[dr, "\odot_a(\mu_{c_{11},\dots,c_{1b_1}},\dots,\mu_{c_{a1},\dots,c_{ab_a}})"] & \\
\odot_{b_1+\cdots+b_a}(\odot_{c_{11}},\dots,\odot_{c_{ab_a}}) \arrow[dr, "\mu_{c_{11},\dots,c_{ab_a}}"'] & & \odot_a(\odot_{c_{11}+\cdots+c_{1b_1}},\dots,\odot_{c_{a1}+\cdots+c_{ab_a}}) \arrow[dl, "\mu_{c_{11}+\cdots+c_{1b_1},\dots,c_{a1}+\cdots+c_{ab_a}}"] \\
& \odot_{c_{11}+\cdots+c_{ab_a}} &
\end{tikzcd}$$

Such a lax monoidal structure is called *normal* if ι is invertible.

Remark. A lax monoidal structure on $\mathscr{C}$ is equivalently a lax algebra structure on $\mathscr{C}$ for the 2-monad on $\underline{\mathrm{Cat}}$ whose strict algebras are the strict monoidal categories.

Proposition 4.5.2. *The Gray tensor product functors $\otimes_a$ are part of a normal lax monoidal structure on $\widehat{\Theta_2}$.*

Proof. Since we chose $\otimes_1$ to be $\mathrm{id}_{\widehat{\Theta_2}}$, we may take $\iota = \mathrm{id}_{\mathrm{id}_{\widehat{\Theta_2}}}$. The transformation μ is defined on the representables as follows. Recall that each κ-cell in

$$\otimes_a\left(\otimes_{b_1}\left(\Theta_2^{\theta_{11}},\dots,\Theta_2^{\theta_{1b_1}}\right),\dots,\otimes_{b_a}\left(\Theta_2^{\theta_{a1}},\dots,\Theta_2^{\theta_{ab_a}}\right)\right)$$

is (non-uniquely) represented by $\zeta_1,\dots,\zeta_a \in \Theta_2$ together with 2-functors

$$\phi : \kappa \to \boxtimes_a(\zeta_1,\dots,\zeta_a)$$
$$\phi_i : \zeta_i \to \boxtimes_{b_i}(\theta_{i1},\dots,\theta_{ib_i})$$

for $1 \le i \le a$. Then $\mu_{b_1,\dots,b_a}$ sends this cell to the κ-cell in

$$\otimes_{b_1+\cdots+b_a}\left(\Theta_2^{\theta_{11}},\dots,\Theta_2^{\theta_{ab_a}}\right)$$

represented by the 2-functor

$$\kappa \xrightarrow{\phi} \boxtimes_a(\zeta_1,\dots,\zeta_a) \xrightarrow{\boxtimes_a(\phi_1,\dots,\phi_a)} \boxtimes_{b_1+\cdots+b_a}(\theta_{11},\dots,\theta_{ab_a}).$$

That μ is well-defined, natural, and satisfies the coherence conditions is all straightforward to check. ◻

4.5.2 The comparison map μ

Fix $a, b_1, \ldots, b_a \in \mathbb{N}$, and let $b = \sum_{u=1}^{a} b_u$. Let $\theta_i \in \Theta_2$ for $1 \leq i \leq b$. We will show that

$$\mu = (\mu_{b_1,\ldots,b_a})_{\Theta_2^{\theta_1},\ldots,\Theta_2^{\theta_b}} : A \to B$$

is a monomorphism and moreover characterise its image, where

$$\begin{aligned} A &= \otimes_a \left(\otimes_{b_1} \left(\Theta_2^{\theta_1}, \ldots, \Theta_2^{\theta_{b_1}} \right), \ldots, \otimes_{b_a} \left(\Theta_2^{\theta_{b-b_a+1}}, \ldots, \Theta_2^{\theta_b} \right) \right), \\ B &= \otimes_b \left(\Theta_2^{\theta_1}, \ldots, \Theta_2^{\theta_b} \right) = N\big(\boxtimes_b(\theta_1, \ldots, \theta_b)\big) \end{aligned}$$

Let $\rho : \{1, \ldots, b\} \to \{1, \ldots, a\}$ denote the unique function such that

$$\sum_{u<\rho(i)} b_u < i \leq \sum_{u \leq \rho(i)} b_u$$

for each $1 \leq i \leq b$. Informally speaking, for each $1 \leq i \leq b$, the i-th factor is contained in the $\rho(i)$-th "subtensor".

Definition 4.5.3. Let ϕ be a $(1; q)$-cell in B with endpoints $\boldsymbol{s}, \boldsymbol{t}$ and underlying shuffles $\leq_p$. We say ϕ is *pure* if, for each pair $(i|k), (j|\ell) \in S(\boldsymbol{s}, \boldsymbol{t})$ with $\rho(i) = \rho(j)$, at least one of the following holds:

(i) $(i|k) \leq_p (j|\ell)$ for all $0 \leq p \leq q$;

(ii) $(i|k) \geq_p (j|\ell)$ for all $0 \leq p \leq q$; or

(iii) for any $(m|n) \in S$ and for any $0 \leq p \leq q$, if

$$(i|k) \leq_p (m|n) \leq_p (j|\ell) \quad \text{or} \quad (i|k) \geq_p (m|n) \geq_p (j|\ell)$$

then $\rho(m) = \rho(i)$.

More generally, call an $(n; \boldsymbol{q})$-cell ϕ in B *pure* if, for each $1 \leq k \leq n$, the $(1; q_k)$-cell $\phi \cdot \eta_h^k$ is pure in the above sense. (See Definition 2.1.11 for the definition of η_h^k.)

If we take $a = 2$, $b_1 = 2$, $b_2 = 1$ and $\theta_1 = \theta_2 = \theta_3 = [1; 0]$ then we recover the example considered at the beginning of this section. In this case, the $(1; 1)$-cell (4.13) which has

$$\begin{aligned} (2|1) \leq_0 (3|1) \leq_0 (1|1), \\ (1|1) \leq_1 (2|1) \leq_1 (3|1) \end{aligned}$$

is not pure; consider $(i|k) = (1|1)$ and $(j|\ell) = (2|1)$.

The rest of this subsection is devoted to proving the following theorem.

Theorem 4.5.4. *The map $\mu : A \to B$ is a monomorphism, and its image consists precisely of the pure cells.*

Proof. Every cell in the image $\mu(A)$ is pure by Lemma 4.5.5. That every pure cell is in the image of $\mu(A)$ follows from Lemmas 4.5.6 and 4.5.7. Finally, the map μ is a monomorphism by Lemma 4.5.8. □

Lemma 4.5.5. *Every cell in the image $\mu(A)$ is pure.*

Proof. It suffices to check the $(1;q)$-cells. So consider a $(1;q)$-cell ϕ in the image of μ; that is, ϕ is a $(1;q)$-cell in B and admits a factorisation

$$\phi : \ [1;q] \xrightarrow{\ \chi\ } \boxtimes_a(\zeta_1, \ldots, \zeta_a) \xrightarrow{\boxtimes_a(\psi_1,\ldots,\psi_a)} \boxtimes_b(\theta_1, \ldots, \theta_b). \tag{4.14}$$

Given $(i|k),(j|\ell) \in S$ with $\rho(i) = \rho(j) = u$, let $x, y \in \zeta_u$ be the unique objects such that

$$\pi_i \circ \psi_u(x-1) < k \le \pi_i \circ \psi_u(x),$$
$$\pi_j \circ \psi_u(y-1) < \ell \le \pi_j \circ \psi_u(y).$$

Then we must have precisely one of the following:

- $x < y$, in which case the pair $(i|k),(j|\ell)$ satisfies Definition 4.5.3(i);
- $x > y$, in which case the pair satisfies (ii); or
- $x = y$, in which case the pair satisfies (iii).

This completes the proof. □

Lemma 4.5.6. *An $(n;\boldsymbol{q})$-cell ϕ in B is contained in $\mu(A)$ if and only if $\phi \cdot \eta_h^k$ is contained in $\mu(A)$ for each $1 \le k \le n$.*

Proof. In this proof, we say a factorisation of the form (4.14) is *nice* if the composite

$$[1;q] \xrightarrow{\ \chi\ } \boxtimes_a(\zeta_1, \ldots, \zeta_a) \xrightarrow{\ \pi_u\ } \zeta_u$$

preserves the first and last objects for each $1 \le u \le a$. Note that any factorisation of the form (4.14) can be made into a nice one by replacing each ζ_u by the appropriate horizontal face.

Now let ϕ be an $(n;\boldsymbol{q})$-cell such that each $\phi \cdot \eta_h^k$ admits a factorisation

$$[1;q_k] \xrightarrow{\ \chi^k\ } \boxtimes_a(\zeta_1^k, \ldots, \zeta_a^k) \xrightarrow{\boxtimes_a(\psi_1^k,\ldots,\psi_a^k)} \boxtimes_b(\theta_1, \ldots, \theta_b)$$

which we may assume to be nice. Then we can factorise ϕ as

$$\phi : \ [n;\mathbf{q}] \xrightarrow{\ \chi\ } \boxtimes_a(\zeta_1, \ldots, \zeta_a) \xrightarrow{\boxtimes_a(\psi_1,\ldots,\psi_a)} \boxtimes_b(\theta_1, \ldots, \theta_b)$$

where ζ_u is obtained by concatenating ζ_u^k's, or more precisely by taking the colimit of

$$\begin{tikzcd}
& {[0]} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & \\
\zeta_u^1 & & \zeta_u^2 & & \zeta_u^n
\end{tikzcd}$$

in 2-Cat, ψ_u is the induced map from this colimit, and χ is obtained by taking the colimit of the top zigzag in the following diagram:

$$\begin{tikzcd}
& {[0]} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & \\
{[1;q_1]} \arrow[d, "\chi^1"'] & & {[1;q_2]} \arrow[d, "\chi^2"'] & & {[1;q_n]} \arrow[d, "\chi^n"] \\
\boxtimes_a(\zeta_1^1, \ldots, \zeta_a^1) \arrow[drr] & & \boxtimes_a(\zeta_1^2, \ldots, \zeta_a^2) \arrow[d] & \cdots & \boxtimes_a(\zeta_1^n, \ldots, \zeta_a^n) \arrow[dll] \\
& & \boxtimes_a(\zeta_1, \ldots, \zeta_a) & &
\end{tikzcd}$$

Thus ϕ is in $\mu(A)$. □

Lemma 4.5.7. *Every pure* $(1;q)$*-cell in* B *is contained in* $\mu(A)$.

We will prove this lemma by constructing a factorisation of the form (4.14) for each pure $(1;q)$-cell ϕ. The intuition behind the construction below is as follows. First, we observe that the condition (iii) in Definition 4.5.3 tells us which elements of

$$S_u \stackrel{\text{def}}{=} \{(i|k) \in S(\boldsymbol{s},\boldsymbol{t}) : \rho(i) = u\}$$

(where $\boldsymbol{s},\boldsymbol{t}$ are the endpoints of ϕ) can be “bundled together”, and moreover the purity of ϕ implies that the collection of these bundles (for fixed u) admits a canonical ordering. The horizontal component of each ζ_u is then taken as the indexing total order for this collection, whereas the vertical components of ζ_u are all $[q]$. The first factor χ is then essentially determined by how the bundles coming from different u's are ordered with respect to each other (by the underlying shuffles of ϕ), and each ψ_u is essentially determined by how the elements are ordered within each bundle in S_u.

Proof. Let ϕ be a pure $(1;q)$-cell in B with endpoints $\boldsymbol{s},\boldsymbol{t}$. Let $\le_0,\dots,\le_q$ be the underlying shuffles of ϕ on the set $S = S(\boldsymbol{s},\boldsymbol{t})$. Define a binary relation $\sim$ on S so that $(i|k) \sim (j|\ell)$ if and only if

- $\rho(i) = \rho(j)$; and
- for any $(m|n) \in S$ and for any $0 \le p \le q$, if

$$(i|k) \le_p (m|n) \le_p (j|\ell) \quad \text{or} \quad (i|k) \ge_p (m|n) \ge_p (j|\ell)$$

  then $\rho(m) = \rho(i)$.

(The second clause is precisely Definition 4.5.3(iii).) It is straightforward to check that $\sim$ is an equivalence relation. We will write $[i|k]$ for the $\sim$-class containing $(i|k) \in S$.

For each $1 \le u \le a$, let $S_u \stackrel{\text{def}}{=} \{(i|k) \in S : \rho(i) = u\}$ and define a binary relation $\le_u$ on the quotient $T_u \stackrel{\text{def}}{=} S_u/\sim$ so that $[i|k] \le_u [j|\ell]$ if and only if

- $(i|k) \sim (j|\ell)$; or
- $(i|k) \le_p (j|\ell)$ for all $0 \le p \le q$.

Before checking that $\le_u$ is well-defined, notice that if $(i|k) \nsim (j|\ell)$ then the purity of ϕ implies that we have either $(i|k) \le_p (j|\ell)$ for all p, or $(i|k) \ge_p (j|\ell)$ for all p. Thus $\le_u$ can be equivalently defined as: $[i|k] \le_u [j|\ell]$ if and only if

- $(i|k) \sim (j|\ell)$; or
- $(i|k) \le_p (j|\ell)$ for *some* $0 \le p \le q$,

or alternatively, for any *fixed* $0 \le p \le q$, we can define: $[i|k] \le_u [j|\ell]$ if and only if

- $(i|k) \sim (j|\ell)$; or
- $(i|k) \le_p (j|\ell)$.

It is easy to see from the third definition that, assuming it is well-defined, $\leq_u$ is a total order

$$S_{u,1} \leq_u S_{u,2} \leq_u \cdots \leq_u S_{u,z_u}$$

on T_u where $z_u \overset{\text{def}}{=} |T_u|$ and each $S_{u,v} \subset S_u$ is a $\sim$-class.

To see that $\leq_u$ is indeed well-defined, consider two $\sim$-related pairs $(i|k) \sim (i'|k')$ and $(j|\ell) \sim (j'|\ell')$ in S_u. If $(i|k) \sim (j|\ell)$ then $(i'|k') \sim (j'|\ell')$ by the transitivity of $\sim$. So consider the case where $(i|k) \nsim (j|\ell)$. Making use of the first and second definitions of $\leq_u$, it suffices to prove that

$$\forall p\big[(i|k) \leq_p (j|\ell)\big] \implies \exists p\big[(i'|k') \leq_p (j'|\ell')\big].$$

So assume that $(i|k) \leq_p (j|\ell)$ for all p. Then $(i|k) \nsim (j|\ell)$ implies that there exist $0 \leq p \leq q$ and $(m|n) \in S$ such that $\rho(m) \neq u$ and $(i|k) \leq_p (m|n) \leq_p (j|\ell)$. Since $(i|k) \sim (i'|k')$ and $(j|\ell) \sim (j'|\ell')$, we can then infer $(i'|k') \leq_p (m|n) \leq_p (j'|\ell')$ as desired.

For each $1 \leq u \leq a$, let $\zeta_u \overset{\text{def}}{=} \big[z_u; q, \ldots, q\big] \in \Theta_2$. Then we can specify a $(1;q)$-cell

$$\bar{\chi} : [1;q] \to \boxtimes_a(\bar{\zeta}_1, \ldots, \bar{\zeta}_a)$$

with endpoints $\mathbf{0}, z$ by specifying shuffles $\trianglelefteq_0 \blacktriangleleft \cdots \blacktriangleleft \trianglelefteq_q$ on $T \overset{\text{def}}{=} S/{\sim} = \coprod_u T_u$. Here a *shuffle* $\trianglelefteq$ on T is a total order on T such that $[i|k] \trianglelefteq_p [j|\ell]$ for any $(i|k), (j|\ell) \in S_u$ with $[i|k] \leq_u [j|\ell]$, and $\trianglelefteq \blacktriangleleft \trianglelefteq'$ if $[i|k] \trianglelefteq [j|\ell]$ implies $[i|k] \trianglelefteq' [j|\ell]$ for any $(i|k) \in S_u$, $(j|\ell) \in S_v$ with $u < v$.

For each $0 \leq p \leq q$, define a binary relation $\trianglelefteq_p$ on T so that $[i|k] \trianglelefteq_p [j|\ell]$ if and only if $(i|k) \sim (j|\ell)$ or $(i|k) \leq_p (j|\ell)$. Note that this agrees with the third definition of $\leq_u$ on each T_u (and hence, assuming it is a well-defined total order, $\trianglelefteq_p$ is a shuffle). Thus to check that $\trianglelefteq_p$ is well-defined, we only need to consider two $\sim$-related pairs $(i|k) \sim (i'|k')$ and $(j|\ell) \sim (j'|\ell')$ such that $\rho(i) \neq \rho(j)$. In this case, it follows from our definition of $\sim$ that $(i|k) \leq_p (j|\ell)$ implies $(i'|k') \leq_p (j'|\ell')$. Hence $\trianglelefteq_p$ is indeed well-defined, and moreover it is a total order since $\leq_p$ is so. Furthermore, it is easy to check that $\leq_p \blacktriangleleft \leq_{p'}$ implies $\trianglelefteq_p \blacktriangleleft \trianglelefteq_{p'}$ for any $0 \leq p \leq p' \leq q$. Thus we obtain the desired map $\bar{\chi}$. There is a map

$$\chi_u = \big[\{0, z_u\}; \mathrm{id}, \ldots, \mathrm{id}\big] : [1;q] \to \zeta_u$$

in Θ_2 for each $1 \leq u \leq a$, and these combine together to induce χ as

$$\begin{tikzcd}
{[1;q]} \arrow[dr, dashed, "\chi"] \arrow[drr, bend left, "\bar{\chi}"] \arrow[ddr, bend right, "\langle \chi_1, \ldots, \chi_a \rangle"'] & & \\
& \boxtimes_a(\zeta_1, \ldots, \zeta_a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \boxtimes_a(\bar{\zeta}_1, \ldots, \bar{\zeta}_a) \arrow[d] \\
& \zeta_1 \times \cdots \times \zeta_a \arrow[r] & \bar{\zeta}_1 \times \cdots \times \bar{\zeta}_a
\end{tikzcd}$$

where the inner square is the pullback square in Lemma 4.1.7.

Now we construct the remaining part of the factorisation (4.14), namely

$$\psi_u : \zeta_u \to \boxtimes_{b_u}(\theta_{u,1}, \ldots, \theta_{u,b_u})$$

for each $1 \leq u \leq a$, where $\theta_{u,i} \overset{\text{def}}{=} \theta_{b_1+\cdots+b_{u-1}+i}$ denotes the i-th factor of the u-th "subtensor". First, define the object part of

$$\bar{\psi}_u : \zeta_u \to \boxtimes_{b_u}(\bar{\theta}_{u,1}, \ldots, \bar{\theta}_{u,b_u})$$

by sending each $0 \leq v \leq z_u$ to the object whose i-th coordinate is given by

$$\max\big(\big\{k : \exists v' \leq v[(i|k) \in S_{u,v'}]\big\} \cup \{s_i\}\big).$$

Its action on $\hom_{\zeta_u}(v-1, v) = [q]$ is given by restricting the $\leq_p$'s to $S_{u,v}$.

Fix $1 \leq i \leq b_u$. Define the horizontal component of

$$\psi_{u,i} : \zeta_u \to \theta_{u,i}$$

by the same formula as above, *i.e.* it sends each $0 \leq v \leq |T_u|$ to

$$\max\big(\big\{k : \exists v' \leq v[(i|k) \in S_{u,v'}]\big\} \cup \{s_i\}\big).$$

If $s_i < k \leq t_i$ then the k-th vertical component of $\psi_{u,i}$ is that of

$$[1; q] \xrightarrow{\ \phi\ } B \xrightarrow{\ \pi_i\ } \theta_i.$$

Finally, we can combine these maps to obtain ψ_u as in

$$\begin{tikzcd}
\zeta_u \arrow[dr, dashed, "\psi_u"] \arrow[drr, bend left, "\bar{\psi}_u"] \arrow[ddr, bend right, "\langle \psi_{u,1},\ldots,\psi_{u,b_u}\rangle"'] & & \\
& \boxtimes_{b_u}(\theta_{u,1},\ldots,\theta_{u,b_u}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \boxtimes_a(\bar{\theta}_{u,1},\ldots,\bar{\theta}_{u,b_u}) \arrow[d] \\
& \theta_{u,1}\times\cdots\times\theta_{u,b_u} \arrow[r] & \bar{\theta}_{u,1}\times\cdots\times\bar{\theta}_{u,b_u}
\end{tikzcd}$$

and one can check that

$$[1; q] \xrightarrow{\ \chi\ } \boxtimes_a(\zeta_1, \ldots, \zeta_a) \xrightarrow{\boxtimes_a(\psi_1,\ldots,\psi_a)} \boxtimes_b(\theta_1, \ldots, \theta_b).$$

is indeed a factorisation of ϕ. □

Lemma 4.5.8. *The map $\mu : A \to B$ is a monomorphism.*

Proof. Consider an $(n; \boldsymbol{q})$-cell ϕ in the image of μ. The proof of Lemma 4.5.7 constructs a factorisation of each $\phi \cdot \eta_h^k$, and then the proof of Lemma 4.5.6 combines them into a factorisation

$$\phi : \ [n; \mathbf{q}] \xrightarrow{\ \chi\ } \boxtimes_a(\zeta_1, \ldots, \zeta_a) \xrightarrow{\boxtimes_a(\psi_1,\ldots,\psi_a)} \boxtimes_b(\theta_1, \ldots, \theta_b)$$

of ϕ. We wish to prove that $(\chi, \psi_1, \ldots, \psi_a)$ represents a unique cell in

$$A = \otimes_a \left(\otimes_{b_1}\left(\Theta_2^{\theta_1}, \ldots, \Theta_2^{\theta_{b_1}}\right), \ldots, \otimes_{b_a}\left(\Theta_2^{\theta_{b-b_a+1}}, \ldots, \Theta_2^{\theta_{b_a}}\right)\right)$$

that is sent to ϕ by μ. So suppose that $(\chi', \psi_1', \ldots, \psi_a')$ also represents such a cell in A, *i.e.*

$$\phi : \ [n; \mathbf{q}] \xrightarrow{\ \chi'\ } \boxtimes_a(\zeta_1', \ldots, \zeta_a') \xrightarrow{\boxtimes_a(\psi_1',\ldots,\psi_a')} \boxtimes_b(\theta_1, \ldots, \theta_b).$$

is another factorisation of ϕ. If ψ'_u can be factored as $\psi'_u = \delta_u \circ \sigma_u$ then

$$\big(\boxtimes_a(\sigma_1, \dots, \sigma_a) \circ \chi'; \delta_1, \dots, \delta_a\big)$$

represents the same cell, and so we may assume without loss of generality that each ψ'_u is a non-degenerate cell in (the nerve of) $\boxtimes_{b_u}(\theta_{u,1}, \dots, \theta_{u,b_u})$. This implies that $\boxtimes_a(\psi'_1, \dots, \psi'_a)$ is a monomorphism. Now for each $1 \le u \le a$ and $1 \le i \le n$, consider the diagram:

$$\begin{tikzcd}
\zeta_u^k \arrow[r, hook] \arrow[ddr, dashed, "\omega_u^k"'] & \zeta_u \arrow[dr, "\psi_u"] \arrow[dd, dashed, "\omega_u"] & \\
& & \boxtimes_{b_u}(\theta_{u,1}, \dots, \theta_{u,b_u}) \\
& \zeta'_u \arrow[ur, "\psi'_u"'] &
\end{tikzcd}$$

By construction of ψ_u^k, the image of ψ'_u contains all of the objects in the image of ψ_u^k. So, at least on the object level, there is ω_u^k as indicated above that renders the perimeter commutative. We can upgrade it to a morphism in Θ_2 by setting its v-th vertical component to be that of

$$[n; \mathbf{q}] \xrightarrow{\chi'} \boxtimes_a(\zeta'_1, \dots, \zeta'_a) \xrightarrow{\pi_u} \zeta'_u$$

for $\pi_u \circ \chi'(k-1) < v \le \pi_u \circ \chi'(k)$. Since ζ_u is the colimit of ζ_u^k's, these induce a unique map ω_u as indicated. Now in the diagram

$$\begin{tikzcd}
& \boxtimes_a(\zeta_1, \dots, \zeta_a) \arrow[dr, "\boxtimes_a(\psi_1,\dots,\psi_a)"] \arrow[dd, "\boxtimes_a(\omega_1,\dots,\omega_a)"] & \\
{[n; \mathbf{q}]} \arrow[ur, "\chi"] \arrow[dr, "\chi'"'] & & \boxtimes_b(\theta_1, \dots, \theta_b) \\
& \boxtimes_a(\zeta'_1, \dots, \zeta'_a) \arrow[ur, "\boxtimes_a(\psi'_1,\dots,\psi'_a)"'] &
\end{tikzcd}$$

the perimeter commutes since both of the two paths compose to ϕ, and the right triangle commutes by construction of ω_u. Moreover we know that the lower right map is a monomorphism, so the left triangle also commutes. This shows that $(\chi', \psi'_1, \dots, \psi'_a)$ and $(\chi, \psi_1, \dots, \psi_a)$ represent the same cell in A, as desired. $\square$

4.5.3 The Leibniz comparison map $\hat{\mu}$

Fix $a, b_1, \dots, b_a \in \mathbb{N}$ and let $b = \sum_{u=1}^{a} b_u$. Note that the natural transformation

$$\mu : \otimes_a\big(\otimes_{b_1}, \dots, \otimes_{b_a}\big) \to \otimes_b$$

may be regarded as a $(b+1)$-ary functor

$$F : \underbrace{\widehat{\Theta_2} \times \cdots \times \widehat{\Theta_2}}_{b \text{ times}} \times \mathbb{2} \to \widehat{\Theta_2}.$$

Definition 4.5.9. We define the *Leibniz comparison map* $\hat{\mu}$ to be the b-ary functor

$$\hat{\mu} \overset{\text{def}}{=} \hat{F}\big(-,\dots,-,0 \to 1\big) : \widehat{\Theta_2}^{\mathbb{2}} \times \cdots \times \widehat{\Theta_2}^{\mathbb{2}} \to \widehat{\Theta_2}^{\mathbb{2}}.$$

The aim of this subsection is to prove the following theorem.

Theorem 4.5.10. *For any monomorphisms $f_1, \dots, f_b$ in $\widehat{\Theta_2}$, the Leibniz comparison map $\hat{\mu}(f_1, \dots, f_b)$ is in* $\mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v)$.

Proof. By Lemma 2.2.11 and Proposition 2.2.16, it suffices to prove the special case where each f_i is the boundary inclusion into a representable cellular set. This follows from Lemmas 4.5.12, 4.5.13, 4.5.15 and 4.5.16 proved below. □

The following corollary of Theorem 4.5.10 states that the Gray tensor product is associative up to homotopy.

Corollary 4.5.11. *For any $X^1, \dots, X^b \in \widehat{\Theta_2}$, the component*

$$\otimes_a\big(\otimes_{b_1}(X^1, \dots, X^{b_1}), \dots, \otimes_{b_a}(X^{b-b_a+1}, \dots, X^b)\big) \to \otimes_b(X^1, \dots, X^b)$$

of μ is in $\mathrm{cell}(\mathcal{H}_h \cup \mathcal{H}_v)$.

Proof. Apply Theorem 4.5.10 to the empty inclusions $f_i : \varnothing \to X^i$. □

Now we complete the proof of Theorem 4.5.10. Fix $\theta_1, \dots, \theta_b \in \Theta_2$, and let $\nu : A^0 \to B$ denote the Leibniz comparison map

$$\nu \overset{\text{def}}{=} \hat{\mu}\left(\begin{array}{c} \partial\Theta_2^{\theta_1} \\ \big\downarrow \\ \Theta_2^{\theta_1} \end{array}, \dots, \begin{array}{c} \partial\Theta_2^{\theta_b} \\ \big\downarrow \\ \Theta_2^{\theta_b} \end{array}\right).$$

Lemma 4.5.12. *The map ν is a monomorphism.*

Proof. By Lemma 2.2.9, it suffices to prove that the functor

$$G : \mathbb{2}^{b+1} \to \widehat{\Theta_2}$$

(defined as in Section 2.2.3) sends each square of the form (2.1) to a pullback square of monomorphisms. The case $i, j \le b$ was treated in Lemma 4.2.1, so we may assume $j = b+1$. Fix $1 \le i \le b$, and let

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ B' & \longrightarrow & B \end{array}$$

be the image of the square (2.1) under G. The horizontal maps are monic by Lemma 4.2.1, and the right vertical one is monic by Corollary 4.5.11. Moreover the commutativity of this square then implies that the left vertical map is also monic.

It remains to prove that this square is a pullback. So consider a pure $(n; \boldsymbol{q})$-cell ϕ contained in the image of the map

$$\boxtimes_b(\theta_1, \dots, \theta_{i-1}, \kappa, \theta_{i+1}, \dots, \theta_b) \to \boxtimes_b(\theta_1, \dots, \theta_b)$$

$$\preceq_p \qquad \boxed{\ldots \preceq_p (i|k) \preceq_p \ldots \preceq_p (m|n) \preceq_p \ldots \preceq_p (j|\ell) \preceq_p \ldots}$$

$$\preceq_0 \qquad \boxed{\ldots \preceq_0 (j|\ell) \preceq_0 \ldots \preceq_0 (i|k) \preceq_0 \ldots}$$

Figure 4.3: A typical upper impurity

induced by some hyperface $\delta : \kappa \longrightarrow \theta_i$. It is straightforward to check that, for the factorisation of ϕ constructed in the proof of Lemma 4.5.8, the map $\psi_{\rho(i)}$ factors through the obvious sub-2-category of the codomain determined by δ. Hence this factorisation specifies a cell in A' as desired. $\square$

Thus we may regard $\nu : A^0 \longrightarrow B$ as a cellular subset inclusion. By Theorem 4.5.4, A^0 is generated by A and the pure cells. Let $A^1 \subset B$ be the cellular subset generated by A^0 and the (potentially) cuttable cells.

Lemma 4.5.13. *The inclusion* $A^0 \hookrightarrow A^1$ *is in* $\mathrm{cell}(\mathcal{H}_h)$.

Proof. Observe that for any potentially cuttable cell χ in B that is not cuttable, χ is pure if and only if its cuttable parent is pure. The rest of the proof is similar to the first part of the proof of Lemma 4.4.2. $\square$

Now consider a non-degenerate cell ϕ in $B \setminus A^1$. Note that ϕ is necessarily a $(1;q)$-cell for some $q \geq 1$ with endpoints $\mathbf{0}, \boldsymbol{t}$ where t_i is the horizontal length of θ_i (*i.e.* $\bar{\theta}_i = [t_i; \mathbf{0}]$). Let $S = S(\mathbf{0}, \boldsymbol{t})$ and let $\preceq_0, \ldots, \preceq_q$ be the underlying shuffles of ϕ. Since ϕ is not pure, ϕ must contain an *impurity* in the following sense.

Definition 4.5.14. An *upper impurity* in ϕ is a quadruple $\mathcal{B} = \langle (i|k), (j|\ell), (m|n), p \rangle$ consisting of $(i|k)$, $(j|\ell)$, $(m|n) \in S$ and $p \in [q]$ such that:

- $i < j$;
- $\rho(i) = \rho(j) \neq \rho(m)$;
- $(i|k) \succeq_0 (j|\ell)$; and
- $(i|k) \preceq_p (m|n) \preceq_p (j|\ell)$.

(See Fig. 4.3.) A *lower impurity* in ϕ is a quadruple $\mathcal{B} = \langle (i|k), (j|\ell), (m|n), p \rangle$ consisting of $(i|k)$, $(j|\ell)$, $(m|n) \in S$ and $p \in [q]$ such that:

- $i < j$;
- $\rho(i) = \rho(j) \neq \rho(m)$;
- $(i|k) \preceq_q (j|\ell)$; and
- $(i|k) \succeq_p (m|n) \succeq_p (j|\ell)$.

We say ϕ is an *upper* cell if it contains no lower impurities.

Let $A^2 \subset B$ be the cellular subset generated by A^1 and the upper cells. Since any face of an upper cell is itself upper, any non-degenerate face in $A^2 \setminus A^1$ must be upper.

Lemma 4.5.15. *The inclusion $A^1 \hookrightarrow A^2$ is in* $\mathrm{cell}(\mathcal{H}_v)$.

Proof. Fix a non-degenerate $(1;q)$-cell ϕ in $A^2 \setminus A^1$ (which is necessarily upper). Define a total order $\leq$ on the set of upper impurities in ϕ so that

$$\langle (i|k),(j|\ell),(m|n),p\rangle \leq \langle (i'|k'),(j'|\ell'),(m'|n'),p'\rangle$$

if and only if:

- $p < p'$;
- $p = p'$ and $(i|k) >_{lex} (i'|k')$;
- $p = p'$, $(i|k) = (i'|k')$ and $(j|\ell) <_{lex} (j'|\ell')$; or
- $p = p'$, $(i|k) = (i'|k')$, $(j|\ell) = (j'|\ell')$ and $(m|n) \leq_{lex} (m'|n')$.

Here $\leq_{lex}$ denotes the lexicographical order so that $(i|k) \leq_{lex} (j|\ell)$ if and only if either:

- $i < j$; or
- $i = j$ and $k \leq \ell$.

This indeed defines a total order on the set of upper impurities in ϕ, hence in particular we have a minimum impurity

$$\mathcal{B}_\phi = \big\langle (i_\phi|k_\phi),(j_\phi|\ell_\phi),(m_\phi|n_\phi),p_\phi\big\rangle.$$

Let $s_\phi \in [q]$ be the largest s satisfying $(i_\phi|k_\phi) \succeq_s (j_\phi|\ell_\phi)$. Note that we must have $s_\phi < p_\phi$ since $(i_\phi|k_\phi) \preceq_{p_\phi} (j_\phi|\ell_\phi)$ and $i_\phi < j_\phi$ imply $(i_\phi|k_\phi) \preceq_p (j_\phi|\ell_\phi)$ for all $p \geq p_\phi$. We will construct the "best approximation" $\preceq$ to $\preceq_{s_\phi}$ such that $(i_\phi|k_\phi) \preceq (j_\phi|\ell_\phi)$ (in the sense of Claim 6 below).

Consider the partition $S = I_1 \cup I_2 \cup I_3 \cup I_4$ where

$$\begin{aligned}
I_1 &= \big\{(x|y) \in S : (x|y) \prec_{s_\phi} (j_\phi|\ell_\phi)\big\}\\
I_2 &= \big\{(x|y) \in S : (j_\phi|\ell_\phi) \preceq_{s_\phi} (x|y) \preceq_{s_\phi} (i_\phi|k_\phi), \quad (x|y) \preceq_{p_\phi} (i_\phi|k_\phi)\big\}\\
I_3 &= \big\{(x|y) \in S : (j_\phi|\ell_\phi) \preceq_{s_\phi} (x|y) \preceq_{s_\phi} (i_\phi|k_\phi), \quad (j_\phi|\ell_\phi) \preceq_{p_\phi} (x|y)\big\}\\
I_4 &= \big\{(x|y) \in S : (i_\phi|k_\phi) \prec_{s_\phi} (x|y)\big\}.
\end{aligned}$$

To see that this is indeed a partition of S, observe that if $(x|y)$ satisfies both

$$(j_\phi|\ell_\phi) \preceq_{s_\phi} (x|y) \preceq_{s_\phi} (i_\phi|k_\phi) \quad \text{and} \quad (i_\phi|k_\phi) \prec_{p_\phi} (x|y) \prec_{p_\phi} (j_\phi|\ell_\phi)$$

then we must have $i_\phi \leq x \leq j_\phi$ since $\preceq_{s_\phi} \blacktriangleleft \preceq_{p_\phi}$. It follows that $\rho(x) = \rho(i_\phi)$. But then either $\big\langle (i_\phi|k_\phi),(x|y),(m_\phi|n_\phi),p_\phi\big\rangle$ or $\big\langle (x|y),(j_\phi|\ell_\phi),(m_\phi|n_\phi),p_\phi\big\rangle$ is an upper impurity strictly smaller than $\mathcal{B}_\phi$, which contradicts our choice of $\mathcal{B}_\phi$.

Now define a total order $\preceq$ on S so that $(x|y) \preceq (z|w)$ if and only if either

- $(x|y) \in I_u$ and $(z|w) \in I_v$ for some $u < v$; or
- $(x|y),(z|w) \in I_u$ for some u and $(x|y) \preceq_{s_\phi} (z|w)$.

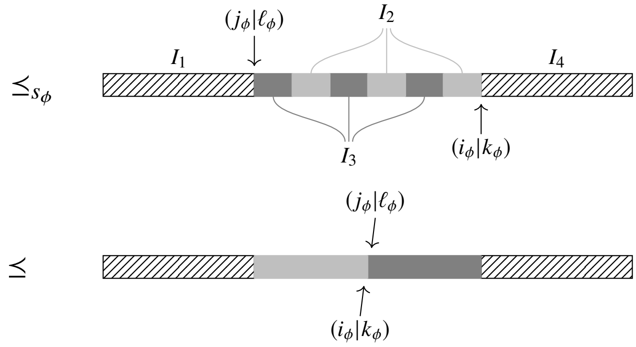


Figure 4.4: $\preceq_{s_\phi}$ and $\preceq$

It is easy to check that $\preceq$ is a shuffle using the fact that $\preceq_{s_\phi}$ and $\preceq_{p_\phi}$ are so.

Observe that $(i_\phi|k_\phi)$ is the $\preceq_{s_\phi}$-maximum element of I_2 and $(j_\phi|\ell_\phi)$ is the $\preceq_{s_\phi}$-minimum element of I_3. Therefore $(j_\phi|\ell_\phi)$ is the immediate $\preceq$-successor of $(i_\phi|k_\phi)$, which in particular implies $\preceq_{s_\phi} \neq \preceq \neq \preceq_{p_\phi}$.

Claim 6. The shuffle $\preceq$ is $\blacktriangleleft$-minimum among those shuffles $\preceq'$ satisfying $\preceq_{s_\phi} \blacktriangleleft \preceq' \blacktriangleleft \preceq_{p_\phi}$ and $(i_\phi|k_\phi) \preceq' (j_\phi|\ell_\phi)$.

Proof of the claim. Suppose that $(x|y), (z|w) \in S$ satisfy $(x|y) \preceq_{s_\phi} (z|w)$ and $x < z$. Then we must have $(x|y) \preceq_{p_\phi} (z|w)$ since $\preceq_{s_\phi} \blacktriangleleft \preceq_{p_\phi}$. Now it follows from our construction of $\preceq$ that $(x|y) \preceq (z|w)$ holds too. This prove $\preceq_{s_\phi} \blacktriangleleft \preceq$.

Now let $\preceq'$ be a shuffle on $S(\mathbf{0}, \boldsymbol{t})$ satisfying $\preceq_{s_\phi} \blacktriangleleft \preceq' \blacktriangleleft \preceq_{p_\phi}$ and $(i_\phi|k_\phi) \preceq' (j_\phi|\ell_\phi)$. Let $(x|y), (z|w) \in S(\mathbf{0}, \boldsymbol{t})$ and suppose that both $(x|y) \preceq (z|w)$ and $x < z$ hold. We wish to show that $(x|y) \preceq' (z|w)$.

- If $(x|y), (z|w) \in I_u$ for some u, then $(x|y) \preceq_{s_\phi} (z|w)$ by the definition of $\preceq$. Thus $\preceq_{s_\phi} \blacktriangleleft \preceq'$ implies $(x|y) \preceq' (z|w)$.
- If $(x|y) \in I_1$ and $(z|w) \in I_u$ for some $u \geq 2$, then $(x|y) \preceq_{s_\phi} (j_\phi|\ell_\phi) \preceq_{s_\phi} (z|w)$. Thus $\preceq_{s_\phi} \blacktriangleleft \preceq'$ implies $(x|y) \preceq' (z|w)$.
- Using $(i_\phi|k_\phi)$ in place of $(j_\phi|\ell_\phi)$ in the previous item, we can prove that if $(z|w) \in I_4$ then $(x|y) \preceq' (z|w)$.
- The remaining case is when $(x|y) \in I_2$ and $(z|w) \in I_3$.
  - If $x < i_\phi$ then $(x|y) \preceq_{s_\phi} (i_\phi|k_\phi)$ and $\preceq_{s_\phi} \blacktriangleleft \preceq'$ imply $(x|y) \preceq' (i_\phi|k_\phi)$.
  - If $x > i_\phi$ then $(x|y) \preceq_{p_\phi} (i_\phi|k_\phi)$ and $\preceq' \blacktriangleleft \preceq_{p_\phi}$ imply $(x|y) \preceq' (i_\phi|k_\phi)$.
  - If $x = i_\phi$ then $(x|y) \preceq_{s_\phi} (i_\phi|k_\phi)$ implies $y \leq k_\phi$ and thus $(x|y) \preceq' (i_\phi|k_\phi)$.

  We can similarly deduce $(j_\phi|\ell_\phi) \preceq' (z|w)$ and hence

$$(x|y) \preceq' (i_\phi|k_\phi) \preceq' (j_\phi|\ell_\phi) \preceq' (z|w).$$

Therefore we indeed have $(x|y) \preceq' (z|w)$, and this shows $\preceq \blacktriangleleft \preceq'$. In particular, by taking $\preceq' = \preceq_{p_\phi}$ we can deduce that $\preceq \blacktriangleleft \preceq_{p_\phi}$. □

Since $\preceq$ is a shuffle, it determines a 1-cell $\bar{g}_\phi$ in $\boxtimes_b(\bar{\theta}_1, \dots, \bar{\theta}_b)$. We can upgrade it to a 1-cell in B as in:

$$
\begin{tikzcd}
{[1;0]} \arrow[d, "\phi\cdot[\mathrm{id};\{s_\phi\}]"'] \arrow[dr, dashed, "g_\phi"] \arrow[drr, bend left, "\bar{g}_\phi"] & & \\
\boxtimes_b(\theta_1, \dots, \theta_b) \arrow[dr, bend right, "\langle \pi_1, \dots, \pi_b \rangle"'] & \boxtimes_b(\theta_1, \dots, \theta_b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \boxtimes_b(\bar{\theta}_1, \dots, \bar{\theta}_b) \arrow[d] \\
& \theta_1 \times \cdots \times \theta_b \arrow[r] & \bar{\theta}_1 \times \cdots \times \bar{\theta}_b
\end{tikzcd}
$$

Consider the following condition on ϕ:

$(*)$ $\phi \cdot \eta_v^{s_\phi + 1} = g_\phi$

where η_v^i is the cellular operator defined in Definition 2.1.12.

Claim 7. Suppose that ϕ is a non-degenerate $(1;q)$-cell in $A^2 \setminus A^1$ not satisfying $(*)$. Then there exists a unique non-degenerate $(1;q+1)$-cell ψ in $A^2 \setminus A^1$ such that ψ satisfies $(*)$ and $\phi = \psi \cdot \delta_v^{1;s_\psi+1}$.

Proof. The cell ψ is the unique one determined by the conditions $\psi \cdot \delta_v^{1;s_\phi+1} = \phi$ and $\psi \cdot \eta_v^{s_\phi+1} = g_\phi$. (Note that we are using s_ϕ and not s_ψ.) These conditions indeed specify a $(1;q+1)$-cell ψ in B by Claim 6. This cell ψ is not in A^1 since it contains ϕ as a face and ϕ is not in A^1.

We show that ψ is an upper cell (and hence contained in A^2). Suppose for contradiction that ψ contains a lower impurity $\mathcal{B}$. Since ϕ contains no lower impurities and $\psi \cdot \eta_v^{q+1} = \phi \cdot \eta_v^q$, this lower impurity $\mathcal{B}$ must be of the form

$$\mathcal{B} = \langle (i|k), (j|\ell), (m|n), s_\phi + 1 \rangle.$$

In other words, we have:

- $i < j$;
- $\rho(i) = \rho(j) \neq \rho(m)$;
- $(i|k) \preceq_q (j|\ell)$; and
- $(i|k) \succeq (m|n) \succeq (j|\ell)$

where $\preceq_p$ are the underlying shuffles of ϕ (and not of ψ) and $\preceq$ is the shuffle constructed above. Note that $i < j$, $(j|\ell) \preceq (i|k)$ and $\preceq_s \blacktriangleleft \preceq$ imply that $(j|\ell) \preceq_{s_\phi} (i|k)$.

Since ϕ is upper, $\langle (i|k), (j|\ell), (m|n), s_\phi \rangle$ is not a lower impurity. Hence we must have either $(i|k) \preceq_{s_\phi} (m|n)$ or $(m|n) \preceq_{s_\phi} (j|\ell)$. In the former case, the assumption $(m|n) \preceq (i|k)$ implies $(m|n) \in I_2$ and $(i|k) \in I_3$. For neither $\langle (i_\phi|k_\phi), (j_\phi|\ell_\phi), (i|k), s_\phi \rangle$ nor $\langle (i_\phi|k_\phi), (j_\phi|\ell_\phi), (m|n), s_\phi \rangle$ to be a lower impurity in ϕ, we must have both $\rho(i_\phi) = \rho(i)$ and $\rho(i_\phi) = \rho(m)$. This contradicts our assumption $\rho(i) \neq \rho(m)$. We can derive a similar contradiction in the case $(m|n) \preceq_{s_\phi} (j|\ell)$ too, and this proves that ψ is upper.

Finally we prove that the minimum impurity $\mathcal{B}_\psi$ in ψ is $\langle (i_\phi|k_\phi), (j_\phi, \ell_\phi), (m_\phi|n_\phi), p_\phi + 1\rangle$. Note that, assuming this fact, it easily follows that ψ satisfies $(*)$ and $\phi = \psi \cdot \delta_v^{1;s_\psi+1}$. Since $\psi \cdot \delta_v^{1;s_\phi+1} = \phi$, if ψ has an impurity $\mathcal{B}$ that is smaller than $\langle (i_\phi|k_\phi), (j_\phi, \ell_\phi), (m_\phi|n_\phi), p_\phi + 1\rangle$ then it must be of the form

$$\mathcal{B} = \langle (i|k), (j|\ell), (m|n), s_\phi + 1\rangle.$$

In other words, we have:

- $i < j$;
- $\rho(i) = \rho(j) \neq \rho(m)$;
- $(i|k) \geq_0 (j|\ell)$; and
- $(i|k) \leq (m|n) \leq (j|\ell)$.

Since $\mathcal{B}_\phi$ is the minimum upper impurity in ϕ, $\langle (i|k), (j|\ell), (m|n), s_\phi\rangle$ is not an upper impurity. Hence we must have either $(m|n) \preceq_{s_\phi} (i|k)$ or $(j|\ell) \preceq_{s_\phi} (m|n)$. In the former case, the assumption $(i|k) \leq (m|n)$ implies $(i|k) \in I_2$ and $(m|n) \in I_3$. For neither $\langle (i_\phi|k_\phi), (j_\phi|\ell_\phi), (i|k), s_\phi\rangle$ nor $\langle (i_\phi|k_\phi), (j_\phi|\ell_\phi), (m|n), s_\phi\rangle$ to be a lower impurity in ϕ, we must have both $\rho(i_\phi) = \rho(i)$ and $\rho(i_\phi) = \rho(m)$. This contradicts our assumption $\rho(i) \neq \rho(m)$. We can derive a similar contradiction in the case $(j|\ell) \preceq_{s_\phi} (m|n)$ too, and this completes the proof of Claim 7. □

We wish to prove that A^2 may be obtained from A^1 by gluing those ϕ satisfying $(*)$ along the inner horn $\Lambda_v^{1;s_\phi+1}$ in lexicographically increasing order of $\mathrm{sil}(\phi)$, $\dim(\phi)$, p_ϕ and s_ϕ where p_ϕ is regarded as an element of $[q]^{\mathrm{op}}$. This conclusion can be deduced from the following analysis of the hyperfaces of ϕ.

Temporary definition. In this proof, if ϕ, ψ are as described in Claim 7 then we say ψ is the $*$-*parent* of ϕ.

Let ϕ be a non-degenerate $(1;q)$-cell in $A^2 \setminus A^1$ satisfying $(*)$. Clearly $\phi \cdot \delta_v^{1;0}$ and $\phi \cdot \delta_v^{1;q}$ have smaller silhouettes than $\mathrm{sil}(\phi)$. The hyperface $\phi \cdot \delta_v^{1;s_\phi+1}$ is treated in Claim 8 below. The hyperface $\phi \cdot \delta_v^{1;s_\phi}$ is:

- contained in A; or
- contained in $A^2 \setminus A^1$ and:
  - it satisfies $(*)$; or
  - it does not satisfy $(*)$, in which case its $*$-parent ψ necessarily has $\mathrm{sil}(\psi) = \mathrm{sil}(\phi)$, $\dim(\psi) = \dim(\phi)$, $\psi \cdot \eta_v^{p_\psi} = \phi \cdot \eta_v^{p_\phi}$ and $s_\psi = s_\phi - 1$.

The hyperface $\phi \cdot \delta_v^{1;p_\phi}$ is:

- contained in A; or
- contained in $A^2 \setminus A^1$ and:
  - it satisfies $(*)$; or
  - it does not satisfy $(*)$, in which case its $*$-parent ψ necessarily has $\mathrm{sil}(\psi) = \mathrm{sil}(\phi)$, $\dim(\psi) = \dim(\phi)$ and $p_\psi > p_\phi$.

For any other value of j, the hyperface $\phi \cdot \delta_v^{1;j}$ is:

- contained in A; or
- contained in $A^2 \setminus A^1$ and it satisfies $(*)$.

Claim 8. The hyperface $\chi = \phi \cdot \delta_v^{1;s_\phi+1}$ is a non-degenerate cell in $A^2 \setminus A^1$ and the minimum upper impurity $\mathcal{B}_\chi$ in χ is

$$\big\langle (i_\phi|k_\phi), (j_\phi|\ell_\phi), (m_\phi|n_\phi), p_\phi - 1 \big\rangle.$$

Consequently χ does not satisfy $(*)$.

Proof of the claim. The cell χ is not contained in A since it admits an impurity

$$\big\langle (i_\phi|k_\phi), (j_\phi|\ell_\phi), (m_\phi|n_\phi), p_\phi - 1 \big\rangle.$$

Now to prove that χ is not in A^0, it suffices to show that χ is not in $B_x(\delta)$ for any $1 \le x \le b$ and for any hyperface $\delta : \kappa \to \theta_x$, where $B_x(\delta) \subset B$ is the image of the map

$$\boxtimes_b(\theta_1, \dots, \theta_{x-1}, \kappa, \theta_{x+1}, \dots, \theta_b) \to \boxtimes_b(\theta_1, \dots, \theta_b)$$

induced by δ.

If δ is either a vertical hyperface or an outer horizontal hyperface, then χ is in $B_x(\delta)$ if and only if the projection $\pi_x(\chi)$ is in (the image of) δ, and similarly for ϕ. Since ϕ is not in $B_x(\delta)$ and $\pi_x(\phi)$ is a degeneracy of $\pi_x(\chi)$, it follows that χ is not in $B_x(\delta)$.

Now consider the case where δ is a y-th horizontal hyperface with $1 \le y \le t_x - 1$. Suppose for contradiction that χ is in $B_x(\delta)$. Then Lemma 4.1.11 implies that $(x|y+1)$ is the immediate $\preceq_p$-successor of $(x|y)$ for all $0 \le p \le q$ with $p \ne s_\phi + 1$. We show that $(x|y+1)$ must then be the immediate successor of $(x|y)$ with respect to $\preceq \, = \, \preceq_{s_\phi+1}$ too. Note that this is automatic if $(x|y), (x|y+1) \in I_u$ for some u by our construction of $\preceq$.

- If $(x|y+1) \prec_{s_\phi} (j_\phi|\ell_\phi)$ then $(x|y), (x|y+1) \in I_1$.
- If $(x|y+1) = (j_\phi|\ell_\phi)$ then $\langle (i_\phi|k_\phi), (x|y), (m_\phi|n_\phi), p_\phi \rangle$ is a strictly smaller impurity than $\mathcal{B}_\phi$, which contradicts our choice of $\mathcal{B}_\phi$.
- Suppose $(j_\phi|\ell_\phi) \preceq_{s_\phi} (x|y) \preceq_{s_\phi} (x|y+1) \preceq_{s_\phi} (i_\phi|k_\phi)$. Since $(x|y+1)$ is the immediate $\preceq_{p_\phi}$-successor of $(x|y)$, it follows that either $(x|y), (x|y+1) \in I_2$ or $(x|y), (x|y+1) \in I_3$.
- The case $(i_\phi|k_\phi) \preceq_{s_\phi} (x|y)$ can be treated similarly to the first two cases.

Therefore $(x|y+1)$ is the immediate $\preceq_p$-successor of $(x|y)$ for all $0 \le p \le q$, including $p = s_\phi + 1$. By Lemma 4.1.11, this implies that ϕ is in $B_x(\delta)$ (for the same δ) which contradicts our assumption that ϕ is not in A^0.

Finally, to see that χ is not contained in A^1, recall that we have $\preceq_{s_\phi} \ne \preceq \ne \preceq_{p_\phi}$ (observed immediately before Claim 6). Since $\phi \cdot \eta_v^{s_\phi+1} = g_\phi$ has $\preceq$ as the underlying shuffle, it follows from $\preceq \blacktriangleleft \preceq_{p_\phi}$ that $s_\phi + 1 < p_\phi$. Thus χ an inner face of ϕ, which implies that χ is absolutely uncuttable (as $\mathrm{sil}(\chi) = \mathrm{sil}(\phi)$ by Proposition 4.3.9).

It is now straightforward to check that

$$\mathcal{B}_\chi = \big\langle (i_\phi|k_\phi), (j_\phi|\ell_\phi), (m_\phi|n_\phi), p_\phi - 1 \big\rangle.$$

This implies that $s_\chi = s_\phi$ and $g_\chi = g_\phi$. Since ϕ is non-degenerate, it follows that

$$\chi \cdot \eta_v^{s_\chi+1} = \left(\phi \cdot \delta_v^{1;s_\phi+1}\right) \cdot \eta_v^{s_\phi+1} = \phi \cdot \eta_v^{s_\phi+2}$$

is not equal to $g_\chi = g_\phi = \phi \cdot \eta_v^{s_\phi+1}$. This shows that χ does not satisfy $(*)$. □

This completes the proof of Lemma 4.5.15. □

Lemma 4.5.16. *The inclusion* $A^2 \hookrightarrow B$ *is in* $\mathrm{cell}(\mathcal{H}_v)$.

Proof. The proof is essentially dual to that of Lemma 4.5.15. □

4.6 Consequences of associativity

We will discuss two consequences of Theorem 4.5.10 in this section.

4.6.1 $\otimes_a$ is left Quillen

First, we generalise Theorem 4.4.1.

Theorem 4.6.1. *The Gray tensor product functor* $\otimes_a$ *is left Quillen for any* $a \geq 1$. *That is, the Leibniz Gray tensor product*

$$\hat{\otimes}_a(f_1, \ldots, f_a)$$

is a monomorphism if each f_i *is, and it is a trivial cofibration if moreover some* f_i *is so.*

Proof. We proceed by induction on a. The case $a = 1$ is trivial, and the case $a = 2$ is Theorem 4.4.1.

Let $a \geq 3$ and suppose that $\hat{\otimes}_{a-1}$ is left Quillen. We already know that $\hat{\otimes}_a$ preserves monomorphisms (Lemma 4.2.1). So let $f_1, \ldots, f_a$ be monomorphisms in $\widehat{\Theta_2}$, and suppose that f_i is a trivial cofibration for some i. We wish to show that $\hat{\otimes}_a(f_1, \ldots, f_a) : A \to B$ is a trivial cofibration. Note that applying the Leibniz construction of $\otimes_2(\otimes_{a-1}, \otimes_1)$ to $f_1, \ldots, f_a$ yields

$$\hat{\otimes}_2\left(\hat{\otimes}_{a-1}(f_1, \ldots, f_{a-1}), f_a\right)$$

by [Our10, Observation 3.22], which we denote by $g : X \to Y$. This map is a trivial cofibration by the inductive hypothesis and Theorem 4.4.1. We can factorise $\hat{\otimes}_a(f_1, \ldots, f_a)$ as:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "\mu"'] & Y \arrow[d] \arrow[ddr, bend left, "\mu"] & \\
A \arrow[r] \arrow[drr, bend right, "\hat{\otimes}_a(f_1,\ldots,f_a)"'] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "h"] & \\
 & & B
\end{tikzcd}$$

A straightforward analysis of the universal property of the unlabelled object reveals that

$$h = \hat{\mu}(f_1, \ldots, f_a).$$

Thus $\hat{\otimes}_a(f_1, \ldots, f_a)$ is a trivial cofibration by Theorem 4.5.10. □

4.6.2 The closed structure

The previous subsection completes the "monoidal" part of the story, and now we consider the "closed" part. By construction of the Gray tensor product, the functor

$$\otimes_{a+1+b}(X^1,\dots,X^a,-,Y^1,\dots,Y^b) : \widehat{\Theta_2} \to \widehat{\Theta_2}$$

admits a right adjoint (which preserves fibrations and trivial fibrations by Theorem 4.6.1) for any $a,b \geq 0$ and for any $X^1,\dots,X^a,Y^1,\dots,Y^b \in \widehat{\Theta_2}$.

Definition 4.6.2. We will write

$$(Y^1,\dots,Y^b) \mathbin{-\!\triangleright} (-) \mathbin{\triangleleft\!-} (X^1,\dots,X^a)$$

or more succinctly

$$\boldsymbol{Y} \mathbin{-\!\triangleright} (-) \mathbin{\triangleleft\!-} \boldsymbol{X}$$

for this right adjoint.

Corollary 4.6.3. *Let $X^1,\dots,X^a,Y^1,\dots,Y^b,Z^1,\dots,Z^c,W^1,\dots,W^d \in \widehat{\Theta_2}$. Then there is a natural transformation*

$$\omega : \big((\boldsymbol{Z},\boldsymbol{W}) \mathbin{-\!\triangleright} (-) \mathbin{\triangleleft\!-} (\boldsymbol{X},\boldsymbol{Y})\big) \longrightarrow \big(\boldsymbol{Z} \mathbin{-\!\triangleright} (\boldsymbol{W} \mathbin{-\!\triangleright} (-) \mathbin{\triangleleft\!-} \boldsymbol{X}) \mathbin{\triangleleft\!-} \boldsymbol{Y}\big).$$

Moreover, the A-component of ω at any 2-quasi-category A is a trivial fibration.

Proof. The natural transformation ω is the mate of μ, *i.e.* the pasting

$$\begin{tikzcd}
& \bullet \arrow[rr, "\mathrm{id}"] \arrow[dr, "\otimes(\boldsymbol{X},\boldsymbol{Y},-,\boldsymbol{Z},\boldsymbol{W})"] \arrow[d, phantom, "\Downarrow\epsilon"] & & \bullet \arrow[rr, "\mathrm{id}"] \arrow[dr, "\otimes(\boldsymbol{X},\otimes(\boldsymbol{Y},-,\boldsymbol{Z}),\boldsymbol{W})"'] \arrow[dl, phantom, "\swarrow\mu"] & & \bullet \arrow[dl, phantom, "\Downarrow\eta"] \\
\bullet \arrow[ur, "(\boldsymbol{Z},\boldsymbol{W}) \mathbin{-\!\triangleright} (-) \mathbin{\triangleleft\!-} (\boldsymbol{X},\boldsymbol{Y})"] \arrow[rr, "\mathrm{id}"'] & & \bullet \arrow[rr, "\mathrm{id}"'] & & \bullet \arrow[ur, "\boldsymbol{Z} \mathbin{-\!\triangleright} (\boldsymbol{W} \mathbin{-\!\triangleright} (-) \mathbin{\triangleleft\!-} \boldsymbol{X}) \mathbin{\triangleleft\!-} \boldsymbol{Y}"'] &
\end{tikzcd}$$

where each vertex is $\widehat{\Theta_2}$ and the 2-cells η, ϵ are the unit and the counit of the appropriate adjunctions. Fix a monomorphism $B \hookrightarrow C$ in $\widehat{\Theta_2}$ and a 2-quasi-category A. We wish to show that any commutative square of the form

$$\begin{tikzcd}
B \arrow[r] \arrow[d, hook] & (\boldsymbol{Z},\boldsymbol{W}) \mathbin{-\!\triangleright} A \mathbin{\triangleleft\!-} (\boldsymbol{X},\boldsymbol{Y}) \arrow[d, "\omega"] \\
C \arrow[r] \arrow[ur, dashed] & \boldsymbol{Z} \mathbin{-\!\triangleright} (\boldsymbol{W} \mathbin{-\!\triangleright} A \mathbin{\triangleleft\!-} \boldsymbol{X}) \mathbin{\triangleleft\!-} \boldsymbol{Y}
\end{tikzcd}$$

admits a diagonal lift as indicated. By construction of ω, such a commutative square corresponds to one of the form

$$\begin{tikzcd}
\otimes(\boldsymbol{X},\boldsymbol{Y},B,\boldsymbol{Z},\boldsymbol{W}) \coprod_{\otimes(\boldsymbol{X},\otimes(\boldsymbol{Y},B,\boldsymbol{Z}),\boldsymbol{W})} \otimes\big(\boldsymbol{X},\otimes(\boldsymbol{Y},C,\boldsymbol{Z}),\boldsymbol{W}\big) \arrow[r] \arrow[d, hook] & A \arrow[d] \\
\otimes(\boldsymbol{X},\boldsymbol{Y},C,\boldsymbol{Z},\boldsymbol{W}) \arrow[r] \arrow[ur, dashed] & 1
\end{tikzcd}$$

and moreover either square admits a diagonal lift if and only if the other does. The latter square indeed admits a lift by Theorem 4.5.10 since the left vertical map is an instance of $\hat{\mu}$ evaluated at the monomorphisms

$$\varnothing \hookrightarrow X^i, \quad \varnothing \hookrightarrow Y^j, \quad B \hookrightarrow C, \quad \varnothing \hookrightarrow Z^k, \text{ and } \varnothing \hookrightarrow W^\ell.$$

This completes the proof. $\square$

A
Appendix

A.1 Braid monoids with zero

In this appendix, we complete the proof of Lemma 4.1.5 using the *braid monoids with zero*. A special case of Lemma 4.1.5 where $\theta_i = [1; 0]$ for each i was first proved by Gray [Gra76, Theorem 2.2] using the *braid groups*. Our argument here is a minor modification of Street's proof of that same special case [Str88, Theorem 1].

Definition A.1.1. A *monoid with zero* is a monoid M with a distinguished element $0 \in M$ such that

$$x0 = 0 = 0x \tag{A.1}$$

for all $x \in M$.

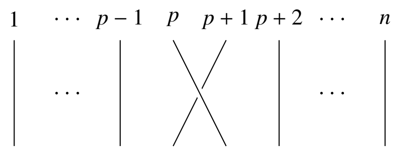
Definition A.1.2. For any $n \geq 1$, let $\mathbb{B}_n$ be the monoid with zero presented by generators $\beta_1, \beta_2, \ldots, \beta_{n-1}$ subject to the relations

$$\beta_q \beta_p = \beta_p \beta_q \qquad \text{for } p + 1 < q, \tag{A.2}$$

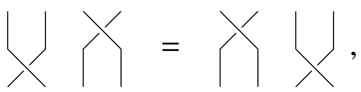
$$\beta_{p+1}\beta_p\beta_{p+1} = \beta_p\beta_{p+1}\beta_p, \quad \text{and} \tag{A.3}$$

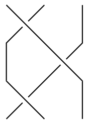
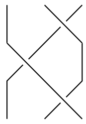
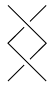
$$\beta_p\beta_p = 0. \tag{A.4}$$

It is called the *braid monoid with zero* since Eqs. (A.2) and (A.3) are precisely the relations in the standard presentation of the *braid group*. The elements of $\mathbb{B}_n$ can be thus visualised as certain braids on n strands where each generator β_p crosses the p-th and the $(p+1)$-th strands:



and the composition is given by vertically stacking the braids. Then omitting the irrelevant strands, Eqs. (A.2) to (A.4) look like

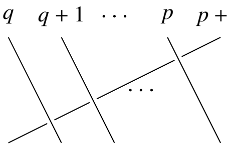


 =  and  = 0

respectively. For $p + 1 \geq q$, let

$$\beta_{p,q} \overset{\text{def}}{=} \beta_p \beta_{p-1} \dots \beta_q$$

so that it looks like:



(We interpret $\beta_{p,p+1}$ to be the identity.)

The following theorem describes a normal form for non-zero elements of $\mathbb{B}_n$.

Theorem A.1.3. *Any non-zero element $x \in \mathbb{B}_n$ can be written uniquely as a product of the form*

$$x = \beta_{1,q_1} \beta_{2,q_2} \dots \beta_{n-1,p_{n-1}}$$

where $p + 1 \geq q_p$ for each p. Conversely, $\beta_{1,q_1} \beta_{2,q_2} \dots \beta_{n-1,p_{n-1}} \neq 0$ for any $p + 1 \geq q_p$.

Remark. This normal form is reminiscent of the sorting algorithm called *insertion sort* in computer science. At the p-th stage, β_{p,q_p} takes the $(p+1)$-th strand at the top and inserts it to the correct position relative to the previously sorted strands.

Proof. We will summarise the proof in [ES, §6] and fill in the gaps therein. We consider the *rewrite system* on the alphabet $\{\beta_1, \dots, \beta_{n-1}, 0\}$ given by the following rewrite rules:

$$\begin{array}{rccll}
t_{p,q}: & \beta_q \beta_p & \rightsquigarrow & \beta_p \beta_q & \text{for } p + 1 < q \\
r_{p,q}: & \beta_{p,q} \beta_p & \rightsquigarrow & \beta_{p-1} \beta_{p,q} & \text{for } p > q \\
s_p: & \beta_p \beta_p & \rightsquigarrow & 0 & \\
y_p: & \beta_p 0 & \rightsquigarrow & 0 & \\
z_p: & 0 \beta_p & \rightsquigarrow & 0 & \\
0: & 00 & \rightsquigarrow & 0 &
\end{array}$$

That is, we consider the process of rewriting a given string in $\{\beta_1, \dots, \beta_{n-1}, 0\}$ by applying these rules to its substrings. Note that none of the rules affects the element of $\mathbb{B}_n$ that the string represents. (That the two sides of $r_{p,q}$ are equal in $\mathbb{B}_n$ is a consequence of Eqs. (A.2) and (A.3)). If a string u can be rewritten to another string v, we say v is a *rewriting* of u.

First we wish to show that this rewrite system is *bounded*, *i.e.* for any given (fixed) string, there is an upper bound on how many times the rewrite rules may be applied. This is done by assigning a natural number to each string in such a way that applying any of these rules decreases that number. Given a string $\beta_{p_1} \dots \beta_{p_m}$, where we interpret β_0 to mean 0, we assign the following natural number:

$$\rho(\beta_{p_1} \dots \beta_{p_m}) \overset{\text{def}}{=} m + \sum_{1 \leq i \leq m} p_i^2 + \left| \{(i,j) \mid i < j \text{ and } 0 < p_j < p_i\} \right|$$

The original formula in [ES] does not have the exponent 2 in the second term, but this exponent is necessary for the rewrite rule $r_{p,q}$ to decrease the value of ρ. (The rule $r_{p,q}$ decreases the second term of ρ by $p^2 - (p-1)^2 = 2p - 1$ and increases the third term by $p - q - 1$. Without the exponent 2, it only decreases the second term by 1.)

Next we need to show that this rewrite system is *locally confluent*, *i.e.* if a given string admits two (possibly overlapping) substrings to each of which some rewrite rule can be applied, then the two resulting strings have a common rewriting. It suffices to check certain special cases (see [ES, Proposition 5.2]), and most of these cases are checked in [ES, Proposition 6.2]. There are a few cases missing in their proof (more precisely, their analysis of the pair (r, t) assumes $j = p$), but these missing cases can be checked easily.

These properties of the rewrite system imply that each string admits a unique *normal form*, *i.e.* a rewriting that admits no further rewritings. It remains to check that a string is in normal form if and only if it is either 0 or of the form described in the theorem. This is done in [ES, Theorem 6.3]. $\square$

Recall that the symmetric group $\mathbb{S}_n$ on n letters $1, \dots, n$ may be presented by generators $\beta_1, \beta_2, \dots, \beta_{n-1}$ subject to Eqs. (A.2) and (A.3) and $\beta_p\beta_p = 1$. Hence we can define a function

$$\sigma_{(-)} : \mathbb{B}_n \setminus \{0\} \to \mathbb{S}_n$$

by assigning the transposition of p and $p+1$ to β_p and then extending this assignation according to $\sigma_{xy} = \sigma_x \circ \sigma_y$. Graphically, $\sigma_x(p) = q$ if the braid x takes the strand in the p-th position at the bottom to the q-th position at the top.

Corollary A.1.4. *The function $\sigma_{(-)}$ is injective.*

Proof. Observe that if

$$x = \beta_{1,q_1}\beta_{2,q_2}\dots\beta_{n-1,p_{n-1}}$$

then q_p is precisely the number of $1 \le r \le p+1$ such that $\sigma_x^{-1}(r) \le \sigma_x^{-1}(p+1)$. This shows that we can recover (the normal form of) x from σ_x. $\square$

Proof of Lemma 4.1.5 continued. It remains to prove that the 2-functor

$$F : \boxtimes_a(\theta_1, \dots, \theta_a) \to \mathscr{T}$$

is locally faithful. Since $\mathscr{T}$ is poset-enriched, this is equivalent to showing that $\boxtimes_a(\theta_1, \dots, \theta_a)$ is also poset-enriched.

Fix two objects $\boldsymbol{s}, \boldsymbol{t}$ and let $n = |S(\boldsymbol{s}, \boldsymbol{t})|$. In this proof, we identify each object $\preceq$ in the hom-category $\boxtimes_a(\theta_1, \dots, \theta_a)(\boldsymbol{s}, \boldsymbol{t})$ with the unique order-preserving bijection

$$f : \big(\{1, \dots, n\}, \le\big) \to \big(S(\boldsymbol{s}, \boldsymbol{t}), \preceq\big).$$

We define an action of the monoid (with zero) $\mathbb{B}_n$ on the set

$$\mathrm{ob}\big(\boxtimes_a(\theta_1, \dots, \theta_a)(\boldsymbol{s}, \boldsymbol{t})\big) \cup \{*\}$$

as follows. The zero element $0 \in \mathbb{B}_n$ sends everything to $*$, and $*$ is fixed by every element in $\mathbb{B}_n$. Given a bijection f as above and $1 \le p < n$, we define:

$$f \cdot \beta_p \overset{\text{def}}{=} \begin{cases} f \circ \sigma_{\beta_p} & \text{if } \pi_1 \circ f(p) > \pi_1 \circ f(p+1), \\ * & \text{otherwise} \end{cases}$$

where the projection $\pi_1 : S(\boldsymbol{s}, \boldsymbol{t}) \to \{1, \dots, a\}$ sends each $(i|k)$ to i.

Claim. This specification indeed extends to an action of $\mathbb{B}_n$. Moreover, for any non-zero element $x \in \mathbb{B}_n$ and any bijection f as above, either $f \cdot x = f \circ \sigma_x$ or $f \cdot x = *$.

Proof of the claim. Assuming the first part, the second part follows from the equation $\sigma_{xy} = \sigma_x \circ \sigma_y$. It suffices to check that, for each of Eqs. (A.2) to (A.4), (the action determined by) either side sends a given bijection f as above to $*$ if and only if the other side does.

For any bijection f as above and any $p + 1 < q$, the following are equivalent:

- $f \cdot \beta_q \neq *$ and $(f \circ \sigma_{\beta_q}) \cdot \beta_p \neq *$;
- $\pi_1 \circ f(p) > \pi_1 \circ f(p+1)$ and $\pi_1 \circ f(q) > \pi_1 \circ f(q+1)$; and
- $f \cdot \beta_p \neq *$ and $(f \circ \sigma_{\beta_p}) \cdot \beta_q \neq *$.

Thus the two sides of Eq. (A.2) determine the same action. A similar analysis can be done for Eq. (A.3), and the action of any β_p applied twice sends any f to $*$. This completes the proof. ◻

If $f(p) = (j|\ell)$, $f(p+1) = (i|k)$ and $j > i$ then there is a morphism $f \to f \circ \sigma_{\beta_p}$ in the hom-category $\boxtimes_a(\theta_1, \dots, \theta_a)(\boldsymbol{s}, \boldsymbol{t})$ which looks like

$$
\begin{tikzcd}
 & & & (k-1,\ell) \arrow[dr] \arrow[dd, Rightarrow] & & & \\
\boldsymbol{s} \arrow[r] & \cdots \arrow[r] & (k-1,\ell-1) \arrow[ur] \arrow[dr] & & (k,\ell) \arrow[r] & \cdots \arrow[r] & \boldsymbol{t} \\
 & & & (k,\ell-1) \arrow[ur] & & &
\end{tikzcd}
\tag{A.5}
$$

where we are suppressing all but the i-th and the j-th coordinates of the middle four objects. We abuse the notation and call this morphism β_p. Since the hom-category $\boxtimes_a(\theta_1, \dots, \theta_a)(\boldsymbol{s}, \boldsymbol{t})$ is generated by the morphisms of the form (A.5), it follows that any morphism $f \to g$ admits a factorisation of the form

$$
f \xrightarrow{\beta_{p_1}} f \circ \sigma_{\beta_{p_1}} \xrightarrow{\beta_{p2}} \dots \xrightarrow{\beta_{pr}} f \circ \sigma_{\beta_1 \dots \beta_{p_r}}. \tag{A.6}
$$

We wish to show that the word $\beta_{p_1} \dots \beta_{p_r}$ determines a non-zero element in $\mathbb{B}_n$. It follows from the proof of Theorem A.1.3 that this word can be reduced either to 0 or to a normal form specified in the theorem by successively applying Eqs. (A.1) to (A.4). We claim that this reduction process may be reproduced in $\boxtimes_a(\theta_1, \dots, \theta_a)(\boldsymbol{s}, \boldsymbol{t})$ with β_p's regarded as morphisms (and concatenation regarded as composition in reverse order). Indeed, Eq. (A.2) corresponds to the interchange law for a 2-category and Eq. (A.3) corresponds to the commutativity of the cube

$$
\begin{tikzcd}[column sep=tiny]
 & & (m,k-1,\ell-1) \arrow[drr] & & \\
(m-1,k-1,\ell-1) \arrow[urr] \arrow[drr] \arrow[dd] \arrow[rrrr, phantom, "\Nearrow"] & & & & (m,k,\ell-1) \arrow[dd] \\
 & & (m-1,k,\ell-1) \arrow[urr] \arrow[dd] & & \\
(m-1,k-1,\ell) \arrow[drr] \arrow[ur, phantom, "\Nearrow"] & & & & (m,k,\ell) \arrow[ul, phantom, "\Nearrow"] \\
 & & (m-1,k,\ell) \arrow[urr] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=tiny]
 & & (m,k-1,\ell-1) \arrow[drr] \arrow[dd] & & \\
(m-1,k-1,\ell-1) \arrow[urr] \arrow[dd] & & & & (m,k,\ell-1) \arrow[dd] \\
 & & (m,k-1,\ell) \arrow[drr] & & \\
(m-1,k-1,\ell) \arrow[urr] \arrow[drr] \arrow[ur, phantom, "\Nearrow"] \arrow[rrrr, phantom, "\Nearrow"] & & & & (m,k,\ell) \arrow[ul, phantom, "\Nearrow"] \\
 & & (m-1,k,\ell) \arrow[urr] & &
\end{tikzcd}
$$

for $(h|m), (i|k), (j|\ell) \in S(\boldsymbol{s}, \boldsymbol{t})$ with $h < i < j$, which follows from Eqs. (4.4) and (4.6). Moreover, Eq. (A.4) (and hence Eq. (A.1)) cannot appear in this process since there is no composable pair of the form $\cdot \xrightarrow{\beta_p} \cdot \xrightarrow{\beta_p} \cdot$ in $\boxtimes_a(\theta_1, \ldots, \theta_a)(\boldsymbol{s}, \boldsymbol{t})$.

Now fix $f, g \in \boxtimes_a(\boldsymbol{\theta})(\boldsymbol{s}, \boldsymbol{t})$. We have shown that any map $f \to g$ admits a factorisation of the form (A.6) such that

$$x = \beta_{p_1} \ldots \beta_{p_r}$$

is a normal form for some $0 \neq x \in \mathbb{B}_n$. Since we must have $\sigma_x = f^{-1} \circ g$, it follows that there is at most one morphism $f \to g$. This completes the proof. □

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