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Enriched Regular Theories

By

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Statement of Originality

This work has not previously been submitted for a degree or diploma at any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

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Abstract

Regular and exact categories were first introduced by Michael Barr in 1971; since then, the theory has developed and found many applications in algebra, geometry, and logic. In particular, a small regular category determines a certain theory, in the sense of logic, whose models are the regular functors into Set. In 1986 Barr showed that each small and regular category can be embedded in a particular category of presheaves; then in 1990 Makkai gave a simple explicit characterization of the essential image of the embedding, in the case where the original regular category is moreover exact. More recently Prest and Rajani, in the additive context, and Kuber and Rosicky, in the ordinary one, described a duality which connects an exact category with its (definable) category of models. Considering a suitable base for enrichment, we define an enriched notion of regularity and exactness, and prove a corresponding version of the theorems of Barr, of Makkai, and of Prest-Rajani/Kuber-Rosicky.

Contents

Introduction

When talking about *theories* we may think of two different approaches, a logical one and a categorical one. From the logical point of view, a theory is given by a list of axioms on a fixed set of operations, and its models are corresponding sets and functions that satisfy those axioms. For instance *algebraic theories* are those whose axioms consist of equations based on the operation symbols of the language (e.g. the axioms for abelian groups or rings). More generally, if the axioms are still equations but the operation symbols are not defined globally, but only on equationally defined subsets, we talk of *essentially algebraic theories.*

Example: *Graphs, seen as sets with a relation, are models of the essentially algebraic theory with two global operations* $s, t : \text{edge} \to \text{vertex}$ *(source and target), a partial operation* $\sigma : \text{edge} \times \text{edge} \to \text{edge}$ *such that* $\sigma(x, y)$ *is defined if and only if* $s(x) = s(y)$ *and* $t(x) = t(y)$*. The axioms of the theory are then:* $\sigma(x, y) = x$*,* $\sigma(x, y) = y$*.*

A further step can be made considering *regular theories*, for which we can allow existential quantification over the usual equations.

Example: *Von Neumann regular rings are models the regular theory with axioms those of rings plus the following:* $\forall x \; \exists y \; x = xyx$*.*

Categorically speaking, we could think of a theory as a category $\mathcal{C}$ with some structure, and of a model of $\mathcal{C}$ as a functor $F : \mathcal{C} \to \mathbf{Set}$ which preserves that structure, this approach was first introduced by Lawvere in [Law63]. Algebraic theories then correspond to categories with finite products, and models are finite product preserving functors. On the other hand a category with finite limits represents an essentially algebraic theory, and functors preserving finite limits are its models [Fre72]. Regular theories correspond instead to *regular categories*: finitely complete ones with coequalizers of kernel pairs, for which regular epimorphisms are pullback stable [MR77]. Models here are functors preserving finite limits and regular epimorphisms; we refer to them as regular functors.

These two notions, categorical and logical, can be recovered from each other: given a logical theory, there is a syntactic way to build a category with the relevant structure for which models of the theory correspond to functors to **Set** preserving this structure, and vice versa. For essentially algebraic theories this translates into a duality between locally finitely presentable and finitely complete categories:

Theorem (Gabriel-Ulmer, [GU71])**.** *The following is a biequivalence of 2-categories:*

$$\mathrm{Lfp}(-,\mathbf{Set}) : \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-,\mathbf{Set}),$$

where **Lfp** *is the 2-category of locally finitely presentable categories, finitary right adjoints, and natural transformations.*

Such a duality can be considered also in the context of regular theories; to describe it let us recall the most important results involving regular categories. First of all, Barr proved in [Bar86] that every small regular category can be regularly embedded in the functor category based on its models:

Theorem (Barr's Embedding)**.** *Let $\mathcal{C}$ be a small regular category; then the evaluation functor* $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C},\mathbf{Set}),\mathbf{Set}]$ *is fully faithful and regular.*

Later Makkai proved in [Mak90] that if the category $\mathcal{C}$ is moreover exact, then it can be recovered from its category of models $\mathrm{Reg}(\mathcal{C},\mathbf{Set})$ as follows:

Theorem (Makkai's Image)**.** *Let $\mathcal{C}$ be a small exact category. The essential image of the embedding* $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C},\mathbf{Set}),\mathbf{Set}]$ *is given by those functors which preserve filtered colimits and small products.*

Then on one side of the duality we can consider the 2-category **Ex** of exact categories, regular functors, and natural transformations. On the other we have the categories of models of regular theories; these can also be described as full subcategories of some locally finitely presentable category which are closed under small products, filtered colimits, and pure subobjects. Equivalently they are finite injectivity classes in some locally finitely presentable category; we refer to them as *definable categories.* A morphism between definable categories is then a functor that preserves filtered colimits and products; denote by **Def** the corresponding 2-category. The duality can hence be expressed as:

Theorem. *The following is a biequivalence of 2-categories:*

$$\mathrm{Def}(-,\mathbf{Set}) : \mathbf{Def} \rightleftarrows \mathbf{Ex}^{op} : \mathrm{Reg}(-,\mathbf{Set})$$

This was first proved in the additive context as Theorem 2.3 of [PR10]; while the ordinary version is Theorem 3.2.5 of [KR18] (though we should mention that the proof appearing in the latter contains a gap, as we explain at the end of Section 4.3).

Gabriel-Ulmer duality has been extended to the enriched context by Kelly in [Kel82b]. The aim of this thesis is to extend the other three theorems, finding a common path that includes both the ordinary and the additive context. Note that an enriched version of Barr's Embedding Theorem already appeared in [Chi11], but the notion of regularity appearing there is different from ours.

First we need to specify our assumptions on the base for enrichment we are going to work with. Start as usual from a symmetric monoidal closed complete and cocomplete $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$; since we want to talk about finite weighted limits and regularity, this should at least be locally finitely presentable as a closed category (in the sense of [Kel82b]) and regular. In fact we ask something more, our bases for enrichment will be (unsorted) *finitary varieties*: categories of the form $\mathrm{FP}(\mathcal{C}, \mathbf{Set})$, consisting of finite product preserving functors for some small category $\mathcal{C}$ with finite products. Equivalently a finitary variety can be described as an exact and cocomplete category with a strong generator made of finitely presentable and projective objects. In addition to this, we ask these finitely presentable and projective objects to respect the monoidal structure (in a sense made clear in Section 2.3). We call a finitary variety with such a structure a *symmetric monoidal finitary variety*, while if the exactness property is dropped we call it a *symmetric monoidal finitary quasivariety* (this, even if not exact, is still a regular category).

In this context we define an enriched version of regularity and exactness (Section 3), which are similar to the ordinary ones but with the additional request that regular epimorphisms should be stable under finite projective powers. This allows us to prove an enriched version of Barr's Embedding Theorem (3.2.4), saying that for each small and regular $\mathcal{V}$-category $\mathcal{C}$ the evaluation functor

$$\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathcal{V}), \mathcal{V}]$$

is a fully faithful regular embedding. If the underlying ordinary category on $\mathcal{C}$ is moreover exact, the essential image of $\mathrm{ev}_{\mathcal{C}}$ is given by those functors that preserve filtered colimits, products, and projective powers (Theorem 3.3.4), recovering a corresponding version of Makkai's

Image Theorem. To obtain these results it's enough to enrich over a symmetric monoidal finitary quasivariety.

An enriched notion of definable $\mathcal{V}$-category is also introduced (Chapter 4); a priori this lies somewhere between the ordinary one, and that of *exactly-definable* $\mathcal{V}$-category, namely categories of the form $\mathrm{Reg}(\mathcal{B},\mathcal{V})$ for an exact $\mathcal{V}$-category $\mathcal{B}$. Then, if our $\mathcal{V}$ is a symmetric monoidal finitary variety, we are able to recover the duality between the 2-category $\mathcal{V}$-**Ex** of small exact $\mathcal{V}$-categories, and $\mathcal{V}$-**Def** of definable $\mathcal{V}$-categories (Theorem 4.3.6), showing that each definable $\mathcal{V}$-category is actually exactly definable. In Section 4.4 we use this to give an explicit description of the free exact completions over finitely complete $\mathcal{V}$-categories and over regular $\mathcal{V}$-categories, while in Section 4.5 we treat the ordinary and additive cases.

Chapter 1

Background

1.1 Regular and Exact Categories

In this section we recall the definitions and the most important results about regular and exact categories. These notions were first introduced and developed by Michael Barr in [BGO71]; a more recent description can also be found in [Bor94].

Definition 1.1.1. A category $\mathcal{C}$ is called *regular* if it has all finite limits, coequalizers of kernel pairs, and regular epimorphisms are pullback stable. A functor $F : \mathcal{C} \to \mathcal{B}$ between regular categories is called regular if it preserves finite limits and regular epimorphisms; write $\mathrm{Reg}(\mathcal{C}, \mathcal{B})$ for the full subcategory of $[\mathcal{C}, \mathcal{B}]$ given by regular functors.

Examples 1.1.2. The following are examples of regular categories:

- the categories **Set** of sets and **Ab** of abelian groups;
- the category $[\mathcal{A}, \mathcal{C}]$ for any small $\mathcal{A}$ and regular $\mathcal{C}$;
- every Grothendieck topos;
- any abelian category;
- categories $\mathcal{C}^T$ of T-algebras for a monad $T : \mathcal{C} \to \mathcal{C}$ that preserves regular epimorphisms, over a regular category $\mathcal{C}$ (for $\mathcal{C} = \mathbf{Set}$ any monad preserves regular epimorphisms since they are all split).

The following is a list of the important properties involving regular epimorphisms in a regular category; this is going to be useful also when we move to the enriched context.

Proposition 1.1.3. *Let $\mathcal{C}$ be a regular category; then:*

1. *each morphism f in $\mathcal{C}$ can be factored as $f = m \circ e$, where e is a regular epimorphism and m a monomorphism; the factorization is unique up to unique isomorphism;*
2. *regular and strong epimorphisms coincide in $\mathcal{C}$;*
3. *if f and g are regular epimorphisms then $f \circ g$ is too;*
4. *if $f = g \circ h$ is a regular epimorphism, then g is too;*
5. *regular epimorphisms are stable under finite products.*

Barr proved, in Theorem 1.3 of [BGO71], that each small regular category can be regularly embedded in a category of presheaves $[\mathcal{A}, \mathbf{Set}]$ for a small $\mathcal{A}$; if we drop the smallness hypothesis on $\mathcal{A}$ we can replace it with $\mathrm{Reg}(\mathcal{C}, \mathbf{Set})$:

Theorem 1.1.4 (Barr's Embedding, Corollary 15 in [Bar86])**.** *Let $\mathcal{C}$ be a small regular category; then the evaluation functor* $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ *is fully faithful and regular.*

Before introducing the notion of exactness we need to recall that of equivalence relation for a morphism in a category:

Definition 1.1.5. An equivalence relation in a category $\mathcal{C}$ is a monomorphism $r = (r_1, r_2) : R \to A \times A$ such that $\mathcal{C}(X, r)$ is an equivalence relation in **Set** (namely, it is reflexive, symmetric, and transitive). Equivalently, r is an equivalence relation if there exist:

- a map $\delta : A \to R$ such that $r_i \circ \delta = id_A$, for $i = 1, 2$;
- a map $\sigma : R \to R$ such that $r_1 \circ \sigma = r_2$ and $r_2 \circ \sigma = r_1$;
- if $\rho_1, \rho_2 : R \times_A R \to R$ denote the pullback of r_1 and r_2 (with ρ_i opposite to r_1); a map $\tau : R \times_A R \to R$ such that $r_i \circ \tau = r_1 \circ \rho_i$, for $i = 1, 2$.

It's easy to see that every kernel pair is an equivalence relation (checking that it is true in **Set**, or using the universal property of the kernel pair to find the maps δ, σ, and τ). We may ask the opposite implication to hold:

Definition 1.1.6. A category $\mathcal{B}$ is called *exact* if it is regular and all equivalence relations in $\mathcal{C}$ are kernel pairs.

Remark 1.1.7. All the examples of 1.1.2 still hold if we replace regular with exact. Moreover in the additive context, an additive category is exact if and only if it is abelian.

For any regular category $\mathcal{C}$, the category $\mathrm{Reg}(\mathcal{C},\mathbf{Set})$ is closed in $[\mathcal{C},\mathbf{Set}]$ under filtered colimits and small products; hence, for each $C\in\mathcal{C}$ the evaluation functor $\mathrm{ev}(C):\mathrm{Reg}(\mathcal{C},\mathbf{Set})\to\mathbf{Set}$ preserves them. Makkai proved that, if $\mathcal{C}$ is moreover exact, then this is enough to describe the essential image of ev:

Theorem 1.1.8 (Makkai, 5.1 in [Mak90])**.** *Let $\mathcal{B}$ be a small exact category. The essential image of the embedding* $\mathrm{ev}:\mathcal{B}\to[\mathrm{Reg}(\mathcal{B},\mathbf{Set}),\mathbf{Set}]$ *is given by those functors which preserve filtered colimits and small products.*

1.2 Enriched Categories

In this section we recall the main features of enriched categories that we are going to use throughout the thesis; the main reference for this is Kelly's book [Kel82a].

Fix henceforth a complete and cocomplete symmetric monoidal closed category $\mathcal{V}=(\mathcal{V}_0,I,\otimes)$; where I is the unit and $\otimes:\mathcal{V}_0\times\mathcal{V}_0\to\mathcal{V}_0$ the tensor product. We denote by $[-,-]$ the internal hom $\mathcal{V}_0\times\mathcal{V}_0\to\mathcal{V}_0$ that makes $\mathcal{V}$ closed, so that $-\otimes Y$ is left adjoint to $[Y,-]$ for each $Y\in\mathcal{V}_0$.

Given a $\mathcal{V}$-category $\mathcal{C}$, which hence has hom-objects $\mathcal{C}(X,Y)$ in $\mathcal{V}_0$, we denote by $\mathcal{C}_0$ the underlying ordinary category of $\mathcal{C}$; this has the same objects as $\mathcal{C}$, but $\mathcal{C}_0(X,Y)=\mathcal{V}_0(I,\mathcal{C}(X,Y))$. Similarly, for any $\mathcal{V}$-functor $F:\mathcal{C}\to\mathcal{B}$ we denote by $F_0:\mathcal{C}_0\to\mathcal{B}_0$ the induced ordinary functor between $\mathcal{C}_0$ and $\mathcal{B}_0$. Note that we allow all our $\mathcal{V}$-categories to be large, unless specified otherwise.

For any two $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{B}$, we denote by $[\mathcal{C},\mathcal{B}]$ the enriched category of $\mathcal{V}$-functors from $\mathcal{C}$ to $\mathcal{B}$. If $\mathcal{C}$ is large this may not exist as a $\mathcal{V}$-enriched category; this problem can be avoided by considering $[\mathcal{C},\mathcal{B}]$ as a $\mathcal{V}'$-category for some extension $\mathcal{V}'$ of $\mathcal{V}$ (as explained in Section 2.6 of [Kel82a]); this allows us still to work with category of functors with a large domain.

Let's now consider the various notion of limits (and colimits) present in the enriched context, starting with powers and copowers:

Definition 1.2.1. Let $\mathcal{C}$ be a $\mathcal{V}$-category, C an object of $\mathcal{C}$, and X an object of $\mathcal{V}$. The *power* of C by X in $\mathcal{C}$, if it exists, is given by an object C^X of $\mathcal{C}$ together with a map $X\to\mathcal{C}(C^X,C)$ inducing a $\mathcal{V}$-natural isomorphism

$$\mathcal{C}(B,C^X)\cong[X,\mathcal{C}(B,C)]$$

in $\mathcal{V}_0$. Dual is the notion of *copower* of C by X, which is denoted by $X \cdot C$.

For an ordinary category $\mathcal{K}$, the power if an object A of $\mathcal{K}$ with a set S is just the product $\prod_{x\in S} A$ of S copies of A.

Given an ordinary locally small category $\mathcal{K}$, we can consider the free $\mathcal{V}$-category $\mathcal{K}_{\mathcal{V}}$ over $\mathcal{K}$; it has the same objects of $\mathcal{K}$ but hom-objects given by $\mathcal{K}_{\mathcal{V}}(A,B) := \mathcal{K}(A,B)\cdot I$ the coproduct of $\mathcal{K}(A,B)$ copies of I in $\mathcal{V}_0$. This will be useful for the next definition.

Definition 1.2.2. Let $\mathcal{C}$ be a $\mathcal{V}$-category, $T : \mathcal{K} \to \mathcal{C}_0$ an ordinary functor, and $T_{\mathcal{V}} : \mathcal{K}_{\mathcal{V}} \to \mathcal{C}$ the induced $\mathcal{V}$-functor. The *conical limit* of $T_{\mathcal{V}}$ in $\mathcal{C}$, if it exists, is given by an object $\mathrm{lim}T_{\mathcal{V}}$ of $\mathcal{C}$ together with a $\mathcal{V}$-natural transformation $\Delta(\mathrm{lim}T_{\mathcal{V}}) \to T_{\mathcal{V}}$ inducing a $\mathcal{V}$-natural isomorphism

$$\mathcal{C}(C, \mathrm{lim}T_{\mathcal{V}}) \cong [\mathcal{K},\mathcal{C}](\Delta(C), T_{\mathcal{V}})$$

in $\mathcal{V}_0$. Dual is the notion of *conical colimit.*

If $\mathrm{lim}T_{\mathcal{V}}$ exists in $\mathcal{C}$, then it coincides with the ordinary limit of $T : \mathcal{K} \to \mathcal{C}_0$ in $\mathcal{C}_0$. The converse doesn't hold in general, but does so if $\mathcal{C}$ has copowers with a strong generator of $\mathcal{V}_0$. Both the notion of power and of conical limit are a particular case of *weighted limits*. For our purposes we don't need to recall their definition (which can be found in Section 3.1 of [Kel82a]); it's important though to know the following fact: a $\mathcal{V}$-category has all weighted limits if and only if it has all conical limits and all powers.

Next we state the enriched version of the Yoneda Lemma for ordinary categories:

Lemma 1.2.3 (Yoneda). *Let $F : \mathcal{C} \to \mathcal{V}$ be a $\mathcal{V}$-functor and C an object of $\mathcal{C}$. Then the map*

$$[\mathcal{C},\mathcal{V}](\mathcal{C}(C,-), F) \longrightarrow FC$$

given by evaluating at 1_C, is invertible.

A direct consequence is:

Theorem 1.2.4 (Yoneda Embedding). *Let $\mathcal{C}$ be a $\mathcal{V}$-category; the $\mathcal{V}$-functor*

$$Y :\ \mathcal{C}^{op} \ \longrightarrow\ [\mathcal{C},\mathcal{V}]$$

sending an object C to $YC = \mathcal{C}(C,-)$, is fully faithful and continuous.

Finally, we recall the notion of the right Kan extension $\mathrm{Ran}_J F$ for a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{V}$ along an embedding $J : \mathcal{C} \to \mathcal{B}$. More generally one could replace $\mathcal{V}$, from the codomain of F, with any $\mathcal{V}$-category $\mathcal{A}$ and J with any $\mathcal{V}$-functor; in this case the existence of the right Kan extension is related to that of some weighted limits in $\mathcal{B}$. Since we are considering only functors to $\mathcal{V}$, which is complete, $\mathrm{Ran}_J F$ always exists and can be defined this way:

Definition 1.2.5. Let $F : \mathcal{C} \to \mathcal{V}$ be a $\mathcal{V}$-functor and $J : \mathcal{C} \to \mathcal{B}$ be fully faithful. The right Kan extension of F along J is the $\mathcal{V}$-functor $\mathrm{Ran}_J F : \mathcal{B} \to \mathcal{V}$ defined on objects by

$$\mathrm{Ran}_J F(B) = [\mathcal{C}, \mathcal{V}](\mathcal{B}(B, J-), F)$$

for each $B \in \mathcal{B}$.

Since J is assumed to be fully faithful, and thanks to Yoneda, the right Kan extension satisfies $\mathrm{Ran}_J F \circ J \cong F$.

1.3 Weak Reflections

Recall the following definitions for ordinary **Set**-enriched categories:

Definition 1.3.1. Given an arrow $h : A \to B$ in a category $\mathcal{L}$, an object $L \in \mathcal{L}$ is said to be *h-injective* if $\mathcal{L}(h, L) : \mathcal{L}(B, L) \to \mathcal{L}(A, L)$ is a surjection of sets. Given a small set $\mathcal{M}$ of arrows in $\mathcal{L}$ write $\mathcal{M}$-inj for the full subcategory of $\mathcal{L}$ consisting of those objects which are h-injective for each $h \in \mathcal{M}$. Categories arising in this way are called *small injectivity classes*.

Definition 1.3.2. Let $\mathcal{D}$ be a full subcategory of $\mathcal{L}$ and $L \in \mathcal{L}$; we say that $p : L \to S$ is a *weak reflection* of L into $\mathcal{D}$ if $S \in \mathcal{D}$ and each $K \in \mathcal{D}$ is p-injective. We say that $\mathcal{D}$ is weakly reflective in $\mathcal{L}$ if each object of $\mathcal{L}$ has a weak reflection into $\mathcal{D}$.

The first result of this section is a well-known one which relates injectivity classes and weakly reflective subcategories:

Theorem 1.3.3. *Each injectivity class $\mathcal{M}$-inj in a locally presentable category is weakly reflective. Moreover, the weak reflections can be taken in the closure of $\mathcal{M}$ under transfinite composition (along filtered colimits) and pushouts.*

The proof we propose is inspired by that in Section III.6 of [AR93] with some changes to make the last part of the statement true. A similar approach can be also found in [Bar86].

Proof. Let $\mathcal{D} = \mathcal{M}$-inj be a small injectivity class in a locally presentable category $\mathcal{L}$. Given $L \in \mathcal{L}$ we want to build a weak reflection into $\mathcal{D}$; we do this by defining a chain of objects $(L_\alpha)_{\alpha \in \mathrm{ORD}}$ indexed on ordinals, with connecting maps $x_{\beta,\alpha} : L_\beta \to L_\alpha$ for $\beta < \alpha$, and proving that this reaches at some point an object of $\mathcal{D}$ with the desired property. Set $L_0 := L$ and $L_\lambda := \mathrm{colim}_{\beta<\lambda} L_\beta$ for each limit λ; for $\alpha = \beta + 1$ we define L_α as the colimit of another chain built as follows. Order the set of diagrams of the form:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & B \\ L_\beta & \end{tikzcd}$$

for any $h \in \mathcal{M}$ and $f : A \to L_\beta$, and call the ordinal ordering this set κ. Then we can define $X_0 := L_\beta$ and $X_\lambda := \mathrm{colim}_{\delta<\lambda} X_\delta$ for each limit $\lambda < \kappa$. Given X_δ, $\delta < \kappa$, define $X_{\delta+1}$ as the pushout

$$\begin{tikzcd} A_\delta \arrow[r, "h_\delta"] \arrow[d, "f_\delta"'] & B_\delta \arrow[dd, dashed] \\ L_\beta \arrow[d] & \\ X_\delta \arrow[r, dashed] & X_{\delta+1} \end{tikzcd}$$

where h_δ and f_δ correspond to δ in the given order. At the end we obtain a chain $(X_\delta)_{\delta<\kappa}$ and we can define $L_{\beta+1}$ as its colimit.

Now let γ be a regular cardinal such that each arrow in $\mathcal{M}$ has γ-presentable domain and codomain; we are going to prove that L_γ is a weak reflection for L (the existence of γ is guaranteed by the fact that $\mathcal{L}$ is locally presentable, Remark after 1.20 in [AR94]).

(1) $L_\gamma \in \mathcal{D} = \mathcal{M}$-inj. For each $h : A \to B$ in $\mathcal{M}$ and $f : A \to L_\gamma$, since A is γ-presentable and $L_\gamma := \mathrm{colim}_{\beta<\gamma} L_\beta$, f factors through some $x_{\beta,\gamma}$ as $f = x_{\beta,\gamma} \circ \bar{f}$. Consider then the diagram:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\bar{f}"'] & B \arrow[d] \arrow[ddr, bend left, ":= f_h"] & \\ L_\beta \arrow[r, "x_{\beta,\beta+1}"] \arrow[drr, bend right, "x_{\beta,\gamma}"'] & L_{\beta+1} \arrow[dr, "x_{\beta+1,\gamma}"'] & \\ & & L_\gamma \end{tikzcd}$$

where the square exists by definition of $L_{\beta+1}$. Then $f = f_h \circ h$ and, as a consequence, L_γ is h-injective for each $h \in \mathcal{M}$.

(2) $p := x_{0,\gamma} : L \to L_\gamma$ is a weak reflection of L in $\mathcal{D}$. Let $K \in \mathcal{D}$ be $\mathcal{M}$-injective; we need to show that it is also p-injective. Given any $g : L \to K$, travelling through the steps of the construction of L_γ, we can prove inductively that g factors compatibly through each $x_{0,\beta}$ and hence through p. Indeed, let's first prove that if it's true for β then it is also true for $\beta + 1$; by definition $L_{\beta+1} = \mathrm{colim}_\delta(X_\delta)$, and assume by induction that g factors through $L \xrightarrow{x_{0,\beta}} L_\beta \to X_\delta$ with a map $g_\delta : X_\delta \to K$, then we can consider the following diagram

$$\begin{tikzcd}
A_\delta \arrow[r, "h_\delta"] \arrow[d, "f_\delta"'] & B_\delta \arrow[dd] \arrow[dddr, bend left, "l"] & \\
L_\beta \arrow[d] & & \\
X_\delta \arrow[r] \arrow[drr, bend right, "g_\delta"'] & X_{\delta+1} \arrow[dr, dashed, "\exists\, g_{\delta+1}"'] & \\
& & K
\end{tikzcd}$$

where l exists since K is $\mathcal{M}$-injective. This gives $g_{\delta+1} : X_{\delta+1} \to K$ and iterating provides a factorization of g through $x_{0,\beta+1}$. For limit ordinals the factorization exists since each step is done compatibly with the colimits $L_\lambda = \mathrm{colim}_{\beta<\lambda} L_\beta$. $\square$

Now we move to the enriched context and consider a corresponding notion of weak reflection. For this, let us fix a symmetric monoidal closed complete and cocomplete category $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ as our base.

Definition 1.3.4. Let $\mathcal{L}$ be a $\mathcal{V}$-category and $\mathcal{D}$ a full subcategory of $\mathcal{L}$. Given $F \in \mathcal{L}$, a *weak reflection* of F into $\mathcal{D}$ is a morphism $p : F \to S$ such that $S \in \mathcal{D}$ and

$$\mathcal{L}(p, T) : \mathcal{L}(S, T) \to \mathcal{L}(F, T)$$

is a regular epimorphism in $\mathcal{V}$ for each $T \in \mathcal{D}$. We say that $\mathcal{D} \subseteq \mathcal{L}$ is weakly reflective if each F in $\mathcal{L}$ has a weak reflection into $\mathcal{D}$.

Since in $\mathcal{V}$ there are several kinds of epimorphisms; one could ask $\mathcal{L}(p, T)$ to be just an epimorphism, or a strong one (or even something else) instead of a regular epimorphism. For instance, to prove the Proposition below it would be enough to consider just epimorphisms. We prefer however, to keep the definition as it is since all the weakly reflective categories we'll consider in the following sections arise in that way.

The proof of the following Proposition is inspired by that of Theorem 9 in [Chi11].

Proposition 1.3.5. *Let $J : \mathcal{D} \hookrightarrow \mathcal{L}$ be the inclusion of a full weakly reflective subcategory for which the weak reflections can be chosen to be regular monomorphisms. Then $\mathcal{D}$ is codense in $\mathcal{L}$; meaning that the functor*

$$\begin{array}{rcl} \widehat{J} := \mathcal{L}(1, J) : \mathcal{L}^{op} & \longrightarrow & [\mathcal{D}, \mathcal{V}] \\ F & \dashrightarrow & \mathcal{L}(F, J-) \end{array}$$

is full and faithful.

Proof. Let us fix F and G in $\mathcal{L}$, we need to prove that $\widehat{J}_{FG} : \mathcal{L}(G, F) \to [\mathcal{D}, \mathcal{V}](\widehat{J}F, \widehat{J}G)$ is an isomorphism. For each T in $\mathcal{D}$ denote by

$$\pi_T : [\mathcal{D}, \mathcal{V}](\widehat{J}F, \widehat{J}G) = \int_S [\mathcal{L}(F, S), \mathcal{L}(G, S)] \longrightarrow [\mathcal{L}(F, T), \mathcal{L}(G, T)]$$

the projection; then $\widehat{J}_{FG}$ is defined by

$$\pi_T \circ \widehat{J}_{FG} := \mathcal{L}(-, T) : \mathcal{L}(G, F) \to [\mathcal{L}(F, T), \mathcal{L}(G, T)].$$

Consider a weak reflection $p : F \rightarrowtail S$ of F in $\mathcal{D}$, which by hypothesis we can assume to be a regular monomorphism in $\mathcal{L}$. Let $u, v : S \to S'$ be such that p is their equalizer; replacing u and v with their composite with a weak reflection of S' we may suppose $S' \in \mathcal{D}$.
Now, for each $f : F \to T$ with $T \in \mathcal{D}$, define the morphism:

$$\overline{f} : [\mathcal{D}, \mathcal{V}](\widehat{J}F, \widehat{J}G) \xrightarrow{\pi_T} [\mathcal{L}(F, T), \mathcal{L}(G, T)] \xrightarrow{ev_f} \mathcal{L}(G, T);$$

it's easy to see that $\mathcal{L}(G, u) \circ \overline{p} = \overline{(u \circ p)} = \overline{(v \circ p)} = \mathcal{L}(G, v) \circ \overline{p}$. But $\mathcal{L}(G, p)$ is the equalizer of $\mathcal{L}(G, u)$ and $\mathcal{L}(G, v)$, then there exists $m : [\mathcal{D}, \mathcal{V}](\widehat{J}F, \widehat{J}G) \to \mathcal{L}(G, F)$ such that

$$\begin{tikzcd} {[\mathcal{D}, \mathcal{V}](\widehat{J}F, \widehat{J}G)} \arrow[rr, "\overline{p}"] \arrow[dr, "m"'] & & \mathcal{L}(G, S) \\ & \mathcal{L}(G, F) \arrow[ur, "\mathcal{L}(G, p)"'] & \end{tikzcd}$$

commutes. We want to prove that m is a section for $\widehat{J}_{FG}$; to do this it's enough to show that the following triangle

$$\begin{tikzcd} {[\mathcal{D}, \mathcal{V}](\widehat{J}F, \widehat{J}G)} \arrow[rr, "\pi_T"] \arrow[dr, "m"'] & & {[\mathcal{L}(F, T), \mathcal{L}(G, T)]} \\ & \mathcal{L}(G, F) \arrow[ur, "\mathcal{L}(-, T)"'] & \end{tikzcd}$$

commutes for each $T \in \mathcal{D}$. Fix then T in $\mathcal{D}$; since p is a weak reflection, $\mathcal{L}(p,T)$ is an epimorphism and hence $[\mathcal{L}(p,T),1] : [\mathcal{L}(F,T),\mathcal{L}(G,T)] \to [\mathcal{L}(S,T),\mathcal{L}(G,T)]$ is a monomorphism. Thus, it is enough to prove that $[\mathcal{L}(p,T),1] \circ \pi_T = [\mathcal{L}(p,T),1] \circ \mathcal{L}(-,T) \circ m$. Consider hence the diagram

$$\begin{tikzcd}
{[\mathcal{D},\mathcal{V}](\widehat{J}F,\widehat{J}G)} \arrow[rr, "m"] \arrow[dr, "\overline{p}"] \arrow[dd, "\pi_T"'] & & \mathcal{L}(G,F) \arrow[dl, "\mathcal{L}(G,p)"'] \arrow[dd, "\mathcal{L}(-,T)"] \\
& \mathcal{L}(G,S) \arrow[dd, "\mathcal{L}(-,T)"'] & \\
{[\mathcal{L}(F,T),\mathcal{L}(G,T)]} \arrow[dr, "{[\mathcal{L}(p,T),1]}"'] & & {[\mathcal{L}(F,T),\mathcal{L}(G,T)]} \arrow[dl, "{[\mathcal{L}(p,T),1]}"] \\
& {[\mathcal{L}(S,T),\mathcal{L}(G,T)]} &
\end{tikzcd}$$

The upper triangle commutes by the definition of m, the square on the right by naturality, that on the left since $\overline{p} = ev_p \circ \pi_S$ and thanks to the commutativity of

$$\begin{tikzcd}
& {[\mathcal{D},\mathcal{V}](\widehat{J}F,\widehat{J}G)} \arrow[dl, "\pi_T"'] \arrow[dr, "\pi_S"] & \\
{[\mathcal{L}(F,T),\mathcal{L}(G,T)]} \arrow[dr, "\sigma_{ST}"] \arrow[dddr, bend right, "{[\mathcal{L}(p,T),1]}"'] & & {[\mathcal{L}(F,S),\mathcal{L}(G,S)]} \arrow[dl, "\rho_{ST}"'] \arrow[dd, "ev_p"] \\
& {[\mathcal{L}(S,T),[\mathcal{L}(F,S),\mathcal{L}(G,T)]]} \arrow[dd, "{[1,ev_p]}"'] & \\
& & \mathcal{L}(G,S) \arrow[dl, "\mathcal{L}(-,T)"] \\
& {[\mathcal{L}(S,T),\mathcal{L}(G,T)]} &
\end{tikzcd}$$

where σ_{ST} and ρ_{ST} are those appearing in the definition of $[\mathcal{D},\mathcal{V}](\widehat{J}F,\widehat{J}G)$ as an end. Consequently m is a right inverse for $\widehat{J}_{FG}$, or equivalently $\widehat{J}_{FG}$ is a spit epimorphism. To conclude observe that

$$\overline{p} \circ \widehat{J}_{FG} = ev_p \circ \pi_S \circ \widehat{J}_{FG} = ev_p \circ \mathcal{L}(-,S) = \mathcal{L}(G,p)$$

and $\mathcal{L}(G,p)$ is a monomorphism since p is. Thus $\widehat{J}_{FG}$ is both a monomorphism and a split epimorphism, and hence is an isomorphism.

□

Chapter 2

Bases for Enrichment

2.1 Locally Finitely Presentable Categories

In this Section we recall the most important properties of locally finitely presentable categories in the ordinary and in the enriched context. The main reference for the first part is [AR94], while for the second [Kel82b].

Definition 2.1.1. Let $\mathcal{L}$ be a category, an object A of $\mathcal{L}$ is called *finitely presentable* if the hom-functor $\mathcal{L}(A, -) : \mathcal{L} \to \mathbf{Set}$ preserves filtered colimits. Denote by $\mathcal{L}_f$ the full subcategory of finitely presentable objects.

The following proposition will be useful later

Proposition 2.1.2 (1.7 in [AR94]). *A category $\mathcal{L}$ has filtered colimits if and only if it has colimits of smooth chains (diagrams $D : \alpha \to \mathcal{L}$, for an ordinal α, such that $D(\lambda) \cong \mathrm{colim} D|_\lambda$ for any limit $\lambda < \alpha$). Any functor between such categories preserves filtered colimits if and only if it preserves colimits of smooth chains.*

Definition 2.1.3. We say that a category $\mathcal{L}$ is *locally finitely presentable* if it is cocomplete and has a small strong generator $\mathcal{G} \subseteq \mathcal{L}_f$.

Let us now state some properties of locally finitely presentable categories:

Proposition 2.1.4. *Let $\mathcal{L}$ be a locally finitely presentable category with strong generator $\mathcal{G} \subseteq \mathcal{L}_f$. Then:*

1. *$\mathcal{L}_f$ is the closure of $\mathcal{G}$ under finite colimits, in particular $\mathcal{L}_f$ is a small and finitely cocomplete category;*

2. *each object of $\mathcal{L}$ can be written as a filtered colimit of objects from $\mathcal{L}_f$;*
3. *filtered colimits commute in $\mathcal{L}$ with finite limits;*
4. *$\mathcal{L}$ is complete.*

A morphism $F : \mathcal{L} \to \mathcal{L}'$ between locally finitely presentable categories is a right adjoint functor that preserves filtered colimits; denote then by **Lfp** the 2-category of locally finitely presentable categories, morphisms between them, and natural transformations. Similarly define **Lex** to be the 2-category of finitely complete categories, finite limit preserving functors, and natural transformations.

Theorem 2.1.5 (Gabriel-Ulmer, [GU71])**.** *The following is a biequivalence of 2-categories:*

$$(-)_f^{op} : \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathbf{Set})$$

The same concepts were introduced in the enriched context by Kelly in [Kel82b]. As always we should consider a symmetric monoidal closed complete and cocomplete category $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$; but we ask in addition $\mathcal{V}_0$ to be locally finitely presentable and that finitely presentable objects respect the monoidal structure in the following sense:

Definition 2.1.6. We say that $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$ is a *locally finitely presentable as a closed category* if:

1. $\mathcal{V}_0$ is locally finitely presentable with strong generator $\mathcal{G} \subseteq (\mathcal{V}_0)_f$;
2. $I \in (\mathcal{V}_0)_f$;
3. if $A, B \in \mathcal{G}$ then $A \otimes B \in (\mathcal{V}_0)_f$.

In this context we can talk about locally finitely presentable $\mathcal{V}$-categories generalising the previously stated results.

Definition 2.1.7. Let $\mathcal{L}$ be a $\mathcal{V}$-category, an object A of $\mathcal{L}$ is called *finitely presentable* if the hom-functor $\mathcal{L}(A, -) : \mathcal{L} \to \mathcal{V}$ preserves conical filtered colimits. Denote by $\mathcal{L}_f$ the full subcategory of finitely presentable objects. We call $\mathcal{L}$ locally finitely presentable as a $\mathcal{V}$-category if it is $\mathcal{V}$-cocomplete and has a small strong generator $\mathcal{G} \subseteq \mathcal{L}_f$.

Note in particular that $\mathcal{V}$ is itself locally finitely presentable as a $\mathcal{V}$-category; moreover by the closure property (2) and (3) in Definition 2.1.6, being finitely presentable in $\mathcal{V}$ or in $\mathcal{V}_0$ is the same: $\mathcal{V}_{0f} = \mathcal{V}_{f0}$.

Definition 2.1.8. We say that a $\mathcal{V}$-category $\mathcal{C}$ has *finite weighted limits*, or that is *finitely complete*, if it has all finite conical limits and finite powers, where with the latter we mean powers with finitely presentable objects of $\mathcal{V}_0$. Denote by $\mathcal{V}$-**Lex** the 2-category of finitely complete $\mathcal{V}$-categories, finite limit preserving $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations.

Finite limits can then be thought as combinations of finite conical limits and finitely presentable powers; a more direct description for general weighted limits can be found in Section 4 of [Kel82b].

Then all the properties of Proposition 2.1.4 still hold if we replace the ordinary notions with the new enriched ones; as a consequence, if $\mathcal{L}$ is locally finitely presentable as a $\mathcal{V}$-category, then $\mathcal{L}_0$ is an ordinary locally finitely presentable category and $\mathcal{L}_{0f} = \mathcal{L}_{f0}$.

Moreover, denoting by $\mathcal{V}$-**Lfp** the 2-category of locally finitely presentable $\mathcal{V}$-categories, right adjoint $\mathcal{V}$-functors that preserve filtered colimits, and $\mathcal{V}$-natural transformations, we obtain once again:

Theorem 2.1.9 (Kelly, [Kel82b])**.** *The following is a biequivalence of 2-categories:*

$$(-)_f^{op} : \mathcal{V}\text{-}\mathbf{Lfp} \rightleftarrows \mathcal{V}\text{-}\mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathcal{V})$$

2.2 Locally Projective Categories

In this section we study the main properties of categories with a strong generator consisting of projective objects:

Definition 2.2.1. Let $\mathcal{K}$ be a category; an object P of $\mathcal{K}$ is called *projective* if the hom-functor $\mathcal{K}(P,-) : \mathcal{K} \to \mathbf{Set}$ preserves all regular epimorphisms existing in $\mathcal{K}$; in other words, if $\mathcal{K}(P,-)$ sends regular epimorphisms to surjections. Denote by $\mathcal{K}_p$ the full subcategory of $\mathcal{K}$ given by the projective objects.

Note that this is not the usual definition of projective object; indeed the normal request would be for $\mathcal{K}(P,-)$ to preserve epimorphisms not regular epimorphisms.

Definition 2.2.2. We say that a category $\mathcal{K}$ is *locally projective* if it has finite limits and coequalizers of kernel pairs, and there exists a (small) strong generator $\mathcal{P} \subseteq \mathcal{K}_p$.

Lemma 2.2.3. *Let $\mathcal{B}$ be a regular category, $\mathcal{A}$ have finite limits and coequalizers of kernel pairs, and $F : \mathcal{A} \to \mathcal{B}$ a functor which preserves finite limits and regular epimorphisms. Then the following are equivalent:*

1. *F is conservative;*
2. *F reflects regular epimorphisms.*

Furthermore $\mathcal{A}$ is then regular.

Proof. (1) $\Rightarrow$ (2). In a category with finite limits and coequalizers of kernel pairs, regular epimorphisms coincide precisely with coequalizers of kernel pairs, and these are preserved by regular functors. Now, since F is conservative, it reflects all limits and colimits that it preserves; in particular it then reflects coequalizers of kernel pairs, namely regular epimorphisms.

(2) $\Rightarrow$ (1). Let f be a morphism in $\mathcal{A}$ such that $F(f)$ is an isomorphism. Then (2) implies that f is a regular epimorphism; thus we only need to prove that it also is monomorphism. Let (u, v) be the kernel pair of f and δ the diagonal:

$$\cdot \xrightarrow{\ \delta\ } \cdot \underset{v}{\overset{u}{\rightrightarrows}} \cdot \xrightarrow{\ f\ } \cdot$$

Since $F(f)$ is an isomorphism and F preserves finite limits, $F(\delta)$ is an isomorphism, and in particular a regular epimorphism; thus (2) implies that δ is a regular epimorphism, which means exactly that the kernel pair is trivial and f is a monomorphism.

Finally, if they hold, to prove that $\mathcal{A}$ is regular it's enough to show that regular epimorphisms in $\mathcal{A}$ are stable under pullback. Consider a regular epimorphism e in $\mathcal{A}$ and the pullback $\bar{e}$ of e along any other morphism in $\mathcal{A}$. The functor F preserves and regular epimorphisms; then $F(\bar{e})$ is such ($\mathcal{B}$ is regular) and hence (2) implies that $\bar{e}$ is a regular epimorphism in $\mathcal{A}$. □

The following result gives a simple way to recognise regular epimorphisms in a locally projective category:

Proposition 2.2.4. *Let $\mathcal{K}$ have finite limits and coequalizers of kernel pairs. The following are equivalent:*

1. *$\mathcal{K}$ is locally projective;*
2. *there exists a small $\mathcal{P} \subseteq \mathcal{K}_p$ such that for every morphism f in $\mathcal{K}$, if $\mathcal{K}(P, f)$ is surjective for each $P \in \mathcal{P}$ then f is a regular epimorphism in $\mathcal{K}$.*

Furthermore, if they hold, $\mathcal{K}$ is a regular category.

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{P} \subseteq \mathcal{K}_p$ be a strong generator for $\mathcal{K}$, we are going to prove that it has the required property. Consider the functor

$$N := \hat{\imath} : \mathcal{K} \longrightarrow [\mathcal{P}^{op}, \mathbf{Set}]$$

such that $N(K) = \mathcal{K}(i-, K)$, where $i : \mathcal{P} \to \mathcal{K}$ is the inclusion (we are seeing $\mathcal{P}$ as a full subcategory of $\mathcal{K}$). Since $\mathcal{P}$ is a strong generator, N is continuous and conservative; thus by the previous Lemma N reflects regular epimorphisms, which is a rephrasing of the property in (2).

(2) $\Rightarrow$ (1). Let $\mathcal{P} \subseteq \mathcal{K}_p$ as in (2); we prove that it strongly generates $\mathcal{K}$. For this, it is enough to prove that $N : \mathcal{K} \to [\mathcal{P}^{op}, Set]$ defined as before, is conservative. This follows again from the previous Lemma since N preserves finite limits and reflects regular epimorphisms by construction. $\square$

Proposition 2.2.5. *Let $\mathcal{K}$ be locally projective; then regular epimorphisms are stable under all small products that exist in $\mathcal{K}$.*

Proof. Let $(e_i)_{i\in I}$ be a set of regular epimorphisms in $\mathcal{K}$ such that $\prod_i e_i$ exists; then for each $P \in \mathcal{P}$, $\mathcal{K}(P, \prod_i e_i) \cong \prod_i \mathcal{K}(P, e_i)$ is a surjection (since they are product stable in **Set**). As a consequence $\prod_i e_i$ is a regular epimorphism in $\mathcal{K}$. $\square$

Proposition 2.2.6. *Let $\mathcal{K}$ be locally projective with strong generator $\mathcal{P}$ and small coproducts; then:*

1. *$\mathcal{K}_p$ is closed under small coproducts;*
2. *$\mathcal{K}$ has enough projectives;*
3. *Q is in $\mathcal{K}_p$ if and only if it is a split subobject of a coproduct from $\mathcal{P}$.*

Proof. (1) Consider a coproduct $\coprod_i P_i$ of projective objects, then $\mathcal{K}(\coprod_i P_i, -) \cong \prod_i \mathcal{K}(P_i, -)$ is surjective because surjections are product stable in **Set**. It follows that $\coprod_i P_i$ is projective.
(2). Let K be an object of $\mathcal{K}$, since $\mathcal{P}$ is strongly generating, there exists a regular epimorphism $P := \coprod_i P_i \twoheadrightarrow K$, with $P_i \in \mathcal{P}$. But $\mathcal{K}_p$ is closed under coproducts, then $P \in \mathcal{K}_p$.
(3). Let $Q \in \mathcal{K}_p$, then as before there is a regular epimorphism $\coprod_i P_i \twoheadrightarrow Q$ with $P_i \in \mathcal{P}$ for each i. Since Q is projective this regular epimorphism splits as desired. On the other hand, consider a coproduct $P := \coprod_i P_i$ of elements from $\mathcal{P}$ ($P \in \mathcal{K}_p$ by the first point), and a split subobject $i : Q \rightarrowtail P$. Let $p : P \to Q$ be such that $p \circ i = id_Q$; then given a regular epimorphism $e : A \to B$ in $\mathcal{K}$ and $f : Q \to B$, since P is projective there is $g' : P \to A$ such that $e \circ g' = f \circ p$. Define then $g := g' \circ i$; it is easy to see that $e \circ g = f$ and hence that $Q \in \mathcal{K}_p$. $\square$

2.3 Finitary Varieties and Quasivarieties

We now merge the two notions of locally finitely presentable and locally projective categories into the following:

Definition 2.3.1. A category $\mathcal{K}$ is called a *finitary quasivariety* if it is cocomplete and has a strong generator formed by finitely presentable and projective objects. If moreover $\mathcal{K}$ is an exact category, it is called a *finitary variety.*
Denote by $\mathcal{K}_{pf}$ the full subcategory of finitely presentable projective objects of $\mathcal{K}$.

By Theorem 3.24 in [AR94] (or actually, the correction appearing in [AR13]), this corresponds to the usual definition of finitary quasivariety and variety. In fact, finitary varieties can be described as the categories of models of multi-sorted Algebraic Theories (whose axioms are systems of linear equations); while finitary quasivarieties are the categories of models of theories whose axioms are implications of linear equations.

Finitary varieties can also be described as some particular categories of functors:

Theorem 2.3.2 (3.16 in [AR94]). *A category $\mathcal{K}$ is a finitary variety if and only if it is equivalent to* $\mathrm{FP}(\mathcal{C}, \mathbf{Set})$ *for some small category $\mathcal{C}$ with finite products. In particular we could take $\mathcal{C}$ to be $(\mathcal{K}_{pf})^{op}$.*

Examples 2.3.3.

- **Set** and **Ab** are finitary varieties; we may take $\mathcal{P}$ equal to $\{1\}$ and $\{\mathbf{Z}\}$ respectively.
- The category **CRng** of commutative rings (with unit) and ring homomorphisms is a finitary variety: note first that a morphism in **CRng** is a regular epimorphism if and only if it is surjective; then if we consider $\mathbf{Z}[x]$, it's easy to see (since morphism $\mathbf{Z}[x] \to R$ are in bijection with elements of R) that it is finitely presentable, projective, and a strong generator. (It's interesting to note that $\mathbf{Z}[x]$ is not projective in the usual algebraic sense.)
- The category **Gra** of graphs seen as sets with a relation, is a finitary quasivariety (but not a finitary variety), with strong generator given by the finite graphs.
- For any small $\mathcal{A}$, the functor category $[\mathcal{A}, \mathbf{Set}]$ is a finitary variety with strong generator $\mathcal{P}$ given by the set af all representable objects.
- More generally, if (T, μ, η) is a monad on a finitary variety (resp. quasivariety) $\mathcal{K}$ and T preserves filtered colimits, then the Eilenberg-Moore category $\mathcal{K}^T$ is a finitary variety

(resp. quasivariety). The strong generator of $\mathcal{K}^T$ is given by the set of free algebras over finitely presentable and projective objects of $\mathcal{K}$.

- All the examples from 3.20 of [AR94].

Note that if $\mathcal{K}$ is a finitary quasivariety then it is both locally finitely presentable and locally projective, and in particular a regular category.

Proposition 2.3.4. *Let $\mathcal{V}$ be a symmetric monoidal closed complete and cocomplete category, and let $\mathcal{K}$ be any cocomplete $\mathcal{V}$-category for which $\mathcal{K}_0$ is a finitary variety (resp. quasivariety). Then for any small $\mathcal{V}$-category $\mathcal{A}$, the category $[\mathcal{A},\mathcal{K}]_0$ of $\mathcal{V}$-functors from $\mathcal{A}$ to $\mathcal{K}$ is a finitary variety (resp. quasivariety).*

Proof. Let $\mathcal{P} \subseteq \mathcal{K}_p$ be a strong generator for $\mathcal{K}$ made of finitely presentable projective objects. Define $\mathcal{P}'$ in $[\mathcal{A},\mathcal{K}]$ as the collection of those functors of the form $\mathcal{A}(d,-)\cdot P$ for each $d \in \mathcal{A}$ and $P \in \mathcal{P}$. These are projective since for any regular epimorphism α in $[\mathcal{A},\mathcal{K}]$ the following isomorphisms hold

$$[\mathcal{A},\mathcal{K}](\mathcal{A}(d,-)\cdot P,\alpha) \cong [\mathcal{A},\mathcal{V}](\mathcal{A}(d,-),\mathcal{K}(P,\alpha-)) \cong \mathcal{K}(P,\alpha_d)$$

and the last is a regular epimorphism since α_d is one and P is projective. The same chain of isomorphisms shows that the elements of $\mathcal{P}'$ are finitely presentable (each $p \in \mathcal{P}$ is finitely presentable and evaluation at d preserves all limits and colimits).

It remains to prove that $\mathcal{P}'$ is a strong generator. Given F in $[\mathcal{A},\mathcal{K}]$, it's enough to prove that for any $d \in \mathcal{A}$ there are $P \in \mathcal{P}$ and $\eta : \mathcal{A}(d,-)\cdot P \to F$ such that η_d is a regular epimorphism; because then we can just take the coproduct of those maps over $d \in \mathcal{A}$. Since $\mathcal{K}$ is locally projective, given d there are $P \in \mathcal{P}$ and a regular epimorphism $f : P \twoheadrightarrow Fd$, define then η as the natural transformation whose transpose $\bar{\eta} : \mathcal{A}(d,-) \to \mathcal{K}(P,F-)$ corresponds, through Yoneda, to f. Consider then the following diagram

$$\begin{tikzcd}
I\cdot P \arrow[d, "id\cdot P"'] \arrow[dr, two heads, "f"] & \\
\mathcal{A}(d,d)\cdot P \arrow[r, "\eta_d"'] & Fd
\end{tikzcd}$$

since $\mathcal{K}_0$ is regular (by Proposition 2.2.5) and f a regular epimorphism, η_d is regular too. □

Example 2.3.5. It follows that, for each commutative ring R, the categories R-Mod, GR-R-Mod of R-modules and graded R-modules, are finitary varieties. Moreover if $\mathcal{A}$ is abelian and a finitary variety (resp. quasivariety) then so is the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes on $\mathcal{A}$.

Definition 2.3.6. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a symmetric monoidal closed category. We say that $\mathcal{V}$ is a *symmetric monoidal finitary quasivariety* if:

1. $\mathcal{V}_0$ is a finitary quasivariety with strong generator $\mathcal{P} \subseteq (\mathcal{V}_0)_{pf}$;
2. $I \in (\mathcal{V}_0)_{pf}$;
3. if $P, Q \in \mathcal{P}$ then $P \otimes Q \in (\mathcal{V}_0)_{pf}$.

We call it a *symmetric monoidal finitary variety* if $\mathcal{V}_0$ is also a finitary variety.

Examples 2.3.7. The following are examples of symmetric monoidal finitary quasivarieties:

- **Set** and **Ab** with the cartesian and group tensor product respectively;
- R-Mod and GR-R-Mod, for each commutative ring R, with the usual algebraic tensor product;
- $[\mathcal{C}^{op}, \mathbf{Set}]$, for any category $\mathcal{C}$ with finite products, equipped with the cartesian product;
- non-positively graded chain complexes $\mathrm{Ch}(\mathcal{A})^-$ for each abelian and symmetric monoidal finitary quasivariety $\mathcal{A}$, with the tensor product inherited from $\mathcal{A}$;
- the category $\mathbf{Ab}_f$ of torsion free abelian groups with the usual tensor product;
- **Gra** with the cartesian product, as well as the category **DGra** of directed graphs;
- the full subcategory **Mono** of all monomorphisms in $\mathbf{Set}^2$.

The first three are always symmetric monoidal finitary varieties, the same holds for the fourth if $\mathcal{A}$ is such. Non examples are: **Cat** with any tensor product (since it is not a quasivariety); the categories **RGra** of reflexive graphs, and **sSet** of simplicial sets with the cartesian product (since the product of two projective objects may not be projective).

Remark 2.3.8. Let $\mathcal{V}$ be a symmetric monoidal finitary quasivariety; then point (3) of the previous definition implies, by Proposition 5.2 in [Kel82b], that $\mathcal{V}_{0f}$ is closed under tensor product. The same holds for $\mathcal{V}_{0p}$: given two projective objects $P, Q \in \mathcal{V}_{0p}$, there are split monomorphisms $P \rightarrowtail \coprod_i P_i$ and $Q \rightarrowtail \coprod_j Q_j$, with $P_i, Q_j \in \mathcal{P}$. Then $P \otimes Q$ is a split subobject of

$$\coprod_i P_i \otimes \coprod_i Q_i \cong \coprod_{i,j} (P_i \otimes Q_j),$$

which is projective; hence $P \otimes Q$ is projective. It follows then that $(\mathcal{V}_0)_{pf}$ is also closed under tensor product.

The following Proposition gives a characterization of the monoidal structures on a finitary variety that make it a symmetric monoidal finitary variety.

Proposition 2.3.9. *Let $\mathcal{C}$ be a category with finite products; there is an equivalence between*

- *symmetric monoidal structures on $\mathcal{C}$ for which $-\otimes- : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ preserves finite products in each variable;*
- *symmetric monoidal structures on $\mathrm{FP}(\mathcal{C},\mathbf{Set})$ which make it a symmetric monoidal finitary variety.*

Moreover, the induced structures make the Yoneda embedding $Y:\mathcal{C}^{op}\to\mathrm{FP}(\mathcal{C},\mathbf{Set})$ a strong monoidal functor.

Proof. On one side, since $\mathrm{FP}(\mathcal{C},\mathbf{Set})_{pf}\simeq\mathcal{C}^{op}$, the remark above implies that every symmetric monoidal structure on $\mathrm{FP}(\mathcal{C},\mathbf{Set})$ which makes it a symmetric monoidal finitary variety, restricts to a symmetric monoidal structure on $\mathcal{C}$. The functor $-\otimes-:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ preserves finite products since $Y:\mathcal{C}^{op}\to\mathrm{FP}(\mathcal{C},\mathbf{Set})$ preserves finite coproducts and these commute with the tensor product.

On the other side, let $(\mathcal{C},\otimes,I)$ be a symmetric monoidal structure on $\mathcal{C}$ as in the first point. It is proven in [Day70] that it induces a symmetric monoidal closed structure on $[\mathcal{C},\mathbf{Set}]$ for which the Yoneda embedding $\mathcal{C}^{op}\to[\mathcal{C},\mathbf{Set}]$ is strong monoidal and for every $F,G:\mathcal{C}\to\mathbf{Set}$ and $c\in\mathcal{C}$

$$(F\otimes G)(c)\cong\int^{c_1,c_2\in\mathcal{C}}\mathcal{C}(c_1\otimes c_2,c)\times F(c_1)\times F(c_2)$$

can be expressed as a coend. Now, if F and G preserve finite products, by Corollary 2.8 of [AR11], we can write them as sifted colimits of representables: $F\cong\mathrm{colim}_iY(c_i)$ and $G\cong\mathrm{colim}_jY(d_j)$. Since sifted colimits commute with products and coends in $\mathbf{Set}$, it follows that

$$F\otimes G\cong\mathrm{colim}_{i,j}Y(c_i)\otimes Y(d_j)\cong\mathrm{colim}_{i,j}Y(c_i\otimes d_j),$$

making $F\otimes G$ a sifted colimits of representables and hence a finite product preserving functor. As a consequence the tensor product on $[\mathcal{C},\mathbf{Set}]$ restricts to $\mathrm{FP}(\mathcal{C},\mathbf{Set})$, and satisfies conditions (2) and (3) of Definition 2.3.6 (with $\mathcal{P}=Y\mathcal{C}^{op}$); we are only left to prove that the symmetric monoidal structure induced on $\mathrm{FP}(\mathcal{C},\mathbf{Set})$ is closed. For this it's enough to show that if F and G preserve finite products, then the internal hom $[G,F]$ (seen in $[\mathcal{C},\mathbf{Set}]$) preserves them too. Write $G\cong\mathrm{colim}_jY(d_j)$ as before; then $[G,F]\cong[\mathrm{colim}_jY(d_j),F]\cong\lim_j[Y(d_j),F]$

and it suffices to show that $[Y(c), F]$ preserves finite products for every $c \in \mathcal{C}$. Fix $d \in \mathcal{C}$, then

$$\begin{aligned}
[Y(c), F](d) &\cong [\mathcal{C}, \mathbf{Set}](Y(d), [Y(c), F]) \\
&\cong [\mathcal{C}, \mathbf{Set}](Y(d) \otimes Y(c), F) \\
&\cong [\mathcal{C}, \mathbf{Set}](Y(d \otimes c), F) \\
&\cong F(d \otimes c);
\end{aligned}$$

in other words $[Y(c), F] \cong F(- \otimes c)$, and this preserves finite products since F does and $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite products in each variable by assumption. $\square$

Remark 2.3.10. Note that if $\mathcal{V}$ is a symmetric monoidal finitary quasivariety it is in particular locally finitely presentable as a closed category, and hence $\mathcal{V}_{0f} = \mathcal{V}_{f0}$.
We can show similar results for the full subcategory of projectives. Denote with $\mathcal{V}_p$ the full subcategory of $\mathcal{V}$ given by the $\mathcal{V}$-projective objects: those $P \in \mathcal{V}$ such that $[P, -] : \mathcal{V} \to \mathcal{V}$ preserves regular epimorphisms. Then the equality $\mathcal{V}_{0p} = \mathcal{V}_{p0}$ holds too; indeed given $P \in \mathcal{V}_{0p}$ and a regular epimorphism e, the function of sets $\mathcal{V}_0(Q, [P, e]) \cong \mathcal{V}_0(Q \otimes P, e)$ is a surjection for each $Q \in \mathcal{P}$ (since $Q \otimes P$ is projective); hence $[P, e]$ is a regular epimorphism (by Proposition 2.2.4) and $P \in \mathcal{V}_{p0}$ (this means exactly that regular epimorphisms are stable in $\mathcal{V}$ under projective powers). Conversely, given $P \in \mathcal{V}_{p0}$, since $I \in \mathcal{V}_{0p}$, the functor $\mathcal{V}_0(P, -) \cong \mathcal{V}_0(I, [P, -])$ preserves regular epimorphisms; thence $P \in \mathcal{V}_{0p}$. As a consequence we can omit the subscript 0 and write $\mathcal{V}_f$ for the full subcategory of finitely presentable objects, $\mathcal{V}_p$ for the full subcategory of projectives, and $\mathcal{V}_{pf}$ for the full subcategory of both projective and finitely presentable ones.

Chapter 3

Regular and Exact $\mathcal{V}$-categories

3.1 Regular $\mathcal{V}$-Categories

From now on we assume that our base category $\mathcal{V}$ is a symmetric monoidal finitary quasivariety with strong generator $\mathcal{P} \subseteq \mathcal{V}_{pf}$.

The following is the notion of regular category we are going to introduce in this context:

Definition 3.1.1. A $\mathcal{V}$-category $\mathcal{C}$ is said to be *regular* if it has all finite weighted limits (equivalently finite conical limits and finite powers), coequalizers of kernel pairs, and is such that regular epimorphisms are stable under pullback and powers with elements of $\mathcal{P}$.
A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ between regular $\mathcal{V}$-categories is called *regular* if it preserves finite weighted limits and regular epimorphisms; we denote by $\mathrm{Reg}(\mathcal{C}, \mathcal{D})$ the $\mathcal{V}$-category of regular functors from $\mathcal{C}$ to $\mathcal{D}$.

Remark 3.1.2. A different notion of regularity appeared before in [Chi11]; there, in a regular $\mathcal{V}$-category, regular epimorphisms need to be stable under all finite powers, instead of just finite-projective ones like in our case. At the same time the base for enrichment can be assumed to be only locally finitely presentable as a closed category, and one can still prove the analogue of 3.2.4. We chose to consider a different approach to recover the usual notions of regularity and exactness for $\mathcal{V} = \mathbf{Ab}$; in fact $\mathbf{Ab}$ itself is not regular as an additive category in the sense of [Chi11], but it is regular in our sense (since $\mathcal{P}$ can be chosen to be just $\mathbb{Z}$).

It follows from the definition that a $\mathcal{V}$-category $\mathcal{C}$ is regular if and only if it has all finite weighted limits, $\mathcal{C}_0$ is an ordinary regular category, and regular epimorphisms are stable under powers with elements of $\mathcal{P}$. Indeed, this is easily checked to be necessary; on the other hand it is sufficient because, considering that $\mathcal{C}$ has $\mathcal{P}$-powers and $\mathcal{P}$ is a strong generator, coequalizers of kernel pairs in $\mathcal{C}$ and $\mathcal{C}_0$ are the same.

Remark 3.1.3. Our notion of regular $\mathcal{V}$-category is a particular case of what are called Φ-exact $\mathcal{V}$-categories in [GL12], Φ being a class of weights. More precisely, there is a suitable choice of Φ for which a $\mathcal{V}$-category is regular if and only if it is Φ-exact; this follows by our embedding Theorem 3.2.4 and Theorem 4.1 of [GL12]. As a consequence, by Corollary 3.7 of the same paper, it follows that every finitely complete $\mathcal{V}$-category $\mathcal{C}$ has a free regular completion $\mathcal{C}_{\text{reg/lex}}$; meaning that there is a lex functor $F : \mathcal{C} \to \mathcal{C}_{\text{reg/lex}}$ which induces an equivalence $\text{Reg}(\mathcal{C}_{\text{reg/lex}}, \mathcal{B}) \simeq \text{Lex}(\mathcal{C}, \mathcal{B})$ for each regular $\mathcal{V}$-category $\mathcal{B}$. See also Remark 3.3.2 and Section 4.4.

Proposition 3.1.4. *Let $\mathcal{C}$ be a regular $\mathcal{V}$-category; then regular epimorphisms are stable under powers with each element of $\mathcal{V}_{pf}$.*

Proof. Let $h : A \to B$ be a regular epimorphism in $\mathcal{C}$ and $P \in \mathcal{V}_{pf}$. By Proposition 2.2.6, P is a split subobject of a coproduct $Q := \coprod_i P_i$ with $P_i \in \mathcal{P}$; write $m : P \to Q$ for the split monomorphism. Since P is also finitely presentable, we can assume the coproduct to be finite; as a consequence Q is finitely presentable and h^Q exists in $\mathcal{C}$. Moreover $h^Q \cong \prod_i (h^{P_i})$ is a regular epimorphism since the h^{P_i} are, and regular epimorphisms are stable under finite products in each ordinary regular category. Consider then the square

$$\begin{tikzcd} A^Q \arrow[r, two heads, "h^Q"] \arrow[d, two heads, "A^m"'] & B^Q \arrow[d, two heads, "B^m"] \\ A^P \arrow[r, "h^P"'] & B^P \end{tikzcd}$$

where A^q and B^q are split epimorphisms, and hence regular. As a consequence, since $\mathcal{C}_0$ is regular, it follows that h^P is a regular epimorphism as desired. □

Remark 3.1.5. $\mathcal{V}$ itself is regular as a $\mathcal{V}$-category since it is both complete and cocomplete, $\mathcal{V}_0$ is regular in the ordinary sense by Proposition 2.2.5, and regular epimorphisms are stable under all projective powers (by Remark 2.3.10).

3.2 Barr's Embedding Theorem

Let us fix a small regular $\mathcal{V}$-category $\mathcal{C}$ and consider $\mathcal{R} := \text{Reg}(\mathcal{C}, \mathcal{V})$ as a full subcategory of $\mathcal{L} := \text{Lex}(\mathcal{C}, \mathcal{V})$.

Lemma 3.2.1. *$\mathcal{L}$ is a coregular $\mathcal{V}$-category.*

Proof. It is shown in Theorem 3 of [Chi11] that regular monomorphisms are stable in $\mathcal{L}$ under pushouts (note that the notion of regular category appearing in the cited paper is different from ours, but the same proof applies to this setting). Hence we only need to show that if H is a regular monomorphism in $\mathcal{L}$ then so is $P \cdot H$ for each $P \in \mathcal{P}$. It is proven in the same Theorem of [Chi11] that each regular monomorphism of $\mathcal{L}$ is a filtered colimit of regular monomorphisms between representables; since $P \cdot -$ preserves colimits it is then enough to consider $H = \mathcal{C}(h, -)$ for a regular epimorphism h in $\mathcal{C}$. Now, the restricted Yoneda embedding $\mathcal{C} \to \mathcal{L}^{op}$ preserves finite limits; hence $P \cdot \mathcal{C}(h, -) \cong \mathcal{C}(h^P, -)$ for each $P \in \mathcal{P}$. But $\mathcal{C}$ is regular, therefore h^P is a regular epimorphism and as a consequence, $P \cdot \mathcal{C}(h, -)$ is a regular monomorphism as claimed. □

The next definition and the following result are useful to understand how $\mathcal{R}$ sits inside $\mathcal{L}$.

Definition 3.2.2. Let $\mathcal{L}$ be a $\mathcal{V}$-category and $\mathcal{M}$ a set of arrows in $\mathcal{L}$; denote by $\mathcal{M}$-inj the full subcategory of $\mathcal{L}$ whose objects are those $L \in \mathcal{L}$ for which $\mathcal{L}(h, L)$ is a regular epimorphism for each $h \in \mathcal{M}$. Subcategories of this form are called *enriched injectivity classes*, or just *injectivity classes* if no confusion will arise. If $\mathcal{L}$ is locally finitely presentable and the arrows in $\mathcal{M}$ have finitely presentable domain and codomain, we call $\mathcal{M}$-inj an *enriched finite injectivity class*.

A more detailed treatment on finite injectivity classes will be given in Section 4.1. We will see, in that section, how they relate to definable categories and categories like $\mathcal{R}$, namely of the form $\mathrm{Reg}(\mathcal{C}, \mathcal{V})$ for a regular $\mathcal{V}$-category $\mathcal{C}$.

Lemma 3.2.3. *$\mathcal{R}$ is an enriched finite injectivity class and a weakly reflective subcategory of $\mathcal{L}$.*

Proof. A lex functor $F \in \mathcal{L}$ is in $\mathcal{R}$ if and only if Fh is a regular epimorphism for each regular epimorphism h in $\mathcal{C}$, as a consequence $\mathcal{R} = \mathcal{M}$-inj where

$$\mathcal{M} := \{\mathcal{C}(h, -) \mid h \text{ regular epimorphism in } \mathcal{C}\},$$

since by Yoneda $\mathrm{Lex}(\mathcal{C}, \mathcal{V})(\mathcal{C}(h, -), F) \cong Fh$. Moreover, $\mathcal{L}(\mathcal{C}(h, -), S)$ is a regular epimorphism in $\mathcal{V}$ if and only if the function $\mathcal{V}_0(P, \mathcal{L}_0(\mathcal{C}(h, -), S)) \cong \mathcal{L}_0(P \cdot \mathcal{C}(h, -), S)$ is surjective for each $P \in \mathcal{P}$ (Proposition 2.2.4), it follows that $\mathcal{D}_0 = \mathcal{M}_0$-inj, with

$$\mathcal{M}_0 := \{P \cdot \mathcal{C}(h, -) \mid h \text{ regular epimorphism in } \mathcal{C}, P \in \mathcal{P}\},$$

is an ordinary finite injectivity class in $\mathcal{L}_0$. By Theorem 1.3.3, $\mathcal{R}_0$ is then an ordinary weakly reflective subcategory of $\mathcal{L}_0$. But given any ordinary weak reflection $s : L \to S$ and $T \in \mathcal{R}$, the function $\mathcal{V}_0(P, \mathcal{L}(s, T)) \cong \mathcal{L}_0(s, T^P)$ is surjective for each $P \in \mathcal{P}$, since $T^P \in \mathcal{R}$ ($\mathcal{R}$ is closed under projective powers in $\mathcal{L}$ by Proposition 4.1.4). It follows then that $\mathcal{L}(s, T)$ is a regular epimorphism and s is an enriched weak reflection. □

This allows us to prove an enriched version of Barr's Embedding Theorem.

Theorem 3.2.4 (Barr's Embedding)**.** *Let $\mathcal{C}$ be a small regular $\mathcal{V}$-category; then the evaluation functor* $\mathrm{ev}_{\mathcal{C}} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathcal{V}), \mathcal{V}]$ *is fully faithful.*

Proof. Let $\mathcal{R} = \mathrm{Reg}(\mathcal{C}, \mathcal{V})$ and $\mathcal{L} = \mathrm{Lex}(\mathcal{C}, \mathcal{V})$ as before. It follows by the previous Lemma that $\mathcal{R}$ is a weakly reflective subcategory of $\mathcal{L}$ and that an ordinary weak reflection is also an enriched one. Remember also, from Theorem 1.3.3, that the ordinary weak reflections can be obtained as filtered colimits and pushouts of the arrows that define $\mathcal{R}_0$ as a finite injectivity class.
Our aim is to apply Proposition 1.3.5, hence we need to prove that our weak reflections can be chosen to be regular monomorphisms. For this let us first note that, since $\mathcal{L}$ is coregular by Lemma 3.2.1, the elements $P \cdot \mathcal{C}(h, -)$ in the class defining $\mathcal{R}_0$ are regular monomorphisms. Now, the fact that $\mathcal{L}$ is locally finitely presentable (since $\mathcal{C}$ is finitely complete) and coregular implies that filtered colimits commute in $\mathcal{L}$ with finite limits, and regular monomorphisms are stable under pushouts. Thus, by Theorem 1.3.3, our weak reflections can actually be chosen to be regular monomorphisms.
In conclusion, note that the functor $\mathrm{ev}_{\mathcal{C}}$ is given by the composite of the restricted Yoneda embedding $Y : \mathcal{C} \to \mathcal{L}^{op}$ and of $\widehat{J} : \mathcal{L}^{op} \to [\mathcal{R}, \mathcal{V}] = [\mathrm{Reg}(\mathcal{C}, \mathcal{V}), \mathcal{V}]$ where $\widehat{J} = \mathcal{L}(1, J)$ (J being the inclusion of $\mathcal{R}$ in $\mathcal{L}$); the first is always fully faithful and the second is so because $\mathcal{R}$ is codense in $\mathcal{L}$. Hence $\mathrm{ev}_{\mathcal{C}}$ is fully faithful. □

Similarly, for each small regular category $\mathcal{C}$, we can find a regular embedding of $\mathcal{C}$ into a category of presheaves over a small base:

Theorem 3.2.5. *Let $\mathcal{C}$ be a small regular $\mathcal{V}$-category. Then there exists a small $\mathcal{V}$-category $\mathcal{A}$ and a fully faithful and regular functor $F : \mathcal{C} \to [\mathcal{A}, \mathcal{V}]$.*

Proof. Let $Y : \mathcal{C} \to \mathcal{L}^{op} = \mathrm{Lex}(\mathcal{C}, \mathcal{V})^{op}$ be the codomain restriction of the Yoneda embedding. For each representable functor $L = \mathcal{C}(C, -)$ in $\mathcal{L}$ we can consider an equalizer

$$L \overset{s}{\rightarrowtail} S_L \underset{v}{\overset{u}{\rightrightarrows}} T_L$$

where S_L and T_L are in $\mathrm{Reg}(\mathcal{C}.\mathcal{V})$, and s is a weak reflection of L into $\mathrm{Reg}(\mathcal{C}.\mathcal{V})$. Consider then the full subcategory $\mathcal{B}$ of $\mathcal{L}$ given by the representable functors and, for each of them, two regular functors S_L and T_L with the property just described. Let $\mathcal{A} \subset \mathcal{B}$ be the full subcategory consisting of all regular functors in $\mathcal{B}$. Then $\mathcal{A}$ is weakly reflective in $\mathcal{B}$ and the weak reflection can be chosen to be regular monomorphisms (if $B \in \mathcal{B}$ is representable, then this is true by construction; if B is one of the new objects, then $B \in \mathcal{A}$ and the identity map is a weak reflection). By construction $\mathcal{A}$ and $\mathcal{B}$ are small categories, and, thanks to Proposition 1.3.5, $\mathcal{A}$ is codense in $\mathcal{B}$. Write $Y' : \mathcal{C} \to \mathcal{B}^{op}$ for the codomain restriction of Y; then we can consider the functor $F : \mathcal{C} \to [\mathcal{A}, \mathcal{V}]$ defined as the composite

$$\mathcal{C} \xrightarrow{Y'} \mathcal{B}^{op} \xrightarrow{\mathcal{B}(1,J)} [\mathcal{A}, \mathcal{V}]$$

where $J : \mathcal{A} \to \mathcal{B}$ is the inclusion. F turns out to be just the evaluation functor restricted to $\mathcal{A}$; thence, since $\mathcal{A} \subseteq \mathrm{Reg}(\mathcal{C}, \mathcal{V})$, the functor F is regular too. Finally F is fully faithful because Y' is, and $\mathcal{A}$ is codense in $\mathcal{B}$. □

3.3 Makkai's Image Theorem

A notion strictly related to that of regularity is exactness:

Definition 3.3.1. A $\mathcal{V}$-category $\mathcal{B}$ is called exact if it is regular and in addition the ordinary category $\mathcal{B}_0$ is exact in the usual sense.

Taking $\mathcal{V} = \mathbf{Set}$ or $\mathcal{V} = \mathbf{Ab}$ this notion coincides with the ordinary one of exact or abelian category. Note moreover that our base $\mathcal{V}$ may not be exact (but only regular).

Remark 3.3.2. If $\mathcal{V}$ is a symmetric monoidal finitary variety, then $\mathcal{V}_0$ is an ordinary exact category and $\mathcal{V}$ is exact as a $\mathcal{V}$-category. Arguing as in Remark 3.1.3, it's easy to see that our notion of exactness coincides with that of Φ'-exactness for a suitable Φ' (different from that defining regularity). It follows then by Theorem 7.7 of [GL12] that each regular $\mathcal{V}$-category has an exact completion $\mathcal{C}_{\mathrm{ex/reg}}$. Similarly each finitely complete $\mathcal{V}$-category $\mathcal{C}$ has an exact completion $\mathcal{C}_{\mathrm{ex/lex}}$. These will be described explicitly in Section 4.4.

Given any exact category $\mathcal{B}$, which is in particular a regular category, we can consider the fully faithful functor

$$\mathrm{ev}_{\mathcal{B}} : \mathcal{B} \to [\mathrm{Reg}(\mathcal{B}, \mathcal{V}), \mathcal{V}]$$

given by Theorem 3.2.4. Moreover $\mathrm{Reg}(\mathcal{B}, \mathcal{V})$ is closed in $[\mathcal{B}, \mathcal{V}]$ under products, projective powers and filtered colimits: this can be checked directly, or will follows from Proposition 4.1.4 plus the fact that $\mathrm{Lex}(\mathcal{B}, \mathcal{V})$ is closed in $[\mathcal{B}, \mathcal{V}]$ under the same (co)limits. It's then easy to see that the essential image of $\mathrm{ev}_{\mathcal{B}}$ is contained in the full subcategory $\mathrm{Def}(\mathrm{Reg}(\mathcal{B}, \mathcal{V}), \mathcal{V})$ of $[\mathrm{Reg}(\mathcal{B}, \mathcal{V}), \mathcal{V}]$ given by those functor preserving products, projective powers and filtered colimits (the notation is that of Section 4.2). We are going to see that it actually is the essential image of $\mathrm{ev}_{\mathcal{B}}$; to do this we use the following result for ordinary categories:

Lemma 3.3.3 (1.4.9 in [MR77]). *Suppose that $F : \mathcal{B} \to \mathcal{C}$ is a conservative, full and regular functor between ordinary regular categories, and $\mathcal{B}$ is exact. If for every object $C \in \mathcal{C}$ there are an object $B \in \mathcal{B}$ and a regular epimorphism $F(B) \twoheadrightarrow C$, then F is an equivalence of categories.*

Then we are ready to prove the following Theorem; the ordinary version appeared originally as Theorem 5.1 of [Mak90]. Another proof of the same result in the ordinary context can be found in Theorem 2.4.2 of [Lur18].

Theorem 3.3.4 (Makkai's Image). *For any exact $\mathcal{V}$-category $\mathcal{B}$; the evaluation map*

$$\mathrm{ev}_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathrm{Def}(\mathrm{Reg}(\mathcal{B}, \mathcal{V}), \mathcal{V})$$

is an equivalence.

Note that the proof relies on two results proven in Section 4.2 below, but of course we will not use it in obtaining these.

Proof. Since $\mathrm{ev}_{\mathcal{B}}$ is fully faithful by Theorem 3.2.4, we only need to prove that it is essentially surjective on objects, or equivalently, that the ordinary functor $(\mathrm{ev}_{\mathcal{B}})_0$ is an equivalence. Thanks to Remark 4.2.2 and Proposition 4.2.7, applied to $\mathrm{Reg}(\mathcal{B}, \mathcal{V})$ as a definable subcategory of $\mathrm{Lex}(\mathcal{B}, \mathcal{V})$, for each $F \in \mathrm{Def}(\mathrm{Reg}(\mathcal{B}, \mathcal{V}), \mathcal{V})$ there are an object $C \in \mathcal{B}$ and a regular epimorphism $\mathrm{ev}_{\mathcal{B}}(C) \twoheadrightarrow F$; hence we can apply the previous Lemma and conclude. $\square$

Chapter 4

Definable $\mathcal{V}$-categories

4.1 Enriched Finite Injectivity Classes

We consider again categories enriched over a base $\mathcal{V}$ which is a symmetric monoidal finitary quasivariety with strong generator $\mathcal{P} \subseteq \mathcal{V}_{pf}$.

The following is the corresponding enriched version of Definition 1.3.1:

Definition 4.1.1. Given an arrow $h : A \to B$ in a $\mathcal{V}$-category $\mathcal{L}$, an object $L \in \mathcal{L}$ is said to be *h-injective* if $\mathcal{L}(h, L) : \mathcal{L}(B, L) \to \mathcal{L}(A, L)$ is a regular epimorphism in $\mathcal{V}$. Given a small set $\mathcal{M}$ of arrows from $\mathcal{L}$ write $\mathcal{M}$-inj for the full subcategory of $\mathcal{L}$ consisting of those objects which are h-injective for each $h \in \mathcal{M}$. $\mathcal{V}$-categories arising in this way are called *enriched injectivity classes*, or just *injectivity classes* if no confusion will arise. If $\mathcal{L}$ is locally finitely presentable and the arrows in $\mathcal{M}$ have finitely presentable domain and codomain, we call $\mathcal{M}$-inj an *enriched finite injectivity class.*

Remark 4.1.2. Injectivity classes in the enriched context were first considered in [LR12]. In that setting a more general notion is introduced: regular epimorphisms are replaced by a suitable class $\mathcal{E}$ of morphisms from $\mathcal{V}$. This way an object L is called $\mathcal{E}$-injective if $\mathcal{L}(h, L) \in \mathcal{E}$, and an $\mathcal{E}$-injectivity class is a full subcategory of $\mathcal{E}$-injective objects with respect to a small set of morphisms.

It's easy to see that, since $\mathcal{V}$ is locally projective, for each enriched injectivity class $\mathcal{D}$ the underlying category $\mathcal{D}_0$ is an ordinary injectivity class: indeed, if $\mathcal{D} = \mathcal{M}$-inj in $\mathcal{L}$, then $\mathcal{D}_0 = \mathcal{M}_0$-inj in $\mathcal{L}_0$ where

$$\mathcal{M}_0 = \{P \cdot h \mid P \in \mathcal{P},\ h \in \mathcal{M}\},$$

this because $\mathcal{L}(h, S)$ is a regular epimorphism in $\mathcal{V}$ if and only if $\mathcal{L}_0(P \cdot h, S) \cong \mathcal{V}_0(P, \mathcal{L}(h, S))$ is surjective for each $P \in \mathcal{P}$. In particular, the underlying ordinary category of each injectivity class $\mathcal{D}$ of a locally finitely presentable $\mathcal{L}$ is accessible and accessibly embedded in $\mathcal{L}_0$ (Theorem 4.8 in [AR94]).

In the ordinary case each finite injectivity class is known to be closed under pure subobjects inside its locally finitely presentable category; let us recall the definition.

Definition 4.1.3. Let $f : X \to Y$ be a morphism in a locally finitely presentable $\mathcal{V}$-category $\mathcal{L}$. We say that f is pure if for each commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with $h : A \to B$ in $\mathcal{L}_f$, there exists $l : B \to X$ such that $l \circ h = u$.

Note that the notion of purity we are considering is the ordinary one; meaning that whenever we consider f pure in $\mathcal{L}$, we are actually seeing it as a pure morphism in the underlying category $\mathcal{L}_0$. Proposition 2.29 in [AR94] shows that each pure morphism is actually a monomorphism, so that we can talk about pure subobjects.

Proposition 4.1.4. *Each finite injectivity class $\mathcal{D} = \mathcal{M}\text{-inj}$ of a locally finitely presentable $\mathcal{V}$-category $\mathcal{L}$ is closed under (small) products, projective powers, filtered colimits and pure subobjects (meaning that if $f : X \to Y$ is pure and $Y \in \mathcal{D}$ then $X \in \mathcal{D}$).*

Proof. Given any arrow $h \in \mathcal{M}$ and $L \in \mathcal{L}_0$, we can see $\mathcal{L}(h, L)$ as an object of the category of arrows $\mathcal{V}_0^2$; since the domain and codomain of h are finitely presentable, the hom-functor $\mathcal{L}(h, -)_0 : \mathcal{L}_0 \to \mathcal{V}_0^2$ preserves filtered colimits as well as products and projective powers (since it preserves all limits). Note moreover that regular epimorphisms are stable in $\mathcal{V}$ under filtered colimits, products, and projective powers (as we saw in Section 2.2). As a consequence, if $S = \mathrm{colim}_i S_i$ is a filtered colimits of objects of $\mathcal{D}$, then $\mathcal{L}(h, S) \cong \mathrm{colim}_i \mathcal{L}(h, S_i)$ is a regular epimorphism; hence $S \in \mathcal{D}$. The same applies if S is a product or a projective power of elements from $\mathcal{D}$.

Now let $f : X \to Y$ be pure with $Y \in \mathcal{D}$, and consider $h \in \mathcal{M}$. By hypothesis $\mathcal{L}(h, Y)$ is a regular epimorphism; thus $\mathcal{L}_0(P \cdot h, Y)$ is surjective for each $P \in \mathcal{P}$ and then, since $P \cdot h$ is still a morphism with finitely presentable domain and codomain, purity implies that $\mathcal{L}_0(P \cdot h, X)$ is surjective as well for each $P \in \mathcal{P}$; thus $\mathcal{L}(h, X)$ is a regular epimorphism and $X \in \mathcal{D}$. $\square$

In the ordinary case, closure under these constructions is enough to characterize finite injectivity classes; indeed it is proven in Theorem 2.2 of [RAB02] that a full subcategory $\mathcal{D}$ of a locally finitely presentable category $\mathcal{L}$ is a finite injectivity class if and only if it is closed in $\mathcal{L}$ under products, filtered colimits and pure subobjects (powers are not necessary since they are a special kind of products). We can obtain a similar result in this context.

Proposition 4.1.5. *Let $\mathcal{D}$ be a full subcategory of a locally finitely presentable $\mathcal{V}$-category $\mathcal{L}$; then $\mathcal{D}$ is a finite injectivity class of $\mathcal{L}$ if and only if it is closed in $\mathcal{L}$ under products, projective powers, filtered colimits, and pure subobjects.*

Proof. One side is given by the previous Proposition. For the other, assume that $\mathcal{D}$ is closed in $\mathcal{L}$ under products, projective powers, filtered colimits, and pure subobjects. By Theorem 2.2 in [RAB02], $\mathcal{D}_0$ is an ordinary finite injectivity class in $\mathcal{L}_0$. Let $\mathcal{M}$ be the set of arrows defining $\mathcal{D}$ as such; we prove that it also defines $\mathcal{D}$ as an enriched finite injectivity class of $\mathcal{L}$. Given $S \in \mathcal{D}$, $\mathcal{L}_0(h, S)$ is surjective for each $h \in \mathcal{M}$; but $\mathcal{D}$ is closed under projective powers, hence $\mathcal{V}_0(P, \mathcal{L}(h, S)) \cong \mathcal{L}_0(h, S^P)$ is surjective for each $P \in \mathcal{P}$; thus $\mathcal{L}(h, S)$ is a regular epimorphism and $S \in \mathcal{M}$-inj. Conversely, given $S \in \mathcal{M}$-inj, $\mathcal{L}_0(h, S) = \mathcal{V}_0(I, \mathcal{L}(h, S))$ is surjective since $I \in \mathcal{V}_{0p}$, and as a consequence, $S \in \mathcal{D}$. $\square$

Proposition 4.1.6. *Each finite injectivity class $\mathcal{D}$ of a locally finitely presentable $\mathcal{V}$-category $\mathcal{L}$ is a weakly reflective subcategory of $\mathcal{L}$ (in the sense of Definition 1.3.4).*

Proof. We saw at the beginning of this section that $\mathcal{D}_0$ is also an ordinary injectivity class in $\mathcal{L}_0$; hence, by Theorem 1.3.3, it is an ordinary weakly reflective subcategory of $\mathcal{L}_0$. It's then enough to show that the weak reflections are actually enriched. Given an ordinary weak reflection $s : L \to S$ and $T \in \mathcal{D}$, the function $\mathcal{V}_0(P, \mathcal{L}(s, T)) \cong \mathcal{L}_0(s, T^P)$ is surjective for each $P \in \mathcal{P}$, since $T^P \in \mathcal{D}$ ($\mathcal{D}$ is closed under projective powers in $\mathcal{L}$). It follows then that $\mathcal{L}(s, T)$ is a regular epimorphism and s is an enriched weak reflection. $\square$

4.2 Definable $\mathcal{V}$-Categories

As we move on to the notion of definable $\mathcal{V}$-category, we should point out that our definition looks different from the usual one introduced in [Pre11] for $\mathcal{V} = \mathbf{Ab}$, and in [KR18] for $\mathcal{V} = \mathbf{Set}$, for which a definable category is just a finite injectivity class.

Definition 4.2.1. Call a $\mathcal{V}$-category $\mathcal{D}$ *definable* if there is a locally finitely presentable $\mathcal{V}$-category $\mathcal{L}$, with $\mathcal{L}_0$ coregular, for which $\mathcal{D}$ is an enriched finite injectivity class of $\mathcal{L}$ with respect to a small set of regular monomorphisms. A morphism between definable $\mathcal{V}$-categories is a $\mathcal{V}$-functor that preserves products, projective powers and filtered colimits. Denote by $\mathcal{V}$-**Def** the 2-category of definable $\mathcal{V}$-categories, morphisms between them, and $\mathcal{V}$-natural transformations.

Remark 4.2.2. For any regular $\mathcal{V}$-category $\mathcal{C}$, the category $\mathrm{Reg}(\mathcal{C},\mathcal{V})$ is a definable subcategory of $\mathrm{Lex}(\mathcal{C},\mathcal{V})$. Indeed, $\mathrm{Lex}(\mathcal{C},\mathcal{V})$ is coregular by Lemma 3.2.1; moreover, by Lemma 3.2.3, $\mathrm{Reg}(\mathcal{C},\mathcal{V}) = \mathcal{M}\text{-inj}$ in $\mathrm{Lex}(\mathcal{C},\mathcal{V})$ where $\mathcal{M} := \{\mathcal{C}(h,-) \mid h \textit{ regular epimorphism in } \mathcal{C}\}$. To conclude it's then enough to note that each $\mathcal{C}(h,-)$ in $\mathcal{M}$ is a regular monomorphism since h is a regular epimorphism in $\mathcal{C}$.

The notion of definable category we are considering lies somewhere between the ordinary one, of finite injectivity class, and that of *exactly-definable* category. We call a $\mathcal{V}$-category $\mathcal{D}$ exactly-definable if $\mathcal{D} \simeq \mathrm{Reg}(\mathcal{B},\mathcal{V})$ for an exact $\mathcal{V}$-category $\mathcal{B}$. Then the previous Remark says that each exactly-definable $\mathcal{V}$-category is definable, and hence a finite injectivity class. If we take $\mathcal{V}$ to be **Set** or **Ab**, then we can prove (see Section 4.5) that each finite injectivity class is conversely an exactly-definable category; showing that in the ordinary and additive context the three notions coincide.

Remark 4.2.3. Note that each locally finitely presentable $\mathcal{V}$-category is definable. Indeed, if $\mathcal{L}$ is locally finitely presentable, then $\mathcal{L} \simeq \mathrm{Lex}(\mathcal{C},\mathcal{V})$ for a finitely complete $\mathcal{V}$-category $\mathcal{C}$; thus by Remark 3.1.3, $\mathcal{L} \simeq \mathrm{Reg}(\mathcal{B},\mathcal{V})$ for some regular $\mathcal{V}$-category $\mathcal{B}$ (this will also follow later from Proposition 4.3.4). Hence $\mathcal{L}$ is definable by the previous Remark.

Proposition 4.2.4. *Let $\mathcal{D}$ be a definable subcategory of $\mathcal{L}$ (which is thence a locally finitely presentable $\mathcal{V}$-category and $\mathcal{L}_0$ is coregular). Then $\mathcal{D}$ is weakly reflective in $\mathcal{L}$ and the weak reflections can be chosen to be regular monomorphisms.*

Proof. By definition $\mathcal{D} = \mathcal{M}\text{-inj}$ for a small set $\mathcal{M}$ of regular monomorphisms in $\mathcal{L}$. That $\mathcal{D}$ is a weakly reflective subcategory of $\mathcal{L}$ follows from Proposition 4.1.6. The fact that the weak reflections can be chosen to be regular monomorphisms follows from Theorem 1.3.3 since filtered colimits commute in $\mathcal{L}$ with finite limits (being locally finitely presentable) and regular monomorphisms are pushout stable ($\mathcal{L}_0$ is coregular). $\square$

Given any $\mathcal{V}$-categories $\mathcal{M}$ and $\mathcal{N}$ with filtered colimits, products, and projective powers, we can consider the full subcategory Def$(\mathcal{M},\mathcal{N})$ of $[\mathcal{M},\mathcal{N}]$ consisting of those $\mathcal{V}$-functors that preserve filtered colimits, products, and powers with projective objects. If $\mathcal{M}$ and $\mathcal{N}$ are definable categories, this is just the hom-category Def$(\mathcal{M},\mathcal{N})$ in the 2-category $\mathcal{V}$-**Def**.

In the remaining part of this section, we are going to see some properties about the $\mathcal{V}$-category Def$(\mathcal{D},\mathcal{V})$ for a definable $\mathcal{V}$-category $\mathcal{D}$.

Lemma 4.2.5 (First part of 3.2.2 in [KR18] for $\mathcal{V}$ = **Set**). *Let $\mathcal{L}$ be a locally finitely presentable $\mathcal{V}$-category and $J : \mathcal{D} \hookrightarrow \mathcal{L}$ an enriched finite injectivity class of $\mathcal{L}$. Given $F \in$ Def$(\mathcal{D},\mathcal{V})$, suppose that for each $L \in \mathcal{D}$ and $x : I \to FL$ there exist $A \in \mathcal{L}_f$ and $\eta : \mathcal{L}(A, J-) \to F$ such that x factors through η_L. Then there is a $B \in \mathcal{L}_f$ and a regular epimorphism $\mathcal{L}(B, J-) \twoheadrightarrow F$.*

Proof. Since $\mathcal{D}_0$ is an ordinary injectivity class, it is also an accessible category. Consider then a regular cardinal λ such that $\mathcal{D}_0$ is λ-accessible, and denote by $\mathcal{D}_\lambda$ the full subcategory of λ-presentable objects in $\mathcal{D}_0$. For each $S \in \mathcal{D}_\lambda$ take $P_S \in \mathcal{V}_p$ and a regular epimorphism $\bar{x}_S : P_S \twoheadrightarrow FS$ which corresponds to an arrow $x_S : I \to (FS)^{P_S} \cong F(S^{P_S})$.
Define $\widehat{S} := \prod_{S\in\mathcal{D}_\lambda} S^{P_S}$ with projection maps $\pi_S : \widehat{S} \to S^{P_S}$; then $\widehat{S} \in \mathcal{D}$ and, since F preserves products and projective powers, $F(\widehat{S}) \cong \prod_{S\in\mathcal{D}_\lambda} F(S)^{P_S}$. Consider then $x : I \to F\widehat{S}$ with components $x_S : I \to F(S)^{P_S}$ for each $S \in \mathcal{D}_\lambda$. By our assumptions there exist $A \in \mathcal{L}_f$ and a natural transformation $\eta : \mathcal{L}(A, J-) \to F$ such that $x = \eta_{\widehat{S}}\circ y$ for some $y : I \to \mathcal{L}(A, \widehat{S})$. For each $S \in \mathcal{D}_\lambda$ we can consider the following diagram:

$$\begin{tikzcd}
 & I \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\mathcal{L}(A,\widehat{S}) \arrow[rr, "\eta_{\widehat{S}}"] \arrow[d, "\mathcal{L}(A,\pi_S)"'] & & F\widehat{S} \arrow[d, "F\pi_S"] \\
\mathcal{L}(A,S^{P_S}) \arrow[rr, "\eta_{(S^{P_S})}"'] \arrow[d, "\cong"'] & & F(S^{P_S}) \arrow[d, "\cong"] \\
\mathcal{L}(A,S)^{P_S} \arrow[rr, "(\eta_S)^{P_S}"'] & & F(S)^{P_S}
\end{tikzcd}$$

Transposing the vertical arrows then we obtain maps $\bar{y}_S : P_S \to \mathcal{L}(A, S)$ such that the diagram

$$\begin{tikzcd}
 & P_S \arrow[dl, "\bar{y}_S"'] \arrow[dr, "\bar{x}_S"] & \\
\mathcal{L}(A,S) \arrow[rr, "\eta_S"] & & FS
\end{tikzcd}$$

commutes.

Since $\bar{x}_S$ is a regular epimorphism, η_S is a regular epimorphism too for each $S \in \mathcal{D}_\lambda$ (remember that $\mathcal{V}_0$ is regular by Proposition 2.2.4), but $\mathcal{D}_\lambda$ generates $\mathcal{D}$ under λ-filtered colimits and $\mathcal{L}(A, J-)$ and F preserve them; then η_T is a regular epimorphism for each T in $\mathcal{D}$. Since regular epimorphisms in $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ are computed pointwise, it follows that η is a regular epimorphism as desired. □

Lemma 4.2.6. *Let $\mathcal{L}$ be a locally finitely presentable $\mathcal{V}$-category with $\mathcal{L}_0$ coregular, and $\mathcal{D}$ a definable subcategory of $\mathcal{L}$; denote by $J : \mathcal{D} \hookrightarrow \mathcal{L}$ be the inclusion. Then for any functor $F : \mathcal{D} \to \mathcal{V}$ preserving filtered colimits, the right Kan extension $\mathrm{Ran}_J F : \mathcal{L} \to \mathcal{V}$ preserves filtered colimits too.*

Proof. First we prove that $\mathrm{Ran}_J F$ preserves some particular limits. For each $L \in \mathcal{L}$ there is a weak reflection $s : L \rightarrowtail S$, with $S \in \mathcal{D}$, which by Proposition 4.2.4 can be chosen to be a regular monomorphism. Consider then the cokernel pair $u, v : S \to M$ of s and a weak reflection $t : M \rightarrowtail T$ associated to M (which again we suppose to be a regular monomorphism):

$$L \overset{s}{\rightarrowtail} S \underset{v}{\overset{u}{\rightrightarrows}} M \overset{t}{\rightarrowtail} T$$

Then $t \circ u$ and $t \circ v$ define L as an equalizer of elements from $\mathcal{D}$; call this a *presentation* for L. We are going to prove that these presentations are J-absolute, in the sense that they are sent to coequalizers by $\mathcal{L}(-, R)$ for each $R \in \mathcal{D}$. For, given $R \in \mathcal{D}$, consider the induced diagram

$$\mathcal{L}(T, R) \xrightarrow{\mathcal{L}(t,R)} \mathcal{L}(M, R) \underset{\mathcal{L}(v,R)}{\overset{\mathcal{L}(u,R)}{\rightrightarrows}} \mathcal{L}(S, R) \xrightarrow{\mathcal{L}(s,R)} \mathcal{L}(L, R).$$

Then $\mathcal{L}(t, R)$ and $\mathcal{L}(s, R)$ are regular epimorphisms since t and s are weak reflections; while $\mathcal{L}(u, R)$ and $\mathcal{L}(v, R)$ form the kernel pair of $\mathcal{L}(s, R)$ since $\mathcal{L}(-, R)$ transforms colimits into limits. As a consequence $\mathcal{L}(s, R)$ is the coequalizer of $\mathcal{L}(t \circ u, R)$ and $\mathcal{L}(t \circ v, R)$ as desired. It follows then that the presentations we are considering are actually codensity presentations, and hence by Theorem 5.29 of [Kel82a], $\mathrm{Ran}_J F$ preserves those limits. In particular, since moreover $\mathrm{Ran}_J F \circ J \cong F$, the object $\mathrm{Ran}_J F(L)$ is defined as the equalizer

$$\mathrm{Ran}_J F(L) \rightarrowtail F(S) \underset{F(t \circ v)}{\overset{F(t \circ u)}{\rightrightarrows}} F(T)$$

To prove that $\mathrm{Ran}_J F$ preserves filtered colimits it is enough to show, by Proposition 2.1.2, that it preserves colimits of smooth chains: diagrams $(L_\beta)_{\beta<\alpha}$ indexed by an ordinal α, such that for each limit $\lambda < \alpha$, $L_\lambda = \mathrm{colim}_{\beta<\lambda} L_\beta$.

Consider hence a smooth chain $(L_\beta)_{\beta<\alpha}$ in $\mathcal{L}$ with connecting maps $d_{\beta,\gamma} : L_\beta \to L_\gamma$; for each $\beta < \alpha$ we define by transfinite induction a presentation

$$\begin{tikzcd} L_\beta \arrow[r, tail, "s_\beta"] & S_\beta \arrow[r, shift left, "u_\beta"] \arrow[r, shift right, "v_\beta"'] & M_\beta \arrow[r, tail, "t_\beta"] & T_\beta \end{tikzcd}$$

for L_β, and smooth chains of such presentations compatibly with $(L_\beta)_{\beta<\alpha}$; meaning that for each $\beta < \gamma < \alpha$ we define a commutative diagram:

$$\begin{tikzcd} L_\beta \arrow[r, tail] \arrow[d, "d_{\beta,\gamma}"'] & S_\beta \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "e_{\beta,\gamma}"'] & M_\beta \arrow[r, tail] \arrow[d, "f_{\beta,\gamma}"'] & T_\beta \arrow[d, "g_{\beta,\gamma}"'] \\ L_\gamma \arrow[r, tail] & S_\gamma \arrow[r, shift left] \arrow[r, shift right] & M_\gamma \arrow[r, tail] & T_\gamma \end{tikzcd}$$

If $\beta = 0$ any presentation for L_0 is fine. Suppose now that everything is defined at level $\beta < \alpha$, then we define a presentation for $L_{\beta+1}$ and the connecting maps

$$\begin{tikzcd} L_\beta \arrow[r, tail, "s_\beta"] \arrow[d, "d_{\beta,\beta+1}"'] & S_\beta \arrow[r, shift left, "u_\beta"] \arrow[r, shift right, "v_\beta"'] \arrow[d, dashed, "e_{\beta,\beta+1}"'] & M_\beta \arrow[r, tail, "t_\beta"] \arrow[d, dashed, "f_{\beta,\beta+1}"'] & T_\beta \arrow[d, dashed, "g_{\beta,\beta+1}"'] \\ L_{\beta+1} \arrow[r, tail, dashed, "s_{\beta+1}"'] & S_{\beta+1} \arrow[r, shift left, dashed, "u_{\beta+1}"] \arrow[r, shift right, dashed, "v_{\beta+1}"'] & M_{\beta+1} \arrow[r, tail, dashed, "t_{\beta+1}"'] & T_{\beta+1} \end{tikzcd}$$

as follows: take the pushout $\tilde{S}_{\beta+1}$ of s_β and $d_{\beta,\beta+1}$ and call the two induced maps $\tilde{s}_{\beta+1} : L_{\beta+1} \rightarrowtail \tilde{S}_{\beta+1}$ and $\tilde{e}_{\beta,\beta+1} : S_\beta \to \tilde{S}_{\beta+1}$, where $\tilde{s}_{\beta+1}$ is a regular monomorphism because $\mathcal{L}_0$ is coregular. Consider now a weak reflection $r_{\beta+1} : \tilde{S}_{\beta+1} \rightarrowtail S_{\beta+1}$; it's then enough to consider $s_{\beta+1} := r_{\beta+1} \circ \tilde{s}_{\beta+1}$, which is still a weak reflection and a regular monomorphism (using projective powers each ordinary weak reflection is an enriched one), and $e_{\beta,\beta+1} := r_{\beta+1} \circ \tilde{e}_{\beta,\beta+1}$. We define $(u_{\beta+1}, v_{\beta+1})$ as the cokernel pair of $s_{\beta+1}$, while $f_{\beta,\beta+1}$ is induced by the universal property of u_β and v_β. Finally define $t_{\beta+1}$ and $g_{\beta,\beta+1}$ as in the first step. This gives a presentation for $L_{\beta+1}$ which is compatible with the chain already defined.

If $\lambda < \alpha$ is a limit ordinal, we take as presentation associated to L_λ the one obtained as the colimit of the presentations defined so far, in other words we consider $x_\lambda := \mathrm{colim}_{\beta<\lambda}(x_\beta)$ for $x = s, u, v, t$. It's easy to check that s_λ and t_λ are still weak reflections: given an arrow $f : L_\lambda \to T$, define by induction compatible arrows $\tilde{f}_\beta : S_\beta \to T$, for $\beta < \lambda$, such that $f \circ d_{\beta,\lambda} = \tilde{f}_\beta \circ s_\beta$; then the colimit of the $\tilde{f}_\beta$ induces a factorization of f through s_λ. This proves that s_λ is an ordinary weak reflection and hence, using projective powers, an enriched one (the same applies for t_λ). As a consequence, since in addition regular monomorphisms and cokernel pairs commute with filtered colimits, the one just defined is a presentation for L_λ. Moreover, by construction, the colimit cocones induce maps $e_{\beta,\lambda}$, $f_{\beta,\lambda}$ and $g_{\beta,\lambda}$ which are compatible with the chains defined so far.

We can then consider the colimit of these chains:

$$\mathrm{colim}_{\beta<\alpha}(L_\beta) \overset{s}{\rightarrowtail} \mathrm{colim}_{\beta<\alpha}(S_\beta) \underset{v}{\overset{u}{\rightrightarrows}} \mathrm{colim}_{\beta<\alpha}(M_\beta) \overset{t}{\rightarrowtail} \mathrm{colim}_{\beta<\alpha}(T_\beta)$$

By the previous arguments this is a presentation for $\mathrm{colim}_{\beta<\alpha}(L_\beta)$; hence it is preserved by $\mathrm{Ran}_J F$, which means that the following is an equalizer

$$\mathrm{Ran}_J F(\mathrm{colim}_{\beta<\alpha} L_\beta) \rightarrowtail F(\mathrm{colim}_{\beta<\alpha} S_\beta) \underset{F(t\circ v)}{\overset{F(t\circ u)}{\rightrightarrows}} F(\mathrm{colim}_{\beta<\alpha} T_\beta)$$

Similarly each $\mathrm{Ran}_J F(L_\beta)$ is given by the equalizer

$$\mathrm{Ran}_J F(L_\beta) \rightarrowtail F(S_\beta) \underset{F(t_\beta\circ v_\beta)}{\overset{F(t_\beta\circ u_\beta)}{\rightrightarrows}} F(T_\beta)$$

In conclusion, since F preserves filtered colimits and equalizers commute with them, the following isomorphisms hold

$$\begin{aligned}
\mathrm{Ran}_J F(\mathrm{colim}_{\beta<\alpha} L_\beta) &\cong \mathrm{eq}(F(t\circ u), F(t\circ v)) \\
&\cong \mathrm{eq}(\mathrm{colim}_{\beta<\alpha} F(t_\beta\circ u_\beta), \mathrm{colim}_{\beta<\alpha} F(t_\beta\circ v_\beta)) \\
&\cong \mathrm{colim}_{\beta<\alpha}(\mathrm{eq}(F(t_\beta\circ u_\beta), F(t_\beta\circ v_\beta))) \\
&\cong \mathrm{colim}_{\beta<\alpha} \mathrm{Ran}_J F(L_\beta)
\end{aligned}$$

as desired. □

By Proposition 4.2.4 and Theorem 1.3.5, each definable category $J : \mathcal{D} \hookrightarrow \mathcal{L}$ is a codense subcategory of $\mathcal{L}$, in other words the functor $\mathcal{L}(1, J) : \mathcal{L}^{op} \to [\mathcal{D}, \mathcal{V}]$ is fully faithful. Hence we can consider the full and faithful functor

$$Y_{\mathcal{D}} : \mathcal{L}_f^{op} \to \mathrm{Def}(\mathcal{D}, \mathcal{V})$$

defined as the composite of the inclusion $\mathcal{L}_f^{op} \hookrightarrow \mathcal{L}^{op}$ and $\mathcal{L}(1, J)$; explicitly, this sends $A \in \mathcal{L}_f^{op}$ to $\mathcal{L}(A, J-) : \mathcal{D} \to \mathcal{V}$. Then the following holds:

Proposition 4.2.7. *Let $\mathcal{D}$ be a definable $\mathcal{V}$-category; for each $F \in \mathrm{Def}(\mathcal{D},\mathcal{V})$ there exist $A, B \in \mathcal{L}_f$ and maps $f, g : B \to A$ such that F is the coequalizer of $Y_\mathcal{D}(f), Y_\mathcal{D}(g)$:*

$$\mathcal{L}(A, J-) \underset{\mathcal{L}(g,J-)}{\overset{\mathcal{L}(f,J-)}{\rightrightarrows}} \mathcal{L}(B, J-) \twoheadrightarrow F$$

In particular $\mathrm{Def}(\mathcal{D},\mathcal{V})$ *is a small $\mathcal{V}$-category.*

Proof. Let us first prove that the hypotheses of Lemma 4.2.5 are satisfied. For this, consider $F \in \mathrm{Def}(\mathcal{D},\mathcal{V})$, $L \in \mathcal{L}$, and $x : I \to FL$, and write L as a filtered colimit of finitely presentable objects $L \cong \mathrm{colim}(A_j)$. By the previous Lemma, $G := \mathrm{Ran}_J F$ preserves filtered colimits, then $GL \cong \mathrm{colim}\ G(A_j)$. Since I is finitely presentable in $\mathcal{V}$, x factors through some colimit map $G(A_j) \to GL$; but $G(A_j) \cong [\mathcal{L},\mathcal{V}](\mathcal{L}(A_j,-), G)$, hence the factorization corresponds to some $\eta : \mathcal{L}(A_j,-) \to G$. Its restriction $\eta J : \mathcal{L}(A_j, J-) \to F$ then satisfies the required property.
Now, thanks to Lemma 4.2.5, for each $F \in \mathrm{Def}(\mathcal{D},\mathcal{V})$ there exists a regular epimorphism $\eta : \mathcal{L}(B, J-) \twoheadrightarrow F$ with $B \in \mathcal{L}_f$; take then the kernel pair $\alpha, \beta : F' \to \mathcal{L}(B, J-)$ of η. Considering again a regular epimorphism $\gamma : \mathcal{L}(A, J-) \twoheadrightarrow F'$ ($A \in \mathcal{L}_f$) and composing it with α and β we obtain F as the coequalizer of restricted representables. Finally, the maps defining F as a coequalizer come from $\mathcal{L}_f$ since the functor $Y_\mathcal{D} : \mathcal{L}_f^{op} \to \mathrm{Def}(\mathcal{D},\mathcal{V})$ is full. □

The same result holds for any locally finitely presentable $\mathcal{V}$-category, even if it is not coregular:

Proposition 4.2.8. *Let $\mathcal{L}$ be a locally finitely presentable $\mathcal{V}$-category and $Y_\mathcal{L} : \mathcal{L}_f^{op} \to$ $\mathrm{Def}(\mathcal{L},\mathcal{V})$ as before. For each $F \in \mathrm{Def}(\mathcal{L},\mathcal{V})$ there exist $A, B \in \mathcal{L}_f$ and maps $f, g : B \to A$ such that F is the coequalizer of $Y_\mathcal{L}(f), Y_\mathcal{L}(g)$:*

$$\mathcal{L}(A, J-) \underset{\mathcal{L}(g,J-)}{\overset{\mathcal{L}(f,J-)}{\rightrightarrows}} \mathcal{L}(B, J-) \twoheadrightarrow F$$

In particular $\mathrm{Def}(\mathcal{L},\mathcal{V})$ *is a small $\mathcal{V}$-category.*

Proof. The same proof of the previous Proposition applies, with the only difference that there is no need to consider the right Kan extension of F. □

Remark 4.2.9. The coequalizers defined in the previous proofs are preserved by regular functors since they are given by the composite of a coequalizer of a kernel pair and a regular epimorphism.

4.3 Duality for Enriched Exact Categories

Let us consider again categories enriched over a base $\mathcal{V}$ which is a symmetric monoidal finitary quasivariety with strong generator $\mathcal{P} \subseteq \mathcal{V}_{pf}$.

We proved in Proposition 4.2.7 that, for a definable $\mathcal{D}$, the category $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ is small; since $\mathcal{V}$ is regular as a $\mathcal{V}$-category, $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ is a regular $\mathcal{V}$-category too (being closed in $[\mathcal{D}, \mathcal{V}]$ under finite limits and coequalizers of kernel pairs). Moreover, given any regular category $\mathcal{B}$, regular functors from $\mathcal{B}$ to $\mathcal{V}$ form a definable category $\mathrm{Reg}(\mathcal{B}, \mathcal{V})$ of $\mathrm{Lex}(\mathcal{B}, \mathcal{V})$ (as shown in Remark 4.2.2). As a consequence, if we denote by $\mathcal{V}$-**Reg** the 2-category of small regular $\mathcal{V}$-categories, regular $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations, we obtain an adjunction

$$\begin{tikzcd} \mathcal{V}\text{-}\mathbf{Def} \arrow[r, shift left, "\mathrm{Def}(-,\mathcal{V})"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Reg}^{op} \arrow[l, shift left, "\mathrm{Reg}(-,\mathcal{V})"] \end{tikzcd}$$

of 2-categories. Indeed for each regular $\mathcal{C}$ and each definable category $\mathcal{D}$ the following holds

$$\mathcal{V}\text{-}\mathbf{Def}(\mathcal{D}, \mathrm{Reg}(\mathcal{C}, \mathcal{V})) \cong \mathcal{V}\text{-}\mathbf{Reg}(\mathcal{C}, \mathrm{Def}(\mathcal{D}, \mathcal{V}))$$

since a functor $\mathcal{D} \to \mathrm{Reg}(\mathcal{C}, \mathcal{V})$ is definable if and only if the corresponding functor $\mathcal{D} \otimes \mathcal{C} \to \mathcal{V}$ is definable on the first variable and regular on the second, if and only if the induced functor $\mathcal{C} \to [\mathcal{D}, \mathcal{V}]$ is regular and takes values in $\mathrm{Def}(\mathcal{D}, \mathcal{V})$.

The counit and unit of this adjunction are given by the evaluation functors:

$$\mathrm{ev}_{\mathcal{C}} : \mathcal{C} \to \mathrm{Def}(\mathrm{Reg}(\mathcal{C}, \mathcal{V}), \mathcal{V})$$

for a regular $\mathcal{C}$, and

$$\mathrm{ev}_{\mathcal{D}} : \mathcal{D} \to \mathrm{Reg}(\mathrm{Def}(\mathcal{D}, \mathcal{V}), \mathcal{V})$$

for a definable $\mathcal{D}$. We already saw in Theorems 3.2.4 and 3.3.4 that the counit is a fully faithful functor, and an equivalence if the category is moreover exact. In the next passages we are going to show that the unit is always an equivalence.

Fix a definable subcategory $\mathcal{D}$ of $\mathcal{L}$ and consider the regular functor $Q : \mathrm{Def}(\mathcal{L}, \mathcal{V}) \to \mathrm{Def}(\mathcal{D}, \mathcal{V})$ given by precomposition with the inclusion $J : \mathcal{D} \hookrightarrow \mathcal{L}$. Remember also that we have defined (just before Proposition 4.2.7) the two fully faithful functors $Y_{\mathcal{L}}$ and $Y_{\mathcal{D}}$ such that the following

$$\begin{tikzcd}
\mathcal{L}_f^{op} \arrow[r, "Y_\mathcal{L}"] \arrow[dr, "Y_\mathcal{D}"'] & \mathrm{Def}(\mathcal{L},\mathcal{V}) \arrow[d, "Q"] \\
& \mathrm{Def}(\mathcal{D},\mathcal{V})
\end{tikzcd}$$

commutes.

Lemma 4.3.1. *Let $f \in \mathrm{Def}(\mathcal{D},\mathcal{V})$ be fixed; consider the category $\mathcal{C}_f$ whose objects are of the form*

$$\begin{tikzcd}
Y_\mathcal{D}a \arrow[r, two heads, "w"] & g \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & Y_\mathcal{D}b \arrow[r, two heads, "z"] & f
\end{tikzcd}$$

with $a, b \in \mathcal{L}_f$, w and z are regular epimorphisms, and (x, y) is the kernel pair of z. Morphisms between such objects are commutative diagrams:

$$\begin{tikzcd}
Y_\mathcal{D}a \arrow[r, two heads] \arrow[d] & g \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Y_\mathcal{D}b \arrow[d] \arrow[dr, two heads] & \\
& & & f \\
Y_\mathcal{D}a' \arrow[r, two heads] & g' \arrow[r, shift left] \arrow[r, shift right] & Y_\mathcal{D}b' \arrow[ur, two heads] &
\end{tikzcd}$$

Then $\mathcal{C}_f$ is connected, in the sense that it is non-empty and given any two objects in $\mathcal{C}_f$ there is a zigzag of arrows connecting them.

Proof. That $\mathcal{C}_f$ is not empty follows by Proposition 4.2.7. Before proceeding note that given a cospan $g \overset{x}{\twoheadrightarrow} f \overset{y}{\leftarrow} h$ in $\mathrm{Def}(\mathcal{D},\mathcal{V})$ with x a regular epimorphism, we can always complete it to a square

$$\begin{tikzcd}
Y_\mathcal{D}a \arrow[r, two heads, "x'"] \arrow[d] & h \arrow[d, "y"] \\
g \arrow[r, two heads, "x"'] & f
\end{tikzcd}$$

with a in $\mathcal{L}_f$ and x' a regular epimorphism. Indeed it's enough to take the pullback l of (x, y) and then a regular epimorphism $Y_\mathcal{D}a \twoheadrightarrow l$ as in Proposition 4.2.7.

Consider now two different objects of $\mathcal{C}_f$

$$\begin{tikzcd}
Y_\mathcal{D}a_i \arrow[r, two heads, "w_i"] & g_i \arrow[r, shift left, "x_i"] \arrow[r, shift right, "y_i"'] & Y_\mathcal{D}b_i \arrow[r, two heads, "z_i"] & f
\end{tikzcd}$$

for $i = 1, 2$, and construct the following diagram:

$$
\begin{tikzcd}
Y_{\mathcal{D}}b_{33} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "(4)"] & Y_{\mathcal{D}}a_{23} \arrow[rr] \arrow[d, two heads] \arrow[drr, phantom, "(3)"] & & Y_{\mathcal{D}}a_2 \arrow[d, two heads, "w_2"] \\
Y_{\mathcal{D}}a_{32} \arrow[r, two heads] \arrow[dd] \arrow[ddr, phantom, "(2)"] & g_{22} \arrow[rr] \arrow[dd] \arrow[dr, shift left, "x_{22}"] \arrow[dr, shift right, "y_{22}"'] & & g_2 \arrow[d, shift right, "y_2"'] \arrow[d, shift left, "x_2"] \\
& & Y_{\mathcal{D}}b_{11} \arrow[r, two heads, "z_{21}"] \arrow[d, two heads, "z_{12}"'] \arrow[dr, phantom, "(1)"] & Y_{\mathcal{D}}b_2 \arrow[d, two heads, "z_2"] \\
Y_{\mathcal{D}}a_1 \arrow[r, two heads, "w_1"'] & g_1 \arrow[r, shift left, "x_1"] \arrow[r, shift right, "y_1"'] & Y_{\mathcal{D}}b_1 \arrow[r, two heads, "z_1"'] & f
\end{tikzcd}
$$

Square (1) is built as explained at the beginning of the proof; the pair (x_{22}, y_{22}) is defined as the kernel pair of $z_1 \circ z_{12}$ (or, which is the same, of $z_2 \circ z_{21}$). The horizontal and vertical arrows from g_{22} are given by the universal property of the kernel pair, while the squares (2), (3), and (4) are obtained with the same argument as (1). To conclude, it's then easy to check that the diagonal of this diagram is an object of $\mathcal{C}_f$ and, together with the remaining horizontal and vertical arrows, gives a zigzag of length 2 between the two fixed objects of $\mathcal{C}_f$. □

Lemma 4.3.2. *The functor*

$$\mathrm{Reg}(\mathrm{Def}(\mathcal{D}, \mathcal{V}), \mathcal{V}) \hookrightarrow \mathrm{Lex}(\mathrm{Def}(\mathcal{D}, \mathcal{V}), \mathcal{V}) \xrightarrow{-\circ Y_{\mathcal{D}}} \mathrm{Lex}(\mathcal{L}_f^{op}, \mathcal{V})$$

is fully faithful. The same holds for any locally finitely presentable $\mathcal{V}$-category $\mathcal{L}'$ in place of $\mathcal{D}$, with respect to $Y_{\mathcal{L}'} : (\mathcal{L}'_f)^{op} \to \mathrm{Def}(\mathcal{L}', \mathcal{V})$.

Proof. Call that functor $(- \circ Y_{\mathcal{D}})'$; since $\mathcal{P} \subseteq \mathcal{V}$ is a strong generator, it's enough to prove that $\mathcal{V}_0(P, (- \circ Y_{\mathcal{D}})'_{F,G})$ is a isomorphism of sets for each $P \in \mathcal{P}$ and $F, G \in \mathrm{Reg}(\mathrm{Def}(\mathcal{D}, \mathcal{V}), \mathcal{V})$. Now, $\widetilde{\mathcal{D}} := \mathrm{Reg}(\mathrm{Def}(\mathcal{D}, \mathcal{V}), \mathcal{V})$ and $\widetilde{\mathcal{L}} = \mathrm{Lex}(\mathcal{L}_f^{op}, \mathcal{V})$ are definable, hence they have powers with elements of $\mathcal{P}$ and $(- \circ Y_{\mathcal{D}})'$ preserves them; therefore

$$\mathcal{V}_0(P, (- \circ Y_{\mathcal{D}})'_{F,G}) \cong \mathcal{V}_0(I, (- \circ Y_{\mathcal{D}})'_{F,G^P}) : \widetilde{\mathcal{D}}_0(F, G^P) \to \widetilde{\mathcal{L}}_0(F \circ Y_{\mathcal{D}}, G^P \circ Y_{\mathcal{D}}).$$

But $\mathcal{V}_0(I, (- \circ Y_{\mathcal{D}})'_{F,G^P})$ is the action of the ordinary functor $(- \circ Y_{\mathcal{D}})'_0$ on morphisms; as a consequence, if the ordinary functor $(- \circ Y_{\mathcal{D}})'_0$ is fully faithful, $\mathcal{V}_0(P, (- \circ Y_{\mathcal{D}})'_{F,G})$ will be an isomorphism for each $P \in \mathcal{P}$, and hence $(- \circ Y_{\mathcal{D}})'$ will be fully faithful. In conclusion, it's enough to show that $(- \circ Y_{\mathcal{D}})'_0$ is fully faithful.

To prove faithfulness, take $\eta, \gamma : F \to G$ such that $\eta Y_{\mathcal{D}} = \gamma Y_{\mathcal{D}}$. For any $f \in \mathrm{Def}(\mathcal{D}, \mathcal{V})$ there is a regular epimorphism $q : Y_{\mathcal{D}} a \twoheadrightarrow f$ with $a \in \mathcal{L}_f$; consider then the square:

$$\begin{tikzcd} FY_{\mathcal{D}}a \arrow[r, shift left, "\eta_{Y_{\mathcal{D}}a}"] \arrow[r, shift right, "\gamma_{Y_{\mathcal{D}}a}"'] \arrow[d, two heads, "Fq"'] & GY_{\mathcal{D}}a \arrow[d, two heads, "Gq"] \\ Ff \arrow[r, shift left, "\eta_f"] \arrow[r, shift right, "\gamma_f"'] & Gf \end{tikzcd}$$

Now, since $\eta_{Y_{\mathcal{D}}a} = \gamma_{Y_{\mathcal{D}}a}$ by hypothesis, and Fq is a regular epimorphism (F being a regular functor), it follows that $\eta_f = \gamma_f$, and so that $\eta = \gamma$.

It remains to prove fullness; given $\eta : FY_{\mathcal{D}} \to GY_{\mathcal{D}}$ in $\mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{V})$, we define $\gamma : F \to G$ such that $\gamma Y_{\mathcal{D}} = \eta$ as follows: for each $f \in \mathrm{Def}(\mathcal{D}, \mathcal{V})$ consider a presentation in $\mathcal{C}_f$

$$\begin{tikzcd} Y_{\mathcal{D}}a \arrow[r, shift left, "Y_{\mathcal{D}}x"] \arrow[r, shift right, "Y_{\mathcal{D}}y"'] & Y_{\mathcal{D}}b \arrow[r, two heads] & f \end{tikzcd}$$

where we are discarding the middle term since it is not needed now. Then, since F and G are regular functors, we obtain a diagram

$$\begin{tikzcd} FY_{\mathcal{D}}a \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\eta_a"'] & FY_{\mathcal{D}}b \arrow[r, two heads] \arrow[d, "\eta_b"'] & Ff \arrow[d, dashed, "\exists!\ \gamma_f"] \\ GY_{\mathcal{D}}a \arrow[r, shift left] \arrow[r, shift right] & GY_{\mathcal{D}}b \arrow[r, two heads] & Gf \end{tikzcd}$$

in which the rows are coequalizers and so there is a unique $\gamma_f : Ff \to Gf$ making the right square commute. This defines $\gamma : F \to G$; it's easy to see that $\gamma Y_{\mathcal{D}} = \eta$; hence we only need to prove that γ is well defined and $\mathcal{V}$-natural.

Given $f \in \mathrm{Def}(\mathcal{D}, \mathcal{V})$ and two different presentations, if there is a morphism in $\mathcal{C}_f$ between them, then the universal property of the coequalizer implies that the induced γ_f is the same for both presentations. In general this follows since, by the previous Lemma, $\mathcal{C}_f$ is connected.

To prove $\mathcal{V}$-naturality, first note that since F and G (being regular functors) preserve powers with finitely presentable objects, it's enough to show that the γ just defined is natural in the ordinary sense. Consider hence $x : f \to g$ in $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ and fix a presentation

$$\begin{tikzcd} Y_{\mathcal{D}}a \arrow[r, shift left] \arrow[r, shift right] & Y_{\mathcal{D}}b \arrow[r, two heads] & g \end{tikzcd}$$

for g; arguing as in the previous Lemma, we can build a presentation for f and a diagram

$$\begin{tikzcd} Y_{\mathcal{D}}a' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Y_{\mathcal{D}}b' \arrow[r, two heads] \arrow[d] & f \arrow[d, "x"] \\ Y_{\mathcal{D}}a \arrow[r, shift left] \arrow[r, shift right] & Y_{\mathcal{D}}b \arrow[r, two heads] & g \end{tikzcd}$$

Now, since $Y_{\mathcal{D}}$ is fully faithful, the two vertical arrows on the left come from $\mathcal{L}_f^{op}$. As a consequence, applying F, G and $\eta : FY_{\mathcal{D}} \to GY_{\mathcal{D}}$ to that diagram, we find a commutative square expressing the naturality of γ at $x : f \to g$.

The last assertion regarding a locally finitely presentable $\mathcal{V}$-category $\mathcal{L}'$, follows with exactly the same proof since, thanks to Proposition 4.2.8, the previous Lemma still holds if we replace $\mathcal{D}$ with $\mathcal{L}'$. $\square$

Corollary 4.3.3. *The functor*

$$- \circ Q : \mathrm{Reg}(\mathrm{Def}(\mathcal{D}, \mathcal{V}), \mathcal{V}) \longrightarrow \mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{V})$$

is fully faithful.

Proof. The functors $(- \circ Y_{\mathcal{D}})'$ and $(- \circ Y_{\mathcal{L}})'$ are fully faithful by the previous Lemma, and $(- \circ Y_{\mathcal{L}})' \circ (- \circ Q) = (- \circ Y_{\mathcal{D}})'$. It follows then that $(- \circ Q)$ is fully faithful. $\square$

Proposition 4.3.4. *For each locally finitely presentable category $\mathcal{L}$ the following is an equivalence:*

$$ev_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{V}).$$

Proof. Write $\mathcal{L} \simeq \mathrm{Lex}(\mathcal{C}, \mathcal{V})$ where the category $\mathcal{C} = \mathcal{L}_f^{op}$ is finitely complete, and consider the evaluation functor $\mathrm{ev} : \mathcal{C} \to \mathrm{Def}(\mathcal{L}, \mathcal{V})$ (which can also be seen as $Y_{\mathcal{L}}$). Then we define $R : \mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{V}) \longrightarrow \mathcal{L}$ as the composite

$$\mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{V}) \hookrightarrow \mathrm{Lex}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{V}) \xrightarrow{-\circ \mathrm{ev}} \mathrm{Lex}(\mathcal{C}, \mathcal{V});$$

this is fully faithful by Lemma 4.3.2. In the other direction, we consider the evaluation map $L := \mathrm{ev}_{\mathcal{L}} : \mathcal{L} \longrightarrow \mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{V})$ as suggested. Take $f \in \mathcal{L} \cong \mathrm{Lex}(\mathcal{C}, \mathcal{V})$ and $C \in \mathcal{C}$, then

$$RLf(C) = (Lf \circ \mathrm{ev})(C) = Lf(\mathrm{ev}(C)) = \mathrm{ev}(C)(f) = f(C).$$

Similarly for each morphism η in $\mathcal{L}$, $RL\eta = \eta$; as a consequence $\mathcal{V}_0(I, (RL)_{fg}) = \mathcal{V}_0(I, id)$ for each $f, g \in \mathcal{L}$. Since both R and L preserve powers from $\mathcal{P}$, this implies that $\mathcal{V}_0(P, (RL)_{fg}) \cong$

$\mathcal{V}_0(P, id)$ for each $P \in \mathcal{P}$; but $\mathcal{P}$ is a strong generator, then $(RL)_{fg} \cong id$, and hence $RL \cong id$. As a consequence R is also essentially surjective; therefore R is an equivalence and L its inverse. □

It remains only to prove the equivalence for each definable category $\mathcal{D}$:

Proposition 4.3.5. *If $\mathcal{D}$ is a definable $\mathcal{V}$-category, then the evaluation map*

$$\mathrm{ev}_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Reg}(\mathrm{Def}(\mathcal{D}, \mathcal{V}), \mathcal{V})$$

is an equivalence.

Proof. $\mathcal{D}$ is a definable subcategory of some locally finitely presentable $\mathcal{L}$; denote by J the inclusion. We can hence consider the commutative square:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\mathrm{ev}_{\mathcal{D}}"] \arrow[d, "J"'] & \mathrm{Reg}(\mathrm{Def}(\mathcal{D}, \mathcal{V}), \mathcal{V}) \arrow[d, "- \circ Q"] \\ \mathcal{L} \arrow[r, "\simeq", "\mathrm{ev}_{\mathcal{L}}"'] & \mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{V}) \end{tikzcd}$$

where $Q : \mathrm{Def}(\mathcal{L}, \mathcal{V}) \to \mathrm{Def}(\mathcal{D}, \mathcal{V})$ is the restriction along J, and we already know that $\mathrm{ev}_{\mathcal{L}}$ is an equivalence. It follows that $\mathrm{ev}_{\mathcal{D}}$ is fully faithful since J, $\mathrm{ev}_{\mathcal{L}}$, and $- \circ Q$ are; therefore we only need to prove that $\mathrm{ev}_{\mathcal{D}}$ is essentially surjective.
Consider a regular functor $F : \mathrm{Def}(\mathcal{D}, \mathcal{V}) \to \mathcal{V}$, then $F \circ Q \cong \mathrm{ev}_{\mathcal{L}}(L)$ some $L \in \mathcal{L}$. It's enough to prove that $L \in \mathcal{D}$, this would imply $\mathrm{ev}_{\mathcal{D}}(L) \circ Q \cong \mathrm{ev}_{\mathcal{L}}(L) \cong F \circ Q$ and hence, by Corollary 4.3.3, $F \cong \mathrm{ev}_{\mathcal{D}}(L)$. Consider then a collection $\mathcal{M}$ of morphisms from $\mathcal{L}_f$ such that $\mathcal{D} = \mathcal{M}$-inj. For each $(h : A \to B) \in \mathcal{M}$, since A and B are finitely presentable $\mathcal{L}(A, -)$ and $\mathcal{L}(B, -)$ are in $\mathrm{Def}(\mathcal{L}, \mathcal{V})$; moreover $Q(\mathcal{L}(h, -))$ is a regular epimorphism in $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ since $Q(\mathcal{L}(h, -))(D) = \mathcal{L}(h, D)$ is for each $D \in \mathcal{D}$. But F is a regular functor, thus $F \circ Q(\mathcal{L}(h, -))$ is still a regular epimorphism. As a consequence $\mathcal{L}(h, L) = \mathrm{ev}_{\mathcal{L}}(L)(\mathcal{L}(h, -)) \cong F \circ Q(\mathcal{L}(h, -))$ is a regular epimorphism for each $h \in \mathcal{M}$, and L is in $\mathcal{D}$. □

Now assume that our base for enrichment $\mathcal{V}$ is a symmetric monoidal finitary variety; then $\mathcal{V}_0$ is an exact category and $\mathcal{V}$ is exact as a $\mathcal{V}$-category. As a consequence $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ is an exact $\mathcal{V}$-category for each definable $\mathcal{D}$; hence the 2-adjunction between $\mathcal{V}$-**Def** and $\mathcal{V}$-**Reg** restricts to

$$\begin{tikzcd} \mathcal{V}\text{-}\mathbf{Def} \arrow[r, shift left, "\mathrm{Def}(-,\mathcal{V})"] \arrow[r, phantom, "\perp"] & \mathcal{V}\text{-}\mathbf{Ex}^{op} \arrow[l, shift left, "\mathrm{Reg}(-,\mathcal{V})"] \end{tikzcd}$$

where $\mathcal{V}$-**Ex** is the 2-category of all small exact $\mathcal{V}$-categories, regular $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. By Theorem 3.3.4 the counit of this adjunction is an equivalence; the same holds for the unit by the previous results. Thence we have proven:

Theorem 4.3.6. *Let $\mathcal{V}$ be a symmetric monoidal finitary variety. Then the 2-adjunction*

$$\mathrm{Def}(-,\mathcal{V}) : \mathcal{V}\text{-}\mathbf{Def} \rightleftarrows \mathcal{V}\text{-}\mathbf{Ex}^{op} : \mathrm{Reg}(-,\mathcal{V})$$

is a biequivalence.

This duality was first shown for the additive case in Theorem 2.3 of [PR10], while the ordinary version appeared more recently in Theorem 3.2.5 of [KR18]. We should mention that the proof of Proposition 3.2.2 from [KR18] contains an unjustified isomorphism which affects the proof of the duality; our Theorem 4.3.6, together with the results in Section 4.5, provides a solution for this.

4.4 Free Exact $\mathcal{V}$-Categories

Consider again $\mathcal{V}$ to be a symmetric monoidal finitary variety as in the last part of the previous section. We are going to use Theorem 4.3.6 to find the free exact categories associated to finitely complete and regular ones.

In the ordinary context, exact completions over regular categories were first considered in [Law73]; while regular and exact completions over finitely complete categories have been dealt with in [CCM82] and [CV98]. A different, but equivalent, description of them has been given in [Hu96] and [HT96], where exact completions are built as certain categories of functors preserving determined limits and colimits.

Proposition 4.4.1. *Let $\mathcal{C}$ be a finitely complete $\mathcal{V}$-category and define $\mathcal{L} = \mathrm{Lex}(\mathcal{C},\mathcal{V})$. Then for each exact $\mathcal{V}$-category $\mathcal{B}$, precomposition with* $\mathrm{ev} : \mathcal{C} \to \mathrm{Def}(\mathcal{L},\mathcal{V})$ *induces an equivalence:*

$$\mathrm{Reg}(\mathrm{Def}(\mathcal{L},\mathcal{V}),\mathcal{B}) \simeq \mathrm{Lex}(\mathcal{C},\mathcal{B}).$$

In other words $\mathrm{Def}(\mathcal{L},\mathcal{V})$ *is the free exact $\mathcal{V}$-category over $\mathcal{C}$ as a $\mathcal{V}$-category with finite limits.*

Proof. By Proposition 4.3.4, the equivalence holds for $\mathcal{B} = \mathcal{V}$ and hence for all functor categories $[\mathcal{A}, \mathcal{V}]$. Let now $\mathcal{B}$ be any exact category; by Theorem 3.2.4 we can assume that $\mathcal{B}$ is a full subcategory of some $[\mathcal{R}, \mathcal{V}]$ with the inclusion $H : \mathcal{B} \to [\mathcal{R}, \mathcal{V}]$ a regular functor. Since the equivalence holds for $[\mathcal{R}, \mathcal{V}]$ we can consider the commutative square

$$\begin{tikzcd}
\mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), \mathcal{B}) \arrow[r, "- \circ \mathrm{ev}"] \arrow[d, "H \circ -"'] & \mathrm{Lex}(\mathcal{C}, \mathcal{B}) \arrow[d, "H \circ -"] \\
\mathrm{Reg}(\mathrm{Def}(\mathcal{L}, \mathcal{V}), [\mathcal{R}, \mathcal{V}]) \arrow[r, "\simeq", "- \circ \mathrm{ev}"'] & \mathrm{Lex}(\mathcal{C}, [\mathcal{R}, \mathcal{V}])
\end{tikzcd}$$

in which the bottom arrow is invertible and the vertical ones fully faithful.
Thus the upper horizontal is fully faithful, and it is enough to prove that, given $F \in \mathrm{Lex}(\mathcal{C}, \mathcal{B})$, the induced extension of HF to $\mathrm{Def}(\mathcal{L}, \mathcal{V})$ takes values in $\mathcal{B}$. Let G be the mentioned extension of HF, then we can consider the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{ev}"'] & \mathcal{B} \arrow[d, "H"] \\
\mathrm{Def}(\mathcal{L}, \mathcal{V}) \arrow[r, "G"'] & [\mathcal{R}, \mathcal{V}]
\end{tikzcd}$$

The commutativity of this square (up to isomorphism) says that G restricted to the evaluation functors $\mathrm{ev}(C)$, for $C \in \mathcal{C}$, takes values in $\mathcal{B}$. Given any other $M \in \mathrm{Def}(\mathcal{L}, \mathcal{V})$ we can write it as a coequalizer:

$$\begin{tikzcd}
\mathrm{ev}(C) \arrow[r, two heads, "\gamma"] & N \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \mathrm{ev}(D) \arrow[r, two heads, "\eta"] & M
\end{tikzcd}$$

where (α, β) is the kernel pair of η and, since ev if fully faithful, $\alpha \circ \gamma = \mathrm{ev}(u)$ and $\beta \circ \gamma = \mathrm{ev}(v)$ for some $u, v : C \to D$ in $\mathcal{C}$. Since G preserves finite limits and coequalizers of kernels pair, the image of the previous diagram through G leads to

$$\begin{tikzcd}
FC \arrow[r, two heads, "G\gamma"] & GN \arrow[r, shift left, "G\alpha"] \arrow[r, shift right, "G\beta"'] & FD \arrow[r, two heads, "G\eta"] & GM
\end{tikzcd}$$

where $G\alpha, G\beta : GN \to FD$ form the kernel pair of $G\eta$. Since $[\mathcal{R}, \mathcal{V}]$ is regular, GN and $(G\alpha, G\beta)$ are given by the image factorization of $(Fu, Fv) : FC \to FD \times FD$ and hence GN is actually in $\mathcal{B}$ and $G\alpha$ and $G\beta$ exist as arrows of $\mathcal{B}$. Moreover, being a kernel pair (in $[\mathcal{R}, \mathcal{V}]$) the pair $(G\alpha, G\beta)$ is an equivalence relation. But $\mathcal{B}$ is exact and hence all equivalence relations are effective; this means that $G\alpha$ and $G\beta$ have a coequalizer in $\mathcal{B}$ which hence coincides with

GM. As a consequence G takes values in $\mathcal{B}$ as claimed.

□

This says that the left biadjoint to the forgetful functor $U_{\text{ex/lex}} : \mathcal{V}\text{-}\mathbf{Ex} \to \mathcal{V}\text{-}\mathbf{Lex}$ is given by the composite

$$\mathcal{V}\text{-}\mathbf{Lex} \xrightarrow{\text{Lex}(-,\mathcal{V})} \mathcal{V}\text{-}\mathbf{Lfp}^{op} \xrightarrow{U_{\text{lfp/def}}} \mathcal{V}\text{-}\mathbf{Def}^{op} \xrightarrow{\text{Def}(-,\mathcal{V})} \mathcal{V}\text{-}\mathbf{Ex}$$

where $U_{\text{lfp/def}} : \mathcal{V}\text{-}\mathbf{Lfp} \to \mathcal{V}\text{-}\mathbf{Def}$ is the forgetful functor. Since the first and the last are actually biequivalences, it follows that $U_{\text{lfp/def}}$ has a left biadjoint too, which is given by

$$\mathcal{V}\text{-}\mathbf{Def} \xrightarrow{\text{Def}(-,\mathcal{V})} \mathcal{V}\text{-}\mathbf{Ex}^{op} \xrightarrow{U_{\text{ex/lex}}} \mathcal{V}\text{-}\mathbf{Lex}^{op} \xrightarrow{\text{Lex}(-,\mathcal{V})} \mathcal{V}\text{-}\mathbf{Lfp}.$$

The next Proposition gives an explicit description of the free exact $\mathcal{V}$-category on a regular one:

Proposition 4.4.2. *Let $\mathcal{C}$ be a regular $\mathcal{V}$-category and define $\mathcal{R} = \text{Reg}(\mathcal{C},\mathcal{V})$. Then for each exact $\mathcal{V}$-category $\mathcal{B}$, precomposition with* $\text{ev} : \mathcal{C} \to \text{Def}(\mathcal{R},\mathcal{V})$ *induces an equivalence:*

$$\text{Reg}(\mathcal{C},\mathcal{B}) \simeq \text{Reg}(\text{Def}(\mathcal{R},\mathcal{V}),\mathcal{B}).$$

In other words $\text{Def}(\mathcal{R},\mathcal{V})$ *is the free exact $\mathcal{V}$-category over $\mathcal{C}$ as a regular $\mathcal{V}$-category.*

Proof. Note that $\mathcal{R}$ is a definable subcategory of $\text{Lex}(\mathcal{C},\mathcal{V})$, hence by Theorem 4.3.6, the equivalence

$$\text{Reg}(\mathcal{C},\mathcal{V}) = \mathcal{R} \simeq \text{Reg}(\text{Def}(\mathcal{R},\mathcal{V}),\mathcal{V})$$

holds and is induced by the evaluation map. Arguing as in the preceding proof we obtain the equivalence for any exact $\mathcal{B}$ in place of $\mathcal{V}$. □

As before, this says that the left biadjoint to the forgetful functor $U_{\text{ex/reg}} : \mathcal{V}\text{-}\mathbf{Ex} \to \mathcal{V}\text{-}\mathbf{Reg}$ is given by the composite

$$\mathcal{V}\text{-}\mathbf{Reg} \xrightarrow{\text{Reg}(-,\mathcal{V})} \mathcal{V}\text{-}\mathbf{Def}^{op} \xrightarrow{\text{Def}(-,\mathcal{V})} \mathcal{V}\text{-}\mathbf{Ex}.$$

4.5 The Ordinary Case

In this last section we give an equivalent characterization of definable categories in the ordinary case, showing how things get easier in this context.

Recall the definition of regular congruence from [Ben89]:

Definition 4.5.1. Let $\mathcal{C}$ be a regular category. A *regular congruence* on $\mathcal{C}$ is a class Σ of maps of $\mathcal{C}$ satisfying:

- every isomorphism belongs to Σ;
- if $f = h \circ g$ and two of the three maps are in Σ, so is the third;
- Σ is pullback stable: for any pullback in $\mathcal{C}$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

  if $f \in \Sigma$, then $f' \in \Sigma$;
- Σ is local: for any pullback in $\mathcal{C}$ as above, for which f is a regular epimorphism, if $g' \in \Sigma$ then $g \in \Sigma$.

The following Lemma states the existence of categories of fractions (for regular congruences) in the 2-category **Reg**:

Lemma 4.5.2 (1.7.3 and 2.2.3 of [Ben89])**.** *Let $\mathcal{C}$ be a regular category and Σ a regular congruence on $\mathcal{C}$. There exists a regular category $\mathcal{C}[\Sigma^{-1}]$ and a regular functor $P_\Sigma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ such that:*

- *a functor $F : \mathcal{C} \to \mathcal{B}$ factors uniquely through P_Σ as $F = F_\Sigma \circ P_\Sigma$ if and only if F inverts the elements of Σ;*
- *F_Σ is a regular functor if and only if F is.*

Next we can proceed with the characterization of definable categories in the ordinary context.

Proposition 4.5.3. *Let $\mathcal{D}$ be a category; the following are equivalent:*

1. *$\mathcal{D}$ is definable in the sense of Definition 4.2.1.*

2. *$\mathcal{D}$ is a finite injectivity class in some locally finitely presentable category $\mathcal{L}$;*
3. *$\mathcal{D}$ is exactly definable: there exists an exact category $\mathcal{B}$ such that $\mathcal{D} \simeq \mathrm{Reg}(\mathcal{B}, \mathbf{Set})$.*

Proof. (1) $\Rightarrow$ (2) follows by definition and (3) $\Leftrightarrow$ (1) is Theorem 4.3.6; hence it suffices to prove (2) $\Rightarrow$ (3). Let then $\mathcal{D} = \mathcal{M}$-inj be a finite injectivity class in a locally finitely presentable category $\mathcal{L}$. Write $\mathcal{L}$ as $\mathrm{Lex}(\mathcal{C}, \mathbf{Set})$ where $\mathcal{C} = \mathcal{L}_f^{op}$; then $\mathcal{M}$ can be identified with a collection of morphisms from $\mathcal{C}$, and $\mathcal{D}$ coincides with the full subcategory of $\mathrm{Lex}(\mathcal{C}, \mathbf{Set})$ given by those functors that send each $h \in \mathcal{M}$ to an epimorphism. Consider now $\mathcal{C}' = \mathcal{C}_{\mathrm{reg/lex}}$ to be the free regular category over $\mathcal{C}$; it follows that $\mathrm{Lex}(\mathcal{C}, \mathbf{Set}) \simeq \mathrm{Reg}(\mathcal{C}', \mathbf{Set})$. Under this equivalence $\mathcal{M}$ corresponds to a small set of arrows in $\mathcal{C}'$, and $\mathcal{D}$ to the full subcategory of $\mathrm{Reg}(\mathcal{C}', \mathbf{Set})$ given by those regular functors that send each $h \in \mathcal{M}$ to an epimorphism. For each $h \in \mathcal{M}$ take its image factorization $h = m_h \circ e_h$ in $\mathcal{C}'$, where e_h is a regular epimorphism and m_h a monomorphism; then a regular functor $F : \mathcal{C}' \to \mathbf{Set}$ sends h to a regular epimorphism in $\mathbf{Set}$ if and only if it sends m_h to an isomorphism. Thus, defining $\mathcal{N} = \{\, m_h \mid h \in \mathcal{M} \,\}$, $\mathcal{D}$ corresponds to the full subcategory of $\mathrm{Reg}(\mathcal{C}', \mathbf{Set})$ given by those regular functors that invert the maps in $\mathcal{N}$. Let now Σ be the saturation of $\mathcal{N}$ with respect to $\mathcal{D}$:

$$\Sigma := \{f \in \mathcal{C}' \mid F(f) \text{ is invertible for each } F \in \mathcal{D}\} \supseteq \mathcal{N},$$

where we are seeing $\mathcal{D}$ in $\mathrm{Reg}(\mathcal{C}', \mathbf{Set})$ as above. It's easy to check that Σ is a regular congruence in $\mathcal{C}'$; hence by the previous Lemma, $\mathcal{C}'' := \mathcal{C}'[\Sigma^{-1}]$ exists and by construction $\mathcal{D} \simeq \mathrm{Reg}(\mathcal{C}'', \mathbf{Set})$. Finally take $\mathcal{B}$ to be the free exact category over $\mathcal{C}''$, then $\mathcal{D} \simeq \mathrm{Reg}(\mathcal{B}, \mathbf{Set})$ as desired. □

Remark 4.5.4. Assuming this result it's easy to prove the duality

$$\mathrm{Def}(-, \mathbf{Set}) : \mathbf{Def} \rightleftarrows \mathbf{Ex}^{op} : \mathrm{Reg}(-, \mathbf{Set}).$$

Indeed the 2-functor $\mathrm{Reg}(-, \mathbf{Set})$ is essentially surjective by the previous Proposition, and (bi)fully faithful since the unit of the adjunction is an equivalence (Theorem 3.3.4). It then follows that $\mathrm{Reg}(-, \mathbf{Set})$ is a biequivalence of 2-categories.

Let us also note that Lemma 4.5.2 has an additive version (corollary 2.6.2 in [Ben89]), saying that if $\mathcal{C}$ is abelian and Σ a regular congruence on $\mathcal{C}$ then $\mathcal{C}[\Sigma^{-1}]$ is abelian too. It follows that the same arguments apply to the additive context, retrieving the characterization of definable categories given in [Pre11].

Chapter 5

Future Directions

There are several ways we could continue this project:

- We strongly believe that the same argument used for $\mathcal{V} = \mathbf{Set}$ in Section 4.5 works also for a general $\mathcal{V}$; so that one could consider a definable $\mathcal{V}$-category to be just a finite injectivity class in a locally finitely presentable $\mathcal{V}$-category. The only thing left to prove would be the analogue of Lemma 4.5.2 for regular $\mathcal{V}$-categories.
- Another possible generalization would be to give an *infinitary* version of each of the obtained results; replacing everywhere "finite" with "less than λ" for a fixed regular cardinal λ. In this context $\mathcal{V}$ would be a λ-(quasi)variety, and the notions of regularity and exactness would be replaced by some notions of λ-regularity and λ-exactness (that should be compared to those appearing in [Mak90]). Similarly, λ-definable categories would be λ-injectivity classes in some locally λ-presentable $\mathcal{V}$-category.
- In the ordinary and the additive context the notion of definable category is strictly related to logic and model theory. In fact, a category $\mathcal{D}$ is definable if and only if there exists a regular logical theory $\mathbb{T}$ whose category of models $\mathrm{Mod}(\mathbb{T})$ is equivalent to $\mathcal{D}$. It would be interesting to recover this equivalence in our context too, introducing suitable notions of *Logical Theories* and *Models* for enriched categories.
- In this thesis we saw two different dualities, one involving finitely complete $\mathcal{V}$-categories (Theorem 2.1.9) and the other about exact $\mathcal{V}$-categories (Theorem 4.3.6). In the ordinary context a further step has been made; Makkai in [Mak87] moved from exact categories to the 2-category of pretoposes, proving a new duality. One could then ask if this has a corresponding enriched version.
- As suggested by the examiners, it would be interesting to find more examples of symmetric monoidal finitary quasivarieties, and in these new cases to spell out in detail how the enriched notions actually differ from the ordinary ones.

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