Oplax actions and enriched icons with applications to coalgebroids and quantum categories

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Summary

This thesis is a formal treatise around the concept of bialgebroids and the alternative ways to describe them. There are two characterizations of bialgebroids due to Szlachányi which play a central role in our investigation; one as cocontinuous opmonoidal monads on the category of two-sided R-modules, and another as certain skew monoidal structures on the category of right R-modules. Lack and Street internalised Szlachányi's characterization to a suitable monoidal bicategory $\mathcal{M}$. In this way, they obtain an equivalence between opmonoidal monads on the enveloping monoidale induced by a biduality and right skew monoidales whose unit has a right adjoint in $\mathcal{M}$. Such equivalence provides a characterization of the quantum categories defined by Day and Street.

In the first two chapters, we focus on the simpler structure of a coalgebroid. In a monoidal bicategory $\mathcal{M}$, coalgebroids generalise as opmonoidal *arrows* between enveloping monoidales. Chapter 2 has two main results. The first one, following Lack and Street's methods, is a characterisation of opmonoidal arrows between enveloping monoidales which leads to the new concept of oplax actions with respect to a skew monoidale. The second result involves the study of comodules. Comodules for coalgebroids are classically defined as comodules for the underlying coring; and in a monoidal bicategory $\mathcal{M}$, opmonoidal arrows and oplax actions each admit a notion of comodule. We prove that these three ways to define comodules are equivalent.

The equivalence between opmonoidal arrows and oplax actions mentioned above is analogous to that of opmonoidal monads and right skew monoidales. We formalise this statement in Chapter 3, and show along the way that monads of oplax actions are right skew monoidales whose unit has a right adjoint.

The last chapter focuses on a different characterisation of bialgebroids: Moerdijk proved that a monad on a monoidal category is an opmonoidal monad if and only if the category of algebras has a monoidal structure such that the forgetful functor is strong monoidal. In other words, the 2-category OpMon of monoidal categories, opmonoidal functors, and opmonoidal natural transformations has Eilenberg-Moore objects for monads. We generalise this theorem in two directions: a multi-object version, and a version enriched in a monoidal bicategory. For the multi-object version, we replace OpMon with the 2-category Icon of bicategories, oplax functors, and icons. And for the version enriched in a monoidal bicategory $\mathcal{M}$, we replace Icon with a bicategory $\mathsf{Icon}(\mathcal{M})$ of $\mathcal{M}$-enriched bicategories, $\mathcal{M}$-enriched oplax functors, and $\mathcal{M}$-enriched icons. At this level of generality, the theorem asserts that the bicategory $\mathsf{Icon}(\mathcal{M})$ has Eilenberg-Moore objects for monads if $\mathcal{M}$ does.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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As anything else in life, doing a Ph.D. comes with all sorts of different feelings; some of which people usually classify as "bad feelings" such as stress, desperation, or disappointment; some feelings are "neutral" such as curiosity or courage; and others which are "good" like joy, enthusiasm, or that overwhelming feeling of fulfilment that you get when you prove something (mathematically). I think that all feelings are enjoyable regardless of how they are generally considered, I will proceed to give a small argument of why I think this is true. Let me start by saying that this is not something that happens naturally, first, one has to have the desire to enjoy a felling. Then, there is an abstraction leap that one must take in which one needs to *feel or think about* oneself kind of abstractly —externally— having the feeling in question. And only then one might be able to enjoy the feeling in question. For example, one might feel happy that one is curious, or passionate, or angry. Needless to say, one can take this abstraction leap as many times as necessary, hence if you did not succeed at enjoying a feeling in the first leap, you might enjoy it in the second leap, or the 3rd leap, or the nth leap for a sufficiently large n. The journey through the Ph.D. has left nothing but joy in the ultimate (or higher) instance.

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1 Introduction

1.1 Aim, Motivation, and Historical Context

Bialgebroids were defined by Takeuchi in [Tak77] as an alternative to the existing theory of $\times$-bialgebras over a commutative algebra due to Sweedler [Swe74], so as to allow a non-commutative base algebra as well. Almost twenty years later in [DS04], Day and Street used 2-dimensional category theory to prove that Takeuchi's bialgebroids and small categories share a common theoretical framework which they call *quantum categories*. It is the main goal of the present thesis to extend aspects of the existing theory of bialgebroids to a more general context, which in particular includes that of quantum categories.

While bialgebras over a commutative ring k consist of a k-algebra and a k-coalgebra interacting in an appropriate way, the elementary description of a bialgebroid from the viewpoint of classical ring and module theory is quite elaborate. If R is a (not necessarily commutative) k-algebra, the data for a R-bialgebroid consists of a k-module B together with suitably compatible R-coring and $(R^\circ \otimes R)$-ring structures; i.e. a comonoid in $R\text{-}\mathsf{Mod}\text{-}R$ and a monoid in $(R^\circ \otimes R)\text{-}\mathsf{Mod}\text{-}(R^\circ \otimes R)$. Some of the symmetry that bialgebras have is now lost for bialgebroids; for example, in the definition of a bialgebra one may exchange the roles of the "algebra" and the "coalgebra" structures and get a bialgebra again, whereas for a bialgebroid swapping the roles of the "ring" and "coring" structures gives a different mathematical object. Notice that there are four R-actions on the same k-module B for which even choosing an adequate notation is not simple and each author does it in a different way.

In the early 2000's Szlachányi made significant contributions towards a simpler description of a bialgebroid based on the work by [Moe02] and [McC02] on opmonoidal monads, and later developing some categorical tools himself; namely *skew monoidal categories*.

Theorem 1.1.1. *For a k-algebra R the following are equivalent,*

(i). A right R-bialgebroid. (original definition $\times_R$-bialgebra [Tak77, Section 4]).

(ii). An $(R^\circ \otimes R)$-ring B for which the category of right B-modules has a monoidal structure such that the forgetful functor is strong monoidal [Sch98, Theorem 5.1].

(iii). A cocontinuous opmonoidal monad on the category R-Mod-R [Szl03, Section 4.2].

(iv). A monoid in a monoidal category of coalgebroids [Szl05, Section 2.1].

(v). A closed right skew monoidal structure on the category Mod-R with skew unit R [Szl12, Theorem 9.1].

Motivated by the work of Szlachányi the Australian school of category theory gives a similar account of Theorem 1.1.1 but in a bicategorical language instead, with the concept of quantum categories for a monoidal category $\mathcal{V}$ taking the place where bialgebroids are. The fact that quantum categories in $\mathcal{V} = \mathsf{Set}$ are small categories, and quantum categories in $\mathcal{V} = \mathsf{Vect}_k^{\mathrm{op}}$ are bialgebroids gives room for a further interpretation; apart from being the non-commutative generalisation of bialgebras, bialgebroids are the several-object generalisation of bialgebras in the same way that a category is the several-object generalisation of a monoid. The following construction builds up more of that intuition.

Example 1.1.2 (The monoid algebra and the category algebroid). Let k be a commutative ring; for every set X one may construct the free k-vector space $k\langle X\rangle$ with X as the basis. Since Set is a cartesian monoidal category, every set has a unique comonoid structure. And this comonoid structure is preserved by the free functor since it is strong braided monoidal, thus $k\langle X\rangle$ is automatically a coalgebra. Similarly for a monoid M we obtain a bialgebra $k\langle M\rangle$, and for a group G we obtain a Hopf algebra $k\langle G\rangle$. Now, if C is a category with a finite set of objects C_0 and set of arrows C_1,

$$\begin{tikzcd} C_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & C_0 \arrow[l, "\mathrm{id}" description] \end{tikzcd}$$

then $k\langle C_1\rangle$ becomes a k-algebra, with product defined on generators by the composition of C if the arrows are composable or 0 otherwise; and with unit the finite sum $\Sigma\,\mathrm{id}_c$ over the set of objects c of C. The identities function id makes $k\langle C_0\rangle$ a commutative subalgebra of $k\langle C_1\rangle$. The source and target functions also induce k-algebra morphisms, thus by restriction of scalars $k\langle C_1\rangle$ inherits four $k\langle C_0\rangle$-module structures, two by the source and two by the target. These module structures make $k\langle C_1\rangle$ into a $k\langle C_0\rangle$-bialgebroid with coproduct taken by duplication $f \longmapsto f \otimes f$ and counit $f \longmapsto 1_k$.

Quantum categories are defined for a monoidal category $\mathcal{V}$, but within the context of the bicategory $\mathsf{Comod}(\mathcal{V})$ of comonoids in $\mathcal{V}$, two sided comodules between them, and their morphisms. The horizontal composition in this bicategory is defined as the tensor product of comodules over comonoids, to ensure this always exists one requires $\mathcal{V}$ to have all equalisers of coreflexive pairs. And if $\mathcal{V}$ is symmetric monoidal then $\mathsf{Comod}(\mathcal{V})$ is a monoidal bicategory with tensor product taken as the underlying product in $\mathcal{V}$. There is a notion of duality amongst the objects of $\mathsf{Comod}(\mathcal{V})$; for each comonoid R there is a comonoid R° obtained by reversing the comultiplication rule of R. These comonoids come equipped with "unit" and "counit" comodules

$$n : I \longrightarrow R^\circ \otimes R \qquad\qquad e : R \otimes R^\circ \longrightarrow I$$

both of which have R as the underlying object and whose actions are the left and right regular actions with respect to R° and R as pictured above. Furthermore, these comodules satisfy the triangle identities in $\mathsf{Comod}(\mathcal{V})$ up to coherent isomorphism. This concept is that of a biduality, and because every object R has a right bidual R° we say that $\mathsf{Comod}(\mathcal{V})$ is right autonomous. Bidualities induce a monoidal product on the object $R^\circ \otimes R$ given by $1 \otimes e \otimes 1 : R^\circ \otimes R \otimes R^\circ \otimes R \longrightarrow R^\circ \otimes R$ and together with the unit n these satisfy the associative and unit laws up to coherent isomorphism. We call this structure the enveloping monoidale of a biduality.

Theorem 1.1.3. *Let $\mathcal{V}$ be a braided monoidal category which has all equalisers of coreflexive pairs, and in which these are preserved by tensoring with objects on either side. For a comonoid R in $\mathcal{V}$ the following are equivalent,*

(i). A quantum category *over R in $\mathcal{V}$ (original definition [DS04, Section 12]); that is a comonad on $R^\circ \otimes R$ in* $\mathsf{Comod}(\mathcal{V})$ *for which the coEilenberg-Moore object has a monoidal structure such that the forgetful arrow is strong monoidal.*

(ii). A monoidal comonad on the enveloping monoidale $R^\circ \otimes R$ in $\mathsf{Comod}(\mathcal{V})$ *[DS04, Proposition 3.3].*

(iii). A left skew monoidal structure on R in $\mathsf{Comod}(\mathcal{V})$ *such that the skew unit is coopmonadic [LS12, Theorem 6.4].*

Each of these equivalences is of great mathematical value, (i)$\Leftrightarrow$(ii) follows from a generalisation of [Moe02, Theorem 7.1] found in [DS04, Lemma 3.2]. It states that for every monad t on a monoidale in a monoidal bicategory $\mathcal{M}$ with Eilenberg-Moore objects for monads, opmonoidal monad structures on t are in bijection with monoidal structures on the Eilenberg-Moore object of t for which the forgetful arrow is strong monoidal. The logical equivalence (i)$\Leftrightarrow$(iii) needs to be handled carefully since much more of the internal structure of $\mathsf{Comod}(\mathcal{V})$ is required, in particular, the structure related to coopmonadic adjunctions. This is addressed further in the next paragraph.

Theorem 1.1.3 is the starting point of this thesis, but we will rather consider its dual statement and for a monoidal bicategory $\mathcal{M}$ taking the role of $\mathsf{Mod}(\mathcal{V})$; this is Theorem 1.1.4 below. In this way, there is less notation and structure to keep track of during the proofs. After this switch of perspective, instead of talking about coopmonadic adjunctions we now talk about opmonadic adjunctions. An opmonadic adjunction in $\mathcal{M}$ (or adjunction of Kleisli-type) is an adjunction with a universal property; in particular, it is an initial adjunction amongst those that have the same associated monad. In the case of Cat, opmonadic adjunctions are those determined by the Kleisli category of algebras for a monad [Mac97, Theorem IV.5.3]. In $\mathsf{Mod}(\mathcal{V})$ opmonadic adjunctions behave quite well; for example, the unit arrow $i : I \longrightarrow R$

of a monoid R in $\mathcal{V}$ induces an opmonadic adjunction as shown.

$$\begin{tikzcd} R \arrow[d, bend left, "i^*"] \\ I \arrow[u, bend left, "i"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

The universal property of this adjunction translates between two descriptions of left-R right-X modules: as arrows $R \longrightarrow X$ in $\mathsf{Mod}(\mathcal{V})$, or as right X-modules A together with a left R-action $R \otimes A \longrightarrow A$ in $\mathcal{V}$.

In general, the monoidal bicategory $\mathcal{M}$ has to satisfy some mild conditions, some of which we collect under the name of *opmonadic-friendly monoidal bicategories* defined in Section 2.2, this is only another way of saying that opmonadic adjunctions behave well with respect to the tensor product and composition. An autonomous monoidal bicategory has right biduals as well as left biduals for every object.

Theorem 1.1.4. *Let $\mathcal{M}$ be an opmonadic-friendly autonomous monoidal bicategory with Eilenberg-Moore objects for monads and an opmonadic adjunction as shown,*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \\ I \arrow[u, bend left, "i_\circ"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

then the following are equivalent:

(i). A monoidale B and a monadic and strong monoidal arrow $B \longrightarrow R^\circ \otimes R$.

(ii). An opmonoidal monad on an enveloping monoidale $R^\circ \otimes R \longrightarrow R^\circ \otimes R$.

(iii). A right skew monoidal structure with skew unit $i : I \longrightarrow R$ the opposite of i°.

Where (i)⇔(ii) is [DS97, Proposition 3.3], and (ii)⇔(iii) is [LS12, Theorem 5.2].

1.2 Structure

We mentioned that bialgebroids are in bijection with monoidal structures on the category of modules over the underlying $(R^\circ \otimes R)$-ring. It is natural to ask if a similar situation holds for the category of comodules. And there are a few things that are known; it is true for a k-coalgebra that bialgebra structures are in bijection with monoidal structures on the category of comodules of the coalgebra. This is because it is possible to see k-coalgebras as coalgebras for a comonad. But in the case of bialgebroids it is only known that the category of comodules has a monoidal structure.

Now, right comodules for an R-bialgebroid are defined as right comodules in $\mathsf{Mod}\text{-}R$ for the underlying R-coring. Hai proves in [Hai08, Lemma 1.4.1] that these comodules bear an extra left R-module structure which he then uses to tensor them together over R to form a monoidal structure. This extra left R-module structure does not need to exist for comodules over an arbitrary R-coring, but it does for what is called an $R|R$-coalgebroid. Coalgebroids were defined by Takeuchi in [Tak87, Definition 3.5], in a slightly more general form; for two k-algebras R and S, an $R|S$-coalgebroid is a module in $(R^\circ \otimes R)$-Mod-$(S^\circ \otimes S)$ with some further structure which in particular includes an underlying S-coring. In [Szl05] Szlachányi proved that these $R|S$-coalgebroids are the arrows of a bicategory whose monads are R-bialgebroids. Thus, from this point of view the more complicated part in the definition of a bialgebroid rests within the coalgebroid. In the first part of Chapter 2 we explore the theory of coalgebroids but in the generalised context of a monoidal bicategory $\mathcal{M}$, hence what we really study are opmonoidal arrow between enveloping monoidales: in the case $\mathcal{M} = \mathsf{Mod}(\mathsf{Vect}_k)$ such opmonoidal arrows are coalgebroids. We prove a theorem similar to Theorem 1.1.4 where in place of right skew monoidales a new structure appears, called *oplax right action*. These oplax right actions are a notion of action with respect to a right skew monoidale, where the associative and unit laws are witnessed by cells that are not necessarily invertible, and satisfy further coherence conditions.

Theorem 1.2.1. *Let $\mathcal{M}$ be an opmonadic-friendly autonomous monoidal bicategory and let $i^\circ \dashv i^\circ$ be an opmonadic adjunction as shown,*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \\ I \arrow[u, bend left, "i_\circ"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

then the following are equivalent:

(i). An opmonoidal arrow between enveloping monoidales $R^\circ \otimes R \longrightarrow S^\circ \otimes S$.

(ii). An oplax right action $S \otimes R \longrightarrow S$, with respect to the skew monoidal structure on R corresponding to the identity opmonoidal monad on $R^\circ \otimes R$ under 1.1.3.

Furthermore, these structures have the same underlying comonad on S.

With this theorem we provide a simpler description of a coalgebroid in the language of classical ring and module theory in Example 2.3.13 which involves only three module structures instead of four, none of which involve algebras with the reversed multiplication.

Theorem 1.2.2. *For two k-algebras R and S the following are equivalent,*

(i). An $R|S$-coalgebroid [Tak87, Original definition 3.5].

(ii). A cocontinuous opmonoidal functor R-Mod-$R \longrightarrow S$-Mod-S.

(iii). A closed oplax right $(\mathsf{Mod}\text{-}R)$*-actegory* $(\mathsf{Mod}\text{-}S) \times (\mathsf{Mod}\text{-}R) \longrightarrow \mathsf{Mod}\text{-}S$.

(iv). A module in $(S \otimes R)$-Mod-S equipped with morphisms $\delta : C \longrightarrow C \otimes_S C$ and $\varepsilon : C \longrightarrow S$ subject to the equations given in Example 2.3.13.

For the rest of Chapter 2 we focus on comodules for the different versions of coalgebroids we developed so far, and apply the same technique as before to show equivalences between them. This procedure generalises Hai's lemma which induces the extra left R-module structure on a comodule for a coalgebroid. At the same time it provides us with three equivalent ways of describing comodules for a coalgebroid, depending on the notion of coalgebroid that we decide to use, see Corollary 2.4.15.

Theorem 1.2.3. *Let $\mathcal{M}$ be an opmonadic-friendly autonomous monoidal bicategory and let $i \dashv i^*$ be an opmonadic adjunction whose dual $i_\circ \dashv i^\circ$ is opmonadic too.*

$$\begin{tikzcd} R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i"] \end{tikzcd} \qquad \begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

Fix a structure of each item in Theorem 1.2.1; then the following are equivalent:

(i). A comodule $R \longrightarrow S$ for the opmonoidal arrow $R^\circ \otimes R \longrightarrow S^\circ \otimes S$ between enveloping monoidales.

(ii). A morphism of oplax right actions $R \longrightarrow S$ from i^*1 into the oplax right action.

(iii). A comodule for the underlying comonad $S \longrightarrow S$.

We may bring this down to the language of classical ring and module theory:

Theorem 1.2.4. *For two k-algebras R and S fix a structure in each item of Theorem 1.1.1, the following are equivalent,*

(i). A comodule for the $R|S$-coalgebroid.

(ii). A comodule for the cocontinuous opmonoidal functor $R\text{-}\mathsf{Mod}\text{-}R \longrightarrow S\text{-}\mathsf{Mod}\text{-}S$.

(iii). An oplax right $(\mathsf{Mod}\text{-}R)$-actegory oplax morphism $(\mathsf{Mod}\text{-}S) \times (\mathsf{Mod}\text{-}R) \longrightarrow \mathsf{Mod}\text{-}S$.

We finish the chapter by showing that if the opmonoidal arrow is an opmonoidal monad then the category of comodules has a monoidal structure such that the forgetful functor is strong monoidal. So in particular we can say that the category of comodules for a quantum category is monoidal.

For Chapter 3 we analyse how these oplax actions take the place of right skew monoidales in the equivalences from Theorems 1.1.4 and 1.2.1. Now, since opmonoidal monads in a monoidal bicategory $\mathcal{M}$ are monads in the bicategory $\mathsf{OpMon}\,\mathcal{M}$ we expect right skew monoidales to be "monads of oplax actions", and this is the motto that motivates the whole chapter. Unfortunately, to define a "monad of oplax actions" is not as straightforward as it seems. The reason is that, for an arbitrary monoidal bicategory $\mathcal{M}$, there is no horizontal composition of oplax actions that we are aware of, hence no bicategory of oplax actions. To get around this problem we fit oplax actions as the 1-simplices of a simplicial object in Cat. Then the 2-simplices may be thought of as encoding generalised horizontal composites of oplax actions;

and this is enough to define monads. We prove that these simplicial-style monads of oplax actions are in bijection with right skew monoidales in $\mathcal{M}$ whose unit has a right adjoint, with no extra assumptions required on the monoidal bicategory $\mathcal{M}$.

Now, simplicial objects in Cat are organised in a 2-category $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$, which apart from the usual notions of equality, isomorphism, and equivalence that exist in any 2-category, there is a notion of *weak equivalence*. A weak equivalence consists of a 2-natural transformation $F : \mathbb{X} \longrightarrow \mathbb{Y}$ whose components are all equivalences of categories $F_n : \mathbb{X}_n \longrightarrow \mathbb{Y}_n$, but the collection of their pseudoinverses satisfies the naturality condition only up to coherent isomorphism, hence constituting a pseudonatural transformation. In the same way that there is a nerve construction which assigns to each category a simplicial set, there are many different nerve-like constructions that assign to each bicategory a simplicial object in Cat or in Set. These are studied in [CCG10], but we are only interested in what we call the lax-2-nerve.

We conclude this chapter by showing how, when $\mathcal{M}$ satisfies the hypothesis of Theorem 1.2.1, our simplicial object in Cat of oplax actions is weakly equivalent to the lax-2-nerve of a bicategory consisting of opmonoidal arrows on enveloping monoidales in $\mathcal{M}$.

In Chapter 4 we turn our attention back to the equivalence between bialgebroid structures on an $(R^\circ \otimes R)$-ring and monoidal structures on the category of modules of the $(R^\circ \otimes R)$-ring. We mentioned at least two generalisations of this fact, one by Moerdijk in [Moe02], which was later described in the context of 2-dimensional category theory by McCrudden in [McC02], reads as follows. The 2-category OpMon of monoidal categories, opmonoidal functors, and opmonoidal natural transformations has Eilenberg-Moore objects for monads. The second generalisation, by Day and Street in [DS04], jumps from Cat to a monoidal bicategory $\mathcal{M}$ and replaces OpMon by $\mathsf{OpMon}\,\mathcal{M}$ the bicategory of monoidales in $\mathcal{M}$, opmonoidal arrows and opmonoidal cells between them. So, it asserts that if $\mathcal{M}$ has Eilenberg-Moore objects for monads then $\mathsf{OpMon}\,\mathcal{M}$ does too.

We prove a multiobject version of both of these theorems. Thus, for example, we replace monoidal categories with bicategories, hence, we require a 2-category with bicategories as objects. In [Lac10b], Lack exhibits such a 2-category of bicategories $\mathsf{Bicat}_{\mathbf{2}}$. It has the property that the full sub-2-category consisting of the one object bicategories is the 2-category Mon of monoidal categories, monoidal functors, and monoidal natural transformations. With little effort we exhibit a similar 2-category that we call Icon, which contains OpMon as the full sub-2-category on the one object bicategories. From here, it does not take much more to generalise the Day-Street version internal to a monoidal bicategory $\mathcal{M}$ to the "multiobject case" once we know what a "multiobject monoidale in $\mathcal{M}$" ought to be. This is covered by the theory of bicategories enriched in a monoidal bicategory $\mathcal{M}$, whose first appearances may be traced back to [Car95]; Garner and Shulman give an excellent account on this topic in [GS16]. So we prove that there exists a bicategory $\mathsf{Icon}\,\mathcal{M}$ of $\mathcal{M}$-enriched bicategories whose full subbicategory determined by the one object $\mathcal{M}$-enriched bicategories is $\mathsf{OpMon}\,\mathcal{M}$. And if $\mathcal{M}$ has Eilenberg-Moore objects for monads, then $\mathsf{Icon}\,\mathcal{M}$ does too.

1.3 Methodology

Apart from the existing division into chapters this thesis has two parts in terms of methodology; the first part comprises Chapters 2 and 3 in which we work in a fixed monoidal bicategory $\mathcal{M}$. The second part consists only of Chapter 4, where we use 3-dimensional category theory methods to prove a statement in 2-dimensional category theory.

1.4 Background, Notation, and Conventions

We assume the axiom of choice for large sets such as the collection of objects of a monoidal bicategory. This allows us to turn certain constructions into pseudofunctors, e.g. $(\_)^\circ$ after Lemma 2.1.23 and EM before Proposition 4.2.7.

1.4.1 Ring and Module Theory

We use the letter k to denote a field or a commutative ring. All k-algebras are associative and unital but not necessarily commutative. For a k-algebra R the k-algebra obtained by reversing the order of the multiplication rule is denoted by R°. For two k-algebras R and S we denote by ${}_RM_S$ a left-R right-S module M. Tensor product over k-algebra R is denoted by $\otimes_R$, the tensor product over the base k is denoted by the tensor symbol $\otimes$ with no decorations. We use Sweedler's notation for the image of an element $c \in C$ under a module morphism $\delta : C \longrightarrow C \otimes C$.

$$\delta(c) = \sum_c c_{(1)} \otimes c_{(2)}$$

1.4.2 Category Theory

Category theory is assumed throughout. We use [Mac97] as the standard reference, but nowadays, there are plenty of resources on the matter. We refer to the data of a category as: objects, arrows, composition, and identities. We reserve the word "morphism" for arrows that preserve some algebraic structure.

R-Mod-S The category of left-R, right-S two-sided modules, and module morphisms between them, for two k-algebras R and S.

ΣM The suspension of a monoid M. This category is obtained by adding a new 0th dimension: ΣM has a single object $\star$ whose hom set is M, composition is the multiplication of M, and identity is the unit element of M.

1.4.3 $1\frac{1}{2}$-dimensional Category Theory: Monoidal Categories

A monoidal category is a category $\mathcal{V}$ with a unit object I and a tensor product functor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ which is associative and unital up to isomorphism. These isomorphisms satisfy further axioms called coherence axioms. The original definition by Mac Lane in [ML63] included five axioms, and later, Kelly proved in [Kel64] that only two of them are required.

According to the microcosm principle [BD98] monoidal categories are the correct context to define monoids in a category. A *monoid* is an object M together with a multiplication $M \otimes M \longrightarrow M$ and a unit $I \longrightarrow M$ which are associative and unital. The dual notion is that of *comonoid*. An object X in a monoidal category $\mathcal{V}$ has a *right dual* X°, abbreviated by $X \dashv X^\circ$, if there are unit and counit arrows $n : I \longrightarrow X^\circ \otimes X$ and $e : X \otimes X^\circ \longrightarrow I$ which satisfy the triangle identities. The object $X^\circ \otimes X$ has a monoid structure with multiplication $1 \otimes e \otimes 1 : X^\circ \otimes X \otimes X^\circ \otimes X \longrightarrow X^\circ \otimes X$ and unit n. We refer to this monoid as the *enveloping monoid* of $X \dashv X^\circ$. If every object in $\mathcal{V}$ has a right dual, we say that $\mathcal{V}$ is a *right autonomous monoidal category*. A *right* $\mathcal{V}$*-actegory* consists of a category $\mathcal{A}$ together with a right action functor $\mathcal{A} \times \mathcal{V} \longrightarrow \mathcal{A}$ which is associative and unital up to coherent isomorphism, see [JK01].

Set The monoidal category of sets and functions. The monoidal product is the cartesian product of sets and the monoidal unit is the terminal object. Monoids in Set are monoids in the classical sense.

Vect_k The monoidal category of k-vector spaces and linear maps. The monoidal product is the tensor product over k and the monoidal unit is the commutative ring k seen as a vector space over itself. Thus, in this class of categories the category of abelian groups is included $\mathsf{Ab} := \mathsf{Vect}_{\mathbb{Z}}$. Monoids in Vect_k are k-algebras, and $\mathbb{Z}$-algebras are rings.

R-Mod-R The monoidal category of two sided R-modules for a k-algebra R. The monoidal product is the tensor product over R and the monoidal unit is the k-algebra R regarded as a module over itself. Monoids in this monoidal category are called R-rings, and comonoids R-corings.

1.4.4 2-dimensional Category Theory: 2-Categories and Bicategories

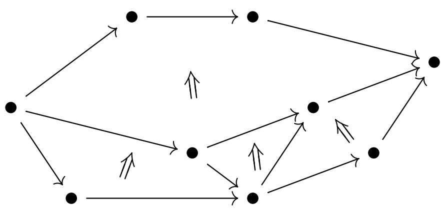
For a more detailed account of 2-dimensional category theory we refer the reader to [Bén67], [KS74], and [Lac10a]. Bicategories were first defined by Bénabou in [Bén67], we denote them with letters $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. A *bicategory* $\mathcal{B}$ consists of several pieces of data: a collection of objects $\mathsf{Ob}\,\mathcal{B}$; a hom category $\mathcal{B}(X, Y)$ for every pair of objects X and Y, which for short we denote by $\mathcal{B}_{X,Y}$ when needed; a composition functor $m : \mathcal{B}_{Y,Z} \times \mathcal{B}_{X,Y} \times \longrightarrow \mathcal{B}_{X,Z}$ for every triple of objects X, Y, and Z; an identity functor $u : I \longrightarrow \mathcal{B}_{X,X}$ for every object X; a natural transformation for every quadruple of objects W, X, Y, and Z, called the associator as shown below;

$$\begin{tikzcd}
\mathcal{B}_{Y,Z} \times \mathcal{B}_{X,Y} \times \mathcal{B}_{W,X} \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] \arrow[dr, phantom, "\cong"] & \mathcal{B}_{X,Z} \times \mathcal{B}_{W,X} \arrow[d, "m"] \\
\mathcal{B}_{X,Z} \times \mathcal{B}_{W,X} \arrow[r, "m"'] & \mathcal{B}_{W,Z}
\end{tikzcd}$$

and two invertible natural transformations for every pair of objects X and Y called the left unitor and the right unitor as shown below.

$$\begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, "u\times 1"] \arrow[dr, "1"'] & \mathcal{B}_{Y,Y}\times\mathcal{B}_{X,Y} \arrow[d, "m"] \arrow[dl, phantom, "\cong"] \\ & \mathcal{B}_{X,Y} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_{X,Y}\times\mathcal{B}_{X,X} \arrow[d, "m"'] \arrow[dr, phantom, "\cong"] & \mathcal{B}_{X,Y} \arrow[l, "1\times u"'] \arrow[dl, "1"] \\ \mathcal{B}_{X,Y} & \end{tikzcd}$$

These are subject to two axioms called the pentagon axiom and the triangle axiom, similar to those for monoidal categories. The objects and arrows of the hom categories $\mathcal{B}(X,Y)$ are called arrows and cells of the bicategory $\mathcal{B}$. We use the plain term “cell” instead of the standard term “2-cell” to avoid referring to two distinct cells as “two 2-cells”. This shall cause no confusion since we do not use higher cells, except in the last section of Chapter 4 where we explicitly call them by the structure they have. A *2-category* is a bicategory whose associator, left unitor and right unitor are the identity natural transformation. Hence, horizontal composition is strictly associative and unital. *Pasting diagrams* are pictorial representations of vertical and horizontal composites of cells in a bicategory $\mathcal{B}$ that ignore all instances of the associator, left unitor, and right unitor. See [Pow90] and [Ver92, Apendix A] for more details, an example is given below.



A priori, pasting diagrams do not have a concrete meaning as a cell within the bicategory $\mathcal{B}$ unless one explicitly states how to place all necessary parenthesis for horizontal composites of arrows and all necessary instances of the the associator, left unitor, and right unitor. But in [MLP85] Mac Lane and Paré proved the coherence of composites of cells involving only instances of the associator and the left and right unitors. This implies that there is a unique way to interpret pasting diagrams once we choose a convention to parenthesise the source and the target arrows [Ver92, Appendix A]. Thus, for any given pasting diagram we assume that its source has the leftmost bracketing and its target has the rightmost bracketing, although the reader is free to use their own favourite convention. Hence all pasting diagrams and proofs are written as if $\mathcal{B}$ was a 2-category. Empty regions of pasting diagrams are assumed to be strictly commutative. Note that the symbol for composition $\circ$ is mostly avoided; this forces us to write more pasting diagrams which makes our proofs more visual. The isomorphism cells sometimes have a preferred direction which we depict with the direction of the isomorphism symbol $\cong$, so an isomorphism cell as below goes from f to f'.

$$\begin{tikzcd} A \arrow[r, bend left, "f'"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\cong"] & B \end{tikzcd}$$

As usual $\mathcal{B}^{\mathrm{op}}$ denotes the bicategory obtained by reversing the direction of the arrows in $\mathcal{B}$ but leaving the cells intact, which of course reverses the order of the horizontal composition as well; $\mathcal{B}^{\mathrm{co}}$ is obtained by reversing the direction of the cells in $\mathcal{B}$ but leaving the arrows intact, this reverses the order of vertical composition. As a pictorial guide, most of the structures that bear the prefixes op- and co- have their cell components pointing up-right, whereas their dual counterparts point down-left, although there might be some exceptions.

$\mathsf{Span}(\mathcal{C})$ The bicategory of spans in a category with pullbacks $\mathcal{C}$. Objects are the same as those of $\mathcal{C}$, arrows from A to B are spans as follows,

$$\begin{tikzcd} & E \arrow[dl, "a"'] \arrow[dr, "b"] & \\ A & & B \end{tikzcd}$$

and cells are span morphisms. Composition is taken by pullback and identities are the spans shown below.

$$\begin{tikzcd} & & E \times_B F \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\ & E \arrow[dl, "a"'] \arrow[dr, "b"] & & F \arrow[dl, "c"'] \arrow[dr, "d"] & \\ A & & B & & C \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "1"'] \arrow[dr, "1"] & \\ A & & A \end{tikzcd}$$

$\Sigma\mathcal{V}$ The suspension of a monoidal category $\mathcal{V}$. It is obtained by adding a new 0th dimension thus $\Sigma\mathcal{V}$ has a single object whose endo-hom category is $\mathcal{V}$.

Lax functors were introduced by Bénabou in [Bén67] under the name of morphisms of bicategories, we denote them with letters F and G. Given bicategories $\mathcal{B}$ and $\mathcal{C}$ a *lax functor* $F : \mathcal{B} \longrightarrow \mathcal{C}$ consists of the following pieces of data: a function $F : \mathsf{Ob}\,\mathcal{B} \longrightarrow \mathsf{Ob}\,\mathcal{C}$; a functor $F : \mathcal{B}(X, Y) \longrightarrow \mathcal{C}(FX, FY)$ for every pair of objects X and Y; a (not necessarily invertible) natural transformation

$$\begin{tikzcd} \mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \arrow[r, "FF"] \arrow[d, "m"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY} \arrow[d, "m"] \\ \mathcal{B}_{X,Z} \arrow[r, "F"'] & \mathcal{C}_{FX,FZ} \end{tikzcd}$$

for every triple of objects X, Y and Z; and a (not necessarily invertible) natural transformation

$$\begin{tikzcd} & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "\swarrow"] & \\ \mathcal{B}_{X,X} \arrow[rr, "F"'] & & \mathcal{C}_{FX,FX} \end{tikzcd}$$

for each object X. These natural transformations are subject to three coherence axioms, and are referred to as the *lax functoriality constraints* of F. A *pseudofunctor* is a lax functor whose lax functoriality constraints are invertible, in which case, we call them *pseudofunctoriality constraints*. A *normal lax functor* is a lax functor that preserves horizontal identities strictly. An *oplax functor* is defined in a similar way as a lax functor with the difference that the constraints point to the other direction, and the axioms are adjusted accordingly.

We may apply a pseudofunctor F to cells not of a globular shape, such as triangles or squares. In order to keep the same shape after applying F one needs to use its pseudofunctoriality constraints; for example, let φ be a square as shown below,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\varphi\Nearrow"] & X' \arrow[d, "t'"] \\ Y \arrow[r, "g"'] & Y' \end{tikzcd}$$

we denote by $F^{\square}\varphi$ the square obtained by precomposing and postcomposing $F\varphi$ with the appropriate instances of the pseudofunctoriality constraints of F with respect to the horizontal composition. These are depicted below as the unnamed isomorphisms.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "Ft"'] \arrow[dr, phantom, "F^{\square}\varphi\Nearrow"] & FX' \arrow[d, "Ft'"] \\ FY \arrow[r, "Fg"'] & FY' \end{tikzcd} \quad := \quad \begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "Ft"'] \arrow[dr, bend left=20] \arrow[dr, phantom, "F\varphi\Nearrow"] & FX' \arrow[d, "Ft'"] \arrow[d, phantom, "\cong"'] \\ FY \arrow[r, "Fg"'] \arrow[u, phantom, "\cong"] & FY' \end{tikzcd}$$

For triangles θ we will use $F^{\triangledown}\theta$. The same notation abbreviates the use of the pseudofunctoriality constraints of F with respect to the horizontal identities to get identity arrows on the outer edges of the diagram where possible.

Lax natural transformations were defined by Gray in [Gra69] under the name of "2-natural transformations" nowadays this terminology is used in a different way, see below. A *lax natural transformation* α between a parallel pair of pseudofunctors F and G as pictured below,

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \end{tikzcd}$$

consists the following data: an arrow $\alpha_X : FX \longrightarrow GX$ in $\mathcal{C}$ for each object X in $\mathcal{B}$, which we refer to as the component of α at X; and a (non necessarily invertible) cell α_f in $\mathcal{C}$ for each arrow $f : X \longrightarrow Y$ in $\mathcal{B}$, which we refer to as the lax naturality constraints of α.

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Swarrow\alpha_f"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

These are subject to two axioms that guarantee the compatibility of the natural transformations α_f and the pseudofunctoriality constraints of F and G. *Pseudonatural transformations* are lax natural transformations whose lax naturality constraints are invertible, in which case we call them pseudonaturality constraints. A *2-natural transformation* is a lax natural transformation whose lax naturality constraints are all identities. The structure obtained by reversing the direction of the lax naturality constraints in the definition of lax natural transformation is called an *oplax natural transformation.*

A *modification* Ξ between a parallel pair of pseudonatural transformations,

$$\begin{tikzcd} \mathcal{B} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] & \Downarrow\alpha \overset{\Xi}{\Rrightarrow} \Downarrow\beta & \mathcal{C} \end{tikzcd}$$

consists of cells Ξ_X for each object X in $\mathcal{B}$

$$\begin{tikzcd} FX \arrow[r, bend left, "\beta_X"] \arrow[r, bend right, "\alpha_X"'] \arrow[r, phantom, "\Xi_X\Uparrow"] & GX \end{tikzcd}$$

subject to an axiom relating the cells Ξ_X with the pseudonaturality constraints of α and β.

1.4.5 $2\frac{1}{2}$-dimensional Category Theory: Monoidal Bicategories

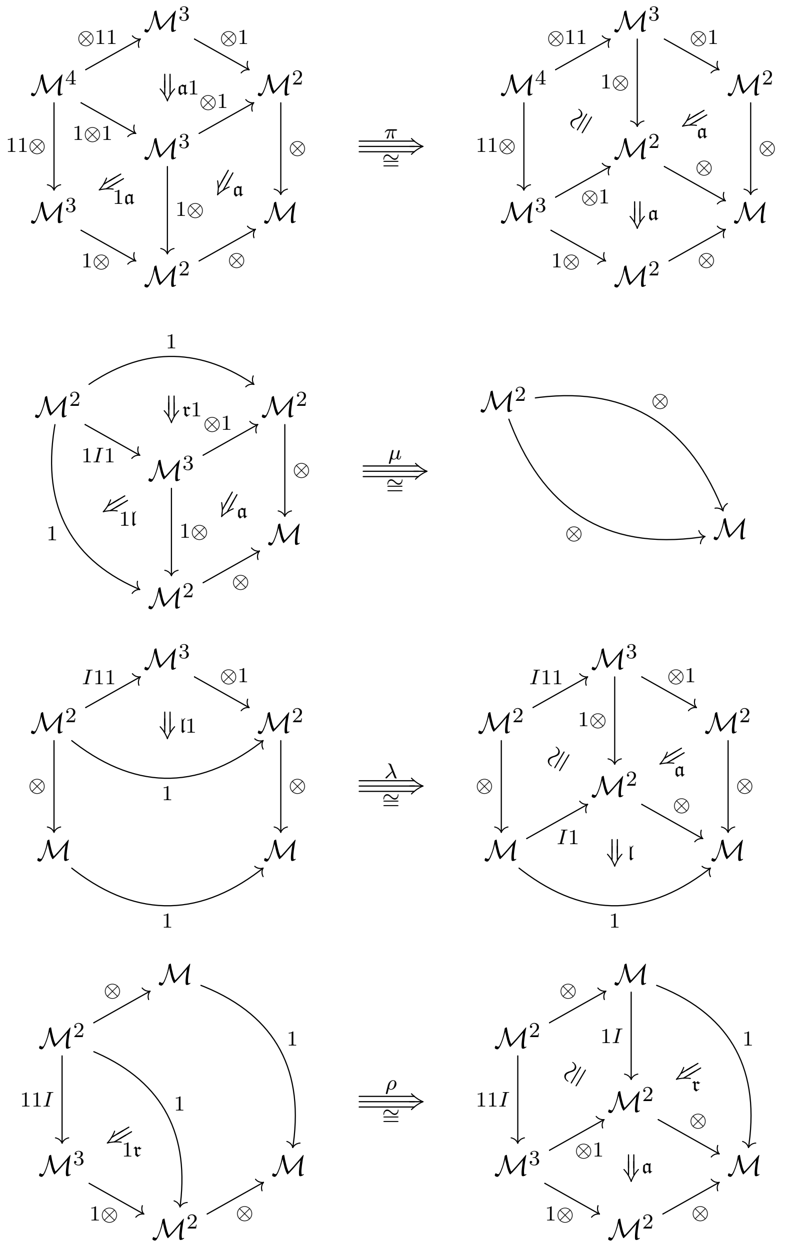
Monoidal bicategories appear in [GPS95] and [Gur13] as a particular case of the concept of tricategory. In our definition we adopt a slight variation of both approaches; we take from [GPS95] the direction of the pseudonatural transformations involved, but we keep the definition algebraic as in [Gur13] by asking for adjoint equivalences instead of mere equivalences where appropriate. Monoidal bicategories are denoted with letters $\mathcal{M}$ and $\mathcal{N}$. A *monoidal bicategory* $\mathcal{M}$ is a bicategory with a tensor product and a unit pseudofunctors;

$$\mathcal{M}\times\mathcal{M} \xrightarrow{\otimes} \mathcal{M} \qquad\qquad \mathbb{1} \xrightarrow{I} \mathcal{M}$$

an associator $\mathfrak{a}$, a left unitor $\mathfrak{l}$, and a right unitor $\mathfrak{r}$ pseudonatural transformations which are adjoint equivalences as pictured below, where for example, $\mathcal{M}^3$ is the 3-fold cartesian product $\mathcal{M}\times\mathcal{M}\times\mathcal{M}$;

$$\begin{tikzcd} \mathcal{M}^3 \arrow[r, "\otimes\times 1"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\cong\Swarrow\mathfrak{a}"] & \mathcal{M}^2 \arrow[d, "\otimes"] \\ \mathcal{M}^2 \arrow[r, "\otimes"'] & \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M} \arrow[r, "I\times 1"] \arrow[dr, "1"'] \arrow[dr, phantom, "\cong\Downarrow\mathfrak{l}"] & \mathcal{M}^2 \arrow[d, "\otimes"] & \mathcal{M} \arrow[l, "1\times I"'] \arrow[dl, "1"] \arrow[dl, phantom, "\mathfrak{r}\Uparrow\cong"] \\ & \mathcal{M} & \end{tikzcd}$$

and four invertible modifications as depicted below, where for example, $11\otimes$ is a shorthand for $1 \times 1 \times \otimes$.



Additionally, these are subject to three axioms which may be found in [GPS95, pp. 10–12]. A *monoidal 2-category* is a monoidal bicategory whose underlying bicategory is a 2-category and the tensor product is a 2-functor. Tensor product of objects, arrows, and cells in a monoidal bicategory $\mathcal{M}$ is denoted by juxtaposition. Similar to the coherence theorem for bicategories, there is a coherence theorem for monoidal bicategories, see [GPS95]. This allows us to draw all pasting diagrams in a monoidal bicategory as if it was a Gray-monoid; that is: the underlying bicategory is a 2-category; the unit and the tensor product with objects in each variable are a 2-functors; $\mathfrak{a}$, $\mathfrak{l}$, and $\mathfrak{r}$ are identities; and π, μ, λ, and ρ are identities. What remains is the interchange law between the tensor product and the horizontal composition which holds up to an isomorphism natural in f and f' that is pictured below.

$$\begin{tikzcd}
XX' \arrow[r, "f1"] \arrow[d, "1f'"'] \arrow[dr, phantom, "\cong"] & YX' \arrow[d, "1f'"] \\
XY' \arrow[r, "f1"'] & YY'
\end{tikzcd}$$

These isomorphisms are subject to three axioms: two which assert that the collection of these isomorphism squares is closed under pasting a pair of squares along one edge; and a coherence axiom as pictured below.

$$\begin{tikzcd}
& YX'X'' \arrow[dr, "1f'1"] & & & & YX'X'' \arrow[dr, "1f'1"] \arrow[dd, "11f''"] & \\
XX'X'' \arrow[ur, "f11"] \arrow[dr, "1f'1"] \arrow[dd, "11f''"'] \arrow[rr, phantom, "\cong"] & & YY'X'' \arrow[dd, "11f''"] & & XX'X'' \arrow[ur, "f11"] \arrow[dd, "11f''"'] \arrow[r, phantom, "\cong"] & \arrow[r, phantom, "\cong"] & YY'X'' \arrow[dd, "11f''"] \\
\arrow[r, phantom, "\cong"] & XY'X'' \arrow[ur, "f11"] \arrow[dd, "11f''"] \arrow[r, phantom, "\cong"] & {} & = & & YX'Y'' \arrow[dr, "1f'1"] & \\
XXY'' \arrow[dr, "1f'1"'] & & YY'Y'' & & XXY'' \arrow[ur, "f11"] \arrow[dr, "1f'1"'] \arrow[rr, phantom, "\cong"] & & YY'Y'' \\
& XY'Y'' \arrow[ur, "f11"'] & & & & XY'Y'' \arrow[ur, "f11"'] &
\end{tikzcd}$$

These axioms are used repeatedly during many of the diagram calculations throughout without explicitly recalling them every time. In general, the tensor product ff' of two arrows $f : X \longrightarrow Y$, $f' : X' \longrightarrow Y'$ in $\mathcal{M}$ may have two possible meanings, namely the source and the target of the isomorphism above. Here ff' always means the following composite.

$$\begin{tikzcd}
XX' \arrow[rr, "ff'"] \arrow[dr, "1f'"'] & & YY' \\
& XY' \arrow[ur, "f1"'] &
\end{tikzcd}$$

Monoidal bicategories have an extra duality; $\mathcal{M}^{\mathrm{rev}}$ is the monoidal bicategory obtained by reversing the order of the tensor product in $\mathcal{M}$. All duality operations commute pairwise and so stacking the superscripts in front of each other as in $\mathcal{M}^{\mathrm{co\ op}}$, $\mathcal{M}^{\mathrm{rev\ op}}$, $\mathcal{M}^{\mathrm{rev\ co}}$, $\mathcal{M}^{\mathrm{rev\ co\ op}}$ should not cause any confusion.

Cat The monoidal 2-category of categories, functors, and natural transformations. The monoidal product is the cartesian product and the monoidal unit is $\mathbb{1}$ the terminal category.

Mod_k The monoidal bicategory of k-algebras, two-sided modules over them, and morphisms of two-sided modules. Their hom categories are the categories of modules defined above $\mathsf{Mod}_k(R, S) = R\text{-}\mathsf{Mod}\text{-}S$. We appropriately call two-sided modules in $R\text{-}\mathsf{Mod}\text{-}S$, modules from R to S [Str07]. Horizontal composition is given by tensor product over k-algebras, while horizontal identities are the two-sided regular modules ${}_RR_R$ of a k-algebra R. The monoidal structure of Mod_k is given by the tensor product over the base k. The monoidal bicategory Mod_k should not to be confused with Vect_k the monoidal category of vector spaces (or modules) over k.

$\mathsf{Mod}(\mathcal{V})$ The monoidal bicategory $\mathsf{Mod}(\mathcal{V})$ of monoids, two sided modules between them, and their morphisms in $\mathcal{V}$. For $\mathcal{V}$ a symmetric monoidal category such that all coequalisers of reflexive pairs exist and are preserved by tensoring on both sides with an object. Of course if $\mathcal{V} = \mathsf{Vect}_k$ then $\mathsf{Mod}(\mathcal{V}) = \mathsf{Mod}_k$.

$\mathsf{Span}(\mathcal{C})$ The monoidal bicategory of spans in a category $\mathcal{C}$ with finite limits. The monoidal product is taken component-wise as the binary product in $\mathcal{C}$, the monoidal unit is the terminal object of $\mathcal{C}$.

$$\begin{tikzcd} & EE' \arrow[dl, "aa'"'] \arrow[dr, "bb'"] & \\ AA' & & BB' \end{tikzcd}$$

Prof The monoidal bicategory of profunctors (also called distributors or modules). Objects are categories, arrows are profunctors, and cells are morphisms of profunctors. A profunctor $\mathcal{C} \longrightarrow \mathcal{D}$ is a functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Set}$, natural transformations between them are morphisms of profunctors. Vertical composition is calculated as the composition of natural transformations. The horizontal composition of profunctors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{E}$ is given by the coend formula, $(G \circ F)(e, c) := \int^{d \in \mathcal{D}} F(d, c) \times G(e, d)$. The identity on a category $\mathcal{C}$ is the hom functor $\mathcal{C}(\_,\_) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Set}$. The monoidal unit is the terminal category $\mathbb{1}$. The monoidal product is given on objects by the cartesian product of categories, while on arrows the monoidal product of two profunctors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $F' : \mathcal{C}' \longrightarrow \mathcal{D}'$ is the composite below,

$$\mathcal{D}^{\mathrm{op}} \times \mathcal{D}'^{\mathrm{op}} \times \mathcal{C} \times \mathcal{C}' \xrightarrow{1 \times \mathsf{twist} \times 1} \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{D}'^{\mathrm{op}} \times \mathcal{C}' \xrightarrow{F \times F'} \mathsf{Set} \times \mathsf{Set} \longrightarrow \mathsf{Set}$$

where the last functor sends a pair of sets to their cartesian product.

A *monoidal pseudofunctor* $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ between a pair of monoidal bicategories $\mathcal{M}$ and $\mathcal{N}$ is a pseudofunctor $\mathcal{F}$ together with the following data: two pseudonatural transformations $\mathcal{F}_2$ and $\mathcal{F}_0$ (not necessarily adjoint equivalences) as pictured below, referred to as the monoidal constraints of $\mathcal{F}$;

$$\begin{tikzcd} \mathcal{M}^2 \arrow[r, "\mathcal{F}\mathcal{F}"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\swarrow \mathcal{F}_2"] & \mathcal{N}^2 \arrow[d, "\otimes"] \\ \mathcal{M} \arrow[r, "\mathcal{F}"'] & \mathcal{N} \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "I"'] \arrow[dr, "I"] & \\ \mathcal{M} \arrow[rr, "\mathcal{F}"'] \arrow[urr, phantom, "\swarrow \mathcal{F}_0"] & & \mathcal{N} \end{tikzcd}$$

and three invertible modifications ω, δ, and γ as follows.

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{N}^3 \arrow[dr, "\otimes 1"] \arrow[dd, phantom, "\Downarrow \mathcal{F}_2\mathcal{F}"] & \\
\mathcal{M}^3 \arrow[ur, "\mathcal{F}\mathcal{F}\mathcal{F}"] \arrow[dd, "1\otimes"'] \arrow[dr, "\otimes 1"] & & \mathcal{N}^2 \arrow[dd, "\otimes"] \\
 & \mathcal{M}^2 \arrow[ur, "\mathcal{F}\mathcal{F}"'] \arrow[dd, "\otimes"] \arrow[l, phantom, "\Leftarrow \mathfrak{a}"] \arrow[r, phantom, "\Leftarrow \mathcal{F}_2"] & \\
\mathcal{M}^2 \arrow[dr, "\otimes"'] & & \mathcal{N} \\
 & \mathcal{M} \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}
\quad \overset{\omega}{\underset{\cong}{\Rrightarrow}} \quad
\begin{tikzcd}[column sep=small]
 & \mathcal{N}^3 \arrow[dr, "\otimes 1"] \arrow[dd, "1\otimes"'] & \\
\mathcal{M}^3 \arrow[ur, "\mathcal{F}\mathcal{F}\mathcal{F}"] \arrow[dd, "1\otimes"'] \arrow[r, phantom, "\mathcal{F}\mathcal{F}^2 \Leftarrow"] & \arrow[r, phantom, "\Leftarrow \mathfrak{a}"] & \mathcal{N}^2 \arrow[dd, "\otimes"] \\
 & \mathcal{N}^2 \arrow[dr, "\otimes"] \arrow[dd, phantom, "\Downarrow \mathcal{F}^2"] & \\
\mathcal{M}^2 \arrow[ur, "\mathcal{F}\mathcal{F}"'] \arrow[dr, "\otimes"'] & & \mathcal{N} \\
 & \mathcal{M} \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{N} \arrow[dddr, bend left, "1"] & \\
\mathcal{M} \arrow[ur, "\mathcal{F}"] \arrow[dd, "1I"'] \arrow[dddr, bend left, "1"] & & \\
 \arrow[r, phantom, "\Leftarrow \mathfrak{r}"] & & \\
\mathcal{M}^2 \arrow[dr, "\otimes"'] & & \mathcal{N} \\
 & \mathcal{M} \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}
\quad \overset{\delta}{\underset{\cong}{\Rrightarrow}} \quad
\begin{tikzcd}[column sep=small]
 & \mathcal{N} \arrow[dd, "1I"'] \arrow[dddr, bend left, "1"] & \\
\mathcal{M} \arrow[ur, "\mathcal{F}"] \arrow[ddd, "1I"'] \arrow[r, phantom, "\mathcal{F}\mathcal{F}^0 \Leftarrow"] & \arrow[r, phantom, "\Leftarrow \mathfrak{r}"] & \\
 & \mathcal{N} \arrow[dr, "\otimes"] \arrow[dd, phantom, "\Downarrow \mathcal{F}^2"] & \\
\mathcal{M}^2 \arrow[ur, "\mathcal{F}\mathcal{F}"'] \arrow[dr, "\otimes"'] & & \mathcal{N} \\
 & \mathcal{M} \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{N} \arrow[dr, "I1"] \arrow[d, phantom, "\Downarrow \mathcal{F}^0\mathcal{F}"] & \\
\mathcal{M} \arrow[ur, "\mathcal{F}"] \arrow[dr, "I1"] \arrow[ddr, bend right, "1"'] & \, & \mathcal{N}^2 \arrow[dd, "\otimes"] \\
 & \mathcal{M}^2 \arrow[ur, "\mathcal{F}\mathcal{F}"'] \arrow[d, "\otimes"] \arrow[l, phantom, "\Leftarrow \mathfrak{l}"] \arrow[r, phantom, "\Leftarrow \mathcal{F}^2"] & \\
 & \mathcal{M} \arrow[r, "\mathcal{F}"'] & \mathcal{N}
\end{tikzcd}
\quad \overset{\gamma}{\underset{\cong}{\Rrightarrow}} \quad
\begin{tikzcd}[column sep=small]
 & \mathcal{N} \arrow[dr, "I1"] \arrow[ddr, bend right, "1"'] & \\
\mathcal{M} \arrow[ur, "\mathcal{F}"] \arrow[ddr, bend right, "1"'] & \arrow[r, phantom, "\Leftarrow \mathfrak{l}"] & \mathcal{N}^2 \arrow[d, "\otimes"] \\
 & & \mathcal{N} \\
 & \mathcal{M} \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}
$$

These are subject to two axioms which may be found in [GPS95, pp. 17–18]. A *monoidal 2-functor* is a monoidal pseudofunctor whose underlying pseudofunctor is a 2-functor.

A *monoidal pseudonatural transformation* between a parallel pair of monoidal pseudofunctors $\mathcal{F}$ and $\mathcal{G} : \mathcal{M} \longrightarrow \mathcal{N}$ consists of a pseudonatural transformation a as shown,

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{N}
\end{tikzcd}
$$

together with two invertible modifications Π and M as follows.

$$
\begin{tikzcd}
\mathcal{M}^2 \arrow[r, bend left, "\mathcal{F}\mathcal{F}"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Leftarrow \mathcal{F}_2"] & \mathcal{N}^2 \arrow[d, "\otimes"] \\
\mathcal{M} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{N}
\end{tikzcd}
\quad \overset{\Pi}{\underset{\cong}{\Rrightarrow}} \quad
\begin{tikzcd}
\mathcal{M}^2 \arrow[r, bend left, "\mathcal{F}\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}\mathcal{G}"'] \arrow[r, phantom, "\Downarrow aa"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Leftarrow \mathcal{G}^2"] & \mathcal{N}^2 \arrow[d, "\otimes"] \\
\mathcal{M} \arrow[r, bend right, "\mathcal{G}"'] & \mathcal{N}
\end{tikzcd}
$$

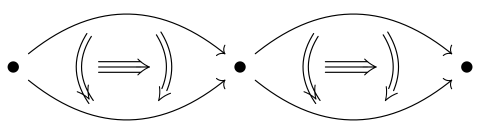
$$\begin{tikzcd}[column sep=small] & I \arrow[dl, "I"'] \arrow[dr, "I"] \arrow[d, phantom, "\Swarrow\mathcal{F}_0"] & \\ \mathcal{M} \arrow[rr, bend left, "\mathcal{F}"] \arrow[rr, bend right, "\mathcal{G}"'] \arrow[rr, phantom, "\Downarrow a"] & & \mathcal{N} \end{tikzcd} \quad \overset{M}{\underset{\cong}{\Rrightarrow}} \quad \begin{tikzcd}[column sep=small] & I \arrow[dl, "I"'] \arrow[dr, "I"] \arrow[d, phantom, "\Swarrow\mathcal{G}^0"] & \\ \mathcal{M} \arrow[rr, bend right, "\mathcal{G}"'] & & \mathcal{N} \end{tikzcd}$$

These are subject to three axioms found in [GPS95, pp. 21–24] or [GG09, Definition 5]. A monoidal 2-natural transformation is a monoidal pseudonatural transformation whose underlying pseudonatural transformation is a 2-transformation.

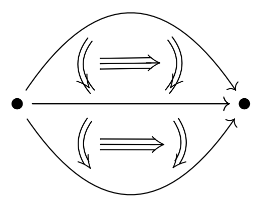
A monoidal modification between a parallel pair of monoidal pseudonatural transformations is a modification between the underlying pseudonatural transformations that satisfies two axioms. These axioms assert the compatibility between the modification itself and the invertible constraints Π and M of its source and target, see [GPS95, pp. 25–26].

1.4.6 3-dimensional Category Theory: 3-categories and Tricategories

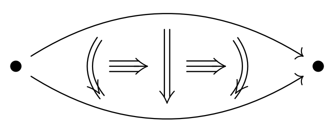
In Chapter 4 several things are organised in various tricategories. We refer the reader to [GP97] or [Gur13] for a full definition of a tricategory which we do not attempt to fit here. Tricategories consist of four levels of data usually called 0-cells, 1-cells, 2-cells, and 3-cells, although we will abstain from using these names outside this subsection. There are three different operations called compositions that we refer to as: the horizontal composition along 0-cells,



the vertical composition along 1-cells,



and the transversal composition along 2-cells.



Along with their three respective kinds of identities, these three operations are associative and unital in an appropriate weak sense; on the nose for the transversal composition, up to isomorphism for the vertical composition, and up to adjoint equivalence for the horizontal composition. It should also be noted that the middle four interchange law between horizontal and vertical composition holds up to an invertible modification.

$\Sigma\mathcal{M}$ The suspension of a monoidal bicategory $\mathcal{M}$. This is a tricategory with precisely one object (0-cell).

$\mathsf{2\text{-}Cat}$ The 3-category of 2-categories, 2-functors, 2-natural transformations, and modifications. Horizontal, vertical, and transversal compositions and identities are strictly associative and unital.

$\mathsf{Mon2\text{-}Cat}$ The 3-category of monoidal 2-categories, monoidal 2-functors, monoidal 2-natural transformations, and monoidal modifications. Horizontal, vertical, and transversal compositions and identities are strictly associative and unital.

Bicat The tricategory of bicategories, pseudofunctors, pseudonatural transformations, and modifications. Horizontal composition and identities of pseudofunctors are strictly associative and unital, vertical composition and identities of pseudonatural transformations are associative and unital up to coherent modification isomorphisms, and transversal composition and identities of modifications are strictly associative and unital. Warning: horizontal composition of pseudonatural transformations is defined up to an invertible modification see [Gur13, Section 5.1].

$\mathsf{MonBicat}$ The tricategory of monoidal bicategories, monoidal pseudofunctors, monoidal pseudonatural transformations, and monoidal modifications. Horizontal composition and identities of monoidal pseudofunctors are strictly associative and unital, vertical composition and identities of monoidal transformations are associative and unital up to coherent monoidal modification isomorphisms, and transversal composition and identities are strictly associative and unital, see [GG09, Corollary 27].

2

Comodules for Coalgebroids

The goal of this chapter is to obtain a deeper understanding of opmonoidal arrows $R^{\circ}R \longrightarrow S^{\circ}S$ between enveloping monoidales in a monoidal bicategory $\mathcal{M}$. If $R = I$, such opmonoidal arrows may be seen as coalgebras in $\mathcal{M}(I, I)$, and as such have a corresponding notion of comodule. More generally, opmonoidal arrows $R^{\circ}R \longrightarrow S^{\circ}S$ also have comodules, and we shall also study these. Before turning to this, however, we present a general summary of the relationship between monads and adjunctions in a bicategory $\mathcal{B}$. We recall the concept of opmonadic adjunction, which plays a key role throughout the chapter. Opmonadic adjunctions in a bicategory $\mathcal{B}$ are adjunctions with a universal property. In the case of $\mathcal{B} = \mathsf{Cat}$ these are precisely the adjunctions determined by the Kleisli category of algebras of a monad. We also present an account of the relationship between monoidales and bidualities in a monoidal bicategory $\mathcal{M}$. Then we dedicate a whole section to analysing the interaction between opmonoidal arrows $R^{\circ}R \longrightarrow N$ and certain opmonadic adjunctions. In particular, we study opmonadic adjunctions where the left adjoint is opmonoidal and the right adjoint is monoidal. This "opmonoidal $\dashv$ monoidal opmonadicity" is one of the most powerful tools used throughout the chapter, providing us with non-trivial equivalences of categories.

Then we explore how a biduality transposes the structure of opmonoidal arrows $M \longrightarrow S^{\circ}S$, we call the resulting structure *oplax actions*. Together with the equivalence in the previous section, this provides an equivalence between opmonoidal arrows on enveloping monoidales and oplax actions with respect to certain skew monoidales.

Opmonoidal Arrows	$R^{\circ}R \longrightarrow S^{\circ}S$
Oplax actions	$SR \longrightarrow S$

Furthermore, both of these structures induce a comonad on S, and corresponding structures induce the same comonad.

We finish by providing a notion of comodule for an opmonoidal arrow, and then use the techniques of opmonadicity and transposition introduced earlier to see these comodules as oplax morphisms of oplax actions. And by a further application of opmonadicity, there is an equivalence between comodules for an opmonoidal arrow $R^{\circ}R \longrightarrow S^{\circ}S$ and comodules for the induced comonad on S.

2.1 Skew Monoidales, Bidualities and Adjunctions

2.1.1 Monads and Adjunctions

Adjunctions and monads appeared independently in 1958. Godement used comonads under the name of *standard constructions* in [God58] to compute sheaf cohomology. In [Kan58] Kan gave a formal account of the theory of adjunctions. Together, monads and adjunctions saw great development during the 60's. Huber proved that every adjunction $F \dashv G$ has an associated monad $(GF, F\varepsilon G, \eta)$. Now, one may arrange adjunctions that have the same associated monad into a category. Moreover, the Eilenberg-Moore category of algebras defines the terminal adjunction, and the Kleisli category of algebras defines the initial adjunction. Beck characterised adjunctions that are isomorphic (or equivalent) to the adjunction determined by the category of Eilenberg-Moore algebras of a given monad in terms of the internal structure of the categories and functors involved. In [Str72] this relationship between monads and adjunctions is abstracted from Cat to what is called the formal theory of monads, where one studies monads and adjunctions internal to an arbitrary bicategory $\mathcal{B}$. With this perspective, adjunctions still induce monads in $\mathcal{B}$, but monads may or may not have the notions in $\mathcal{B}$ that correspond to the Kleisli and Eilenberg-Moore adjunctions in Cat. One may characterize the constructions of Kleisli and Eilenberg-Moore algebras for a monad in Cat up to isomorphism (or up to equivalence) by even stronger universal properties than the ones mentioned earlier. These universal properties are described as bicategorical colimit and limit notions. In [LS02] Lack and Street describe the completion for a bicategory $\mathcal{B}$ under the limit notion that corresponds to the category of Eilenberg-Moore algebras, as well as the free cocompletion under the colimit notion corresponding to the Kleisli category of algebras.

Definition 2.1.1. A *monad* on an object S in a bicategory $\mathcal{B}$ consists of an endo-arrow $t : S \longrightarrow S$, and two cells

$$\begin{tikzcd} S \arrow[rr, bend left, "t"] \arrow[dr, "t"'] & \arrow[d, phantom, "\mu\Uparrow"] & S \\ & S \arrow[ur, "t"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, bend left, "t"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\eta\Uparrow"] & S \end{tikzcd}$$

called multiplication and unit, and satisfying the associative and unit laws.

$$\begin{tikzcd} S \arrow[r, "t"'] \arrow[rr, bend left, "t"] \arrow[rrr, bend left=60, "t"] & S \arrow[r, "t"'] \arrow[phantom, u, "\mu\Uparrow"] & S \arrow[r, "t"'] \arrow[phantom, u, "\mu\Uparrow"] & S \end{tikzcd} = \begin{tikzcd} S \arrow[r, "t"'] \arrow[rrr, bend left=60, "t"] & S \arrow[r, "t"'] \arrow[rr, bend left, "t"] \arrow[phantom, u, "\mu\Uparrow"] & S \arrow[r, "t"'] \arrow[phantom, u, "\mu\Uparrow"] & S \end{tikzcd}$$

$$
\begin{tikzcd}
S \arrow[r, "t"] \arrow[r, bend right=60, "1"'] \arrow[r, phantom, bend right=30, "\eta\Uparrow"] \arrow[rr, bend left=60, "t"] & S \arrow[r, "t"'] \arrow[u, phantom, "\mu\Uparrow"] & S
\end{tikzcd}
=
\begin{tikzcd}
S \arrow[r, bend left, "t"] \arrow[r, bend right, "t"'] & S
\end{tikzcd}
$$

$$
\begin{tikzcd}
S \arrow[r, "t"'] \arrow[rr, bend left=60, "t"] & S \arrow[r, "t"] \arrow[r, bend right=60, "1"'] \arrow[r, phantom, bend right=30, "\eta\Uparrow"] \arrow[u, phantom, "\mu\Uparrow"] & S
\end{tikzcd}
=
\begin{tikzcd}
S \arrow[r, bend left, "t"] \arrow[r, bend right, "t"'] & S
\end{tikzcd}
$$

Definition 2.1.2. An *adjunction* in a bicategory $\mathcal{B}$ consists of two objects R and S, two arrows $f : S \longrightarrow R$ and $g : R \longrightarrow S$, two cells called and unit and counit,

$$
\begin{tikzcd}
S \arrow[r, "f"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\eta\Nearrow"] & R \arrow[d, "g"] \\
& S
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[d, "g"'] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3, "\varepsilon\Nearrow"] & \\
S \arrow[r, "f"'] & R
\end{tikzcd}
$$

satisfying the triangle (or snake) equations below.

$$
\begin{tikzcd}
S \arrow[r, "f"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\eta\Nearrow"] & R \arrow[d, "g"] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3, "\varepsilon\Nearrow"] & \\
& S \arrow[r, "f"'] & R
\end{tikzcd}
=
\begin{tikzcd}
S \arrow[d, bend left, "f"] \arrow[d, bend right, "f"'] \\
R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[d, "g"'] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3, "\varepsilon\Nearrow"] & \\
S \arrow[r, "f"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\eta\Nearrow"] & R \arrow[d, "g"] \\
& S
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[d, bend left, "g"] \arrow[d, bend right, "g"'] \\
S
\end{tikzcd}
$$

This situation is denoted by $f \dashv g$ or more explicitly by the diagram below.

$$
\begin{tikzcd}
R \arrow[d, bend left, "g"] \arrow[d, phantom, "\dashv"] \\
S \arrow[u, bend left, "f"]
\end{tikzcd}
$$

Example 2.1.3. In Cat Freyd's adjoint functor theorems provide necessary and sufficient conditions for a functor to have a left adjoint. In Mod_k a module $M : R \longrightarrow S$ has a right adjoint if and only if it is finitely generated and projective as an R-module. In this case, M is also called a Cauchy module, or a module with a right dual [Str07, Section 5].

Every adjunction $f \dashv g$ induces a monad on S on the composite $t : S \xrightarrow{f} R \xrightarrow{g} S$ with unit η and multiplication as below.

$$
\begin{tikzcd}
S \arrow[r, "f"] & R \arrow[rr, bend left, "1"] \arrow[dr, "g"'] & \arrow[d, phantom, "\varepsilon\Uparrow"] & R \arrow[r, "g"] & S \\
& & S \arrow[ur, "f"'] & &
\end{tikzcd}
$$

We often abbreviate this situation with the following diagram.

$$\begin{tikzcd} S_t \arrow[d, bend left, "g"] \arrow[d, phantom, "\dashv"] \\ S \arrow[u, bend left, "f"] \arrow[loop right, "t"] \end{tikzcd}$$

Remark 2.1.4. Every monad $t : S \longrightarrow S$ in a bicategory $\mathcal{B}$ induces two monads in Cat for each object X in $\mathcal{B}$. These are obtained by using the covariant and contravariant hom functors based at X.

$$\mathcal{B}(t, X) : \mathcal{B}(S, X) \longrightarrow \mathcal{B}(S, X)$$
$$\mathcal{B}(X, t) : \mathcal{B}(X, S) \longrightarrow \mathcal{B}(X, S)$$

Now, one may consider the categories of Eilenberg-Moore algebras for each of these monads $\mathcal{B}(S, X)^{\mathcal{B}(t,X)}$ and $\mathcal{B}(X, S)^{\mathcal{B}(X,t)}$; we call their objects modules for the monad t based at X. For example, an object in $\mathcal{B}(S, X)^{\mathcal{B}(t,X)}$ consists of an arrow $x : S \longrightarrow X$ together with an *action cell* χ that is associative and unital with respect to the monad structure of t.

$$\begin{tikzcd} S \arrow[rr, bend left, "x"] \arrow[dr, "t"'] & \arrow[d, phantom, "\chi\Uparrow"] & X \\ & S \arrow[ur, "x"'] & \end{tikzcd}$$

To avoid confusion we may specify which hom functor to use, or use the following notation $(x, \chi) : S \longrightarrow X$ to refer to a module for the monad t as above. Some authors call them the right and left modules for the monad t, but it is not clear which ones to name left modules and which to name right modules, and it seems that one may give a good argument for either name. On the one hand, we may say that modules for a monad induced by the contravariant hom functor based at X are "right t-actions" with respect to the horizontal composition $\chi : x \circ t \longrightarrow x$. And on the other hand, in bicategories like $\mathcal{B} = \mathsf{Mod}_{\mathbb{Z}}$ a monad is a ring, and a module for a monad induced by the contravariant hom functor based at X is a left module with respect to a ring.

For a monad in $\mathcal{B} = \mathsf{Cat}$, we can construct the categories of Kleisli and Eilenberg-Moore algebras, these provide us with adjunctions whose associated monad is the one we started with. It is possible to generalise these constructions for monads in an arbitrary bicategory $\mathcal{B}$. But in this case we can not guarantee their existence since the construction is given in terms of a universal property.

Definition 2.1.5. A *Kleisli object* for a monad $t : S \longrightarrow S$ in a bicategory $\mathcal{B}$, if it exists, is the universal object S_t that represents up to equivalence the modules $(x, \chi) : S \longrightarrow X$ for the monad t based at X for every object X in $\mathcal{B}$. In other words, there is an equivalence of categories as shown.

$$\mathcal{B}(S_t, X) \simeq \mathcal{B}(S, X)^{\mathcal{B}(t,X)}$$

Dually, an *Eilenberg-Moore object* is defined by using the covariant hom functors $\mathcal{B}(X, \_)$ instead of the contravariant ones, thus Eilenberg-Moore objects in $\mathcal{B}$ are Kleisli objects in $\mathcal{B}^{\mathrm{op}}$.

Concretely, a Kleisli object for a monad t is an object S_t and a “universal module” φ for the monad t,

$$\begin{tikzcd} S \arrow[rr, bend left, "f"] \arrow[dr, "t"'] & \arrow[d, phantom, "\varphi\Uparrow"] & S_t \\ & S \arrow[ur, "f"'] & \end{tikzcd}$$

in the sense that for every other module $(x, \chi) : S \longrightarrow X$ for the monad t there exists a arrow $\bar{x} : S_t \longrightarrow X$ and an isomorphism as shown,

$$\begin{tikzcd} & S_t \arrow[dr, "\bar{x}"] \arrow[d, phantom, "\cong"] & \\ S \arrow[ur, "f"] \arrow[rr, "x"'] & {} & X \end{tikzcd}$$

that satisfies the following equation.

$$\begin{tikzcd} S \arrow[rr, bend left, "f"] \arrow[dr, "t"'] & \arrow[d, phantom, "\varphi\Uparrow"] & S_t \arrow[dr, "\bar{x}"] & \\ & S \arrow[ur, "f"] \arrow[rr, bend right, "x"'] & \arrow[u, phantom, "\cong"] & X \end{tikzcd} \quad = \quad \begin{tikzcd} & S_t \arrow[dr, "\bar{x}"] \arrow[d, phantom, "\cong"] & \\ S \arrow[ur, "f"] \arrow[rr, "x"] \arrow[dr, "t"'] & \arrow[d, phantom, "\chi\Uparrow"] & X \\ & S \arrow[ur, "x"'] & \end{tikzcd}$$

And for every morphism of modules for the monad t in $\mathcal{B}(S, X)^{\mathcal{B}(t,X)}$ a corresponding condition. Which together assert that the equivalence in the definition is given by precomposition with (f, φ).

$$\mathcal{B}(S_t, X) \xrightarrow[\simeq]{} \mathcal{B}(S, X)^{\mathcal{B}(t,X)}$$

$$S_t \xrightarrow{\bar{x}} X \quad \longmapsto \quad \begin{tikzcd} S \arrow[rr, bend left, "f"] \arrow[dr, "t"'] & \arrow[d, phantom, "\varphi\Uparrow"] & S_t \arrow[r, "\bar{x}"] & X \\ & S \arrow[ur, "f"'] & & \end{tikzcd}$$

Remark 2.1.6. Since Kleisli objects are defined by a universal property, they are unique up to equivalence when they exist. It is no surprise that the definition can be stated as some sort of limit notion; a Kleisli object for a monad t is the lax bicolimit of the diagram that the monad t depicts in $\mathcal{B}$, and an Eilenberg-Moore object is the lax bilimit of the same diagram. In [Str72] the universal property Eilenberg-Moore for objects holds up to isomorphism, thus in this case Eilenberg-Moore objects are a lax pseudolimit.

As part of the structure that comes together with a Kleisli object, one finds an adjunction $\mathsf{free} \dashv \mathsf{forget}$ that has t as its associated monad [Str72, §1].

$$\begin{tikzcd} S_t \arrow[d, bend left, "\mathsf{forget}"] \arrow[d, phantom, "\dashv"] \\ S \arrow[u, bend left, "\mathsf{free}"] \arrow[loop right, "t"] \end{tikzcd}$$

And in the same way as in Cat, for every adjunction $f \dashv g$ whose induced monad is t,

$$\begin{tikzcd} S_t \arrow[d, bend left, "g"] \arrow[d, phantom, "\dashv"] \\ S \arrow[u, bend left, "f"] \arrow[loop right, "t"] \end{tikzcd}$$

there is a comparison arrow $S_t \longrightarrow R$ that commutes with the left and right adjoints up to isomorphism. Then, a classic question is; when is an adjunction $f \dashv g$ with associated monad t equivalent or isomorphic (in the sense that the comparison is an equivalence or an isomorphism) to the adjunction induced by the Kleisli object of t? Adjunctions in Cat which are equivalent to the adjunction induced by the category of Eilenberg-Moore algebras of the associated monad are characterised by Beck's monadicity theorem in terms of the internal structure of the categories, functors, and natural transformations involved.

Definition 2.1.7. An adjunction $f \dashv g$ in a bicategory $\mathcal{B}$ (or a left adjoint) is called *opmonadic* (or of *Kleisli type*), if for every object X in $\mathcal{B}$ the adjunction obtained by applying the representable functor $\mathcal{B}(\_, X)$ is monadic in Cat (in the up to equivalence sense).

$$\begin{tikzcd} R \arrow[d, bend left, "g"] \arrow[d, phantom, "\dashv"] \\ S \arrow[u, bend left, "f"] \arrow[loop right, "t"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{B}(R,X) \arrow[d, bend left, "\mathcal{B}(g,X)"] \arrow[d, phantom, "\dashv"] \\ \mathcal{B}(S,X) \arrow[u, bend left, "\mathcal{B}(f,X)"] \arrow[loop right, "\mathcal{B}(t,X)"] \end{tikzcd}$$

In other words, if t is the monad associated to the adjunction $f \dashv g$, being opmonadic means that for every object X in $\mathcal{M}$ the comparison functor is an equivalence of categories.

$$\mathcal{B}(R, X) \xrightarrow[\simeq]{} \mathcal{B}(S, X)^{\mathcal{B}(t,X)}$$

Ergo, R is the Kleisli object of the monad t.

Dually, a *monadic adjunction* is one such that for every X the adjunctions obtained by applying the representable functor $\mathcal{B}(X, \_)$ are monadic in Cat.

Example 2.1.8. In Cat, the Kleisli category of algebras K for a monad t on S determines an adjunction, and this adjunction is opmonadic (in the up to isomorphism sense) [Str72, Theorem 13]. More generally, an adjunction in Cat is opmonadic if and only if the left adjoint is essentially surjective on objects. Then the comparison functor with respect to the Kleisli category of algebras for the associated monad is an equivalence that commutes with the left adjoints and with the right adjoints up to isomorphism. And in Mod_k, all adjunctions are monadic and opmonadic. For example, the opmonadicity of adjunctions $i \dashv i^*$ where $i : k \longrightarrow R$ is the unit of the ring translates between left R-modules seen either as arrows with source R in Mod_k or as modules M with a left R-action $R \otimes M \longrightarrow M$.

2.1.2 Monoidales and Bidualities

We continue by introducing and giving basic properties of monoidales [DS97, Section 3] and skew monoidales [LS12, Section 4] which play a central role. To do this we need to upgrade our bicategory $\mathcal{B}$ to a monoidal bicategory $\mathcal{M}$, see Subsection 1.4.5. We also construct some monoidales and skew monoidales from bidualities and adjunctions in different ways, some are not standard.

Definition 2.1.9. A *right skew monoidale* in $\mathcal{M}$ consists of the following items:

- An object M.
- A product arrow $m : MM \longrightarrow M$.
- A unit arrow $u : I \longrightarrow M$.
- An associator cell α (not necessarily invertible).

$$\begin{tikzcd}
MMM \arrow[r, "m1"] \arrow[d, "1m"'] \arrow[dr, phantom, "\overset{\alpha}{\Nearrow}"] & MM \arrow[d, "m"] \\
MM \arrow[r, "m"'] & M
\end{tikzcd}$$

- A left unitor cell λ and a right unitor cell ρ (not necessarily invertible).

$$\begin{tikzcd}
M \arrow[r, "u1"] \arrow[dr, "1"'] & MM \arrow[d, "m"] \arrow[dl, phantom, "\lambda\Nearrow"] \arrow[dr, phantom, "\Searrow\rho"] & M \arrow[l, "1u"'] \arrow[dl, "1"] \\
& M &
\end{tikzcd}$$

Satisfying the following five axioms: in order, the pentagon, the triangle, α-λ, α-ρ, and λ-ρ compatibilities.

$$\begin{tikzcd}[column sep=small]
& & MMM \arrow[drr, "m1"] \arrow[d, phantom, "\alpha1\Uparrow"] & & \\
MMMM \arrow[urr, "m11"] \arrow[drr, "1m1"] \arrow[dd, "11m"'] & & {} & & MM \arrow[dd, "m"] \\
& & MMM \arrow[urr, "m1"] \arrow[dd, "1m"] \arrow[dll, phantom, "1\alpha\Nearrow"] \arrow[drr, phantom, "\alpha\Nearrow"] & & \\
MMM \arrow[drr, "1m"'] & & & & M \\
& & MM \arrow[urr, "m"'] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& & MMM \arrow[drr, "m1"] \arrow[dd, "1m"] & & \\
MMMM \arrow[urr, "m11"] \arrow[dd, "11m"'] \arrow[dr, phantom, "\cong"] & & & \arrow[d, phantom, "\alpha\Nearrow"] & MM \arrow[dd, "m"] \\
& & MM \arrow[drr, "m"] & {} & \\
MMM \arrow[urr, "m1"] \arrow[drr, "1m"'] & & \alpha\Uparrow & & M \\
& & MM \arrow[urr, "m"'] & &
\end{tikzcd}
\tag{SKM1}$$

$$\begin{tikzcd}[column sep=small]
MM \arrow[rrrr, bend left, "1"] \arrow[drr, "1u1"] \arrow[ddd, bend right, "1"'] & & \rho1\Uparrow & & MM \arrow[dd, "m"] \\
& & MMM \arrow[urr, "m1"] \arrow[dd, "1m"] \arrow[dll, phantom, "1\lambda\Nearrow"] \arrow[drr, phantom, "\alpha\Nearrow"] & & \\
& & & & M \\
& & MM \arrow[urr, "m"'] & &
\end{tikzcd}
=
\begin{tikzcd}
MM \arrow[dr, bend left, "m"] \arrow[dr, bend right, "m"'] & \\
& M
\end{tikzcd}
\tag{SKM2}$$

$$
\begin{tikzcd}
& MMM \arrow[dr, "m1"] \arrow[d, phantom, "\lambda 1\Uparrow"] & \\
MM \arrow[ur, "u11"] \arrow[rr, bend right, "1"'] \arrow[d, "m"'] & {} & MM \arrow[d, "m"] \\
M \arrow[rr, bend right, "1"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
& MMM \arrow[dd, "1m"] \arrow[dr, "m1"] & \\
MM \arrow[ur, "u11"] \arrow[dd, "m"'] \arrow[r, phantom, "\cong"] & {} & MM \arrow[dd, "m"] \arrow[l, phantom, "\alpha\Nearrow"] \\
& MM \arrow[dr, "m"] & \\
M \arrow[ur, "u1"] \arrow[rr, bend right, "1"'] \arrow[r, phantom, "\lambda\Uparrow"] & {} & M
\end{tikzcd}
\tag{SKM3}
$$

$$
\begin{tikzcd}
& M \arrow[ddr, bend left, "1"] & \\
MM \arrow[ur, "m"] \arrow[d, "11u"'] \arrow[ddr, bend left, "1"] & & \\
MMM \arrow[dr, "1m"'] \arrow[r, phantom, "1\rho\Nearrow"] & {} & M \\
& MM \arrow[ur, "m"'] &
\end{tikzcd}
=
\begin{tikzcd}
& M \arrow[dd, "1u"] \arrow[dddr, bend left, "1"] & \\
MM \arrow[ur, "m"] \arrow[dd, "11u"'] \arrow[r, phantom, "\cong"] & {} & \\
& MM \arrow[dr, "m"] \arrow[r, phantom, "\rho\Nearrow"] & {} \\
MMM \arrow[ur, "m1"] \arrow[dr, "1m"'] \arrow[r, phantom, "\alpha\Uparrow"] & {} & M \\
& MM \arrow[ur, "m"'] &
\end{tikzcd}
\tag{SKM4}
$$

$$
\begin{tikzcd}
I \arrow[dr, bend left, "u"] \arrow[dr, bend right, "u"'] & \\
& M
\end{tikzcd}
=
\begin{tikzcd}
& M \arrow[dd, "1u"] \arrow[dddr, bend left, "1"] & \\
I \arrow[ur, "u"] \arrow[dd, "u"'] \arrow[r, phantom, "\cong"] & {} & \\
& MM \arrow[dr, "m"] \arrow[r, phantom, "\rho\Nearrow"] & {} \\
M \arrow[ur, "u1"] \arrow[rr, bend right, "1"'] \arrow[r, phantom, "\lambda\Uparrow"] & {} & M
\end{tikzcd}
\tag{SKM5}
$$

Remark 2.1.10. *Left skew monoidales* are defined similarly but the associator and unitor cells point in the opposite direction; that means these are right skew monoidales in $\mathcal{M}^{\text{co}}$. But one may also get left skew monoidales by reversing the tensor product; that is, by taking right skew monoidales in $\mathcal{M}^{\text{rev}}$. When α is invertible we speak of a Hopf right skew monoidale; if instead λ or ρ are invertible we speak of a left or right normal right skew monoidale; and we speak of a *monoidale* when α, λ and ρ are isomorphisms, and in this case, a well known argument by Kelly [Kel64] implies that the axioms may be reduced from five to two: the pentagon (SKM1) and the triangle (SKM2). For example, in Cat (left/right skew) monoidales are (left/right skew) monoidal categories.

Definition 2.1.11. An *opmonoidal arrow* $C : M \longrightarrow N$ between right skew monoidales M and N in $\mathcal{M}$ consists of an arrow $C : M \longrightarrow N$ in $\mathcal{M}$ equipped with an opmonoidal composition constraint cell C^2 and an opmonoidal unit constraint cell C^0 as shown below,

$$
\begin{tikzcd}
MM \arrow[r, "CC"] \arrow[d, "m"'] \arrow[dr, phantom, "C^2\Nearrow"] & NN \arrow[d, "m"] \\
M \arrow[r, "C"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "C^0\Nearrow"] & \\
M \arrow[rr, "C"'] & {} & N
\end{tikzcd}
$$

satisfying three axioms.

$$
\begin{tikzcd}
 & & NNN \arrow[drr, "m1"] & & \\
MMM \arrow[urr, "CCC"] \arrow[drr, "m1"'] \arrow[dd, "1m"'] \arrow[rr, phantom, "C^2C\Uparrow"] & & & & NN \arrow[dd, "m"] \\
 & & MM \arrow[urr, "CC"'] \arrow[dd, "m"] \arrow[ll, phantom, "\alpha\Rrightarrow"] \arrow[rr, phantom, "C^2\Rrightarrow"] & & \\
MM \arrow[drr, "m"'] & & & & N \\
 & & M \arrow[urr, "C"'] & &
\end{tikzcd}
=
\begin{tikzcd}
 & & NNN \arrow[drr, "m1"] \arrow[dd, "1m"] & & \\
MMM \arrow[urr, "CCC"] \arrow[dd, "1m"'] \arrow[r, phantom, "CC^2\Rrightarrow"] & & & & NN \arrow[dd, "m"] \arrow[l, phantom, "\alpha\Rrightarrow"'] \\
 & & NN \arrow[drr, "m"] \arrow[dd, phantom, "C^2\Uparrow"] & & \\
MM \arrow[urr, "CC"] \arrow[drr, "m"'] & & & & N \\
 & & M \arrow[urr, "C"'] & &
\end{tikzcd}
\tag{OM1}
$$

$$
\begin{tikzcd}
 & & N \arrow[dddrr, bend left, "1"] & & \\
M \arrow[urr, "C"] \arrow[dd, "1u"'] \arrow[dddrr, bend left, "1"] & & & & \\
 & & & & \\
MM \arrow[drr, "m"'] \arrow[ur, phantom, "\rho\Rrightarrow"] & & & & N \\
 & & M \arrow[urr, "C"'] & &
\end{tikzcd}
=
\begin{tikzcd}
 & & N \arrow[dd, "1u"] \arrow[dddrr, bend left, "1"] \arrow[dr, phantom, "\rho\Rrightarrow"] & & \\
M \arrow[urr, "C"] \arrow[dd, "1u"'] \arrow[r, phantom, "CC^0\Rrightarrow"] & & & & \\
 & & NN \arrow[drr, "m"] \arrow[dd, phantom, "C^2\Uparrow"] & & \\
MM \arrow[urr, "CC"] \arrow[drr, "m"'] & & & & N \\
 & & M \arrow[urr, "C"'] & &
\end{tikzcd}
\tag{OM2}
$$

$$
\begin{tikzcd}
 & & N \arrow[drr, "u1"] & & \\
M \arrow[urr, "C"] \arrow[drr, "u1"'] \arrow[dddrr, bend right, "1"'] \arrow[rr, phantom, "C^0C\Uparrow"] & & & & NN \arrow[dd, "m"] \\
 & & MM \arrow[urr, "CC"'] \arrow[dd, "m"] \arrow[ll, phantom, "\lambda\Rrightarrow"] \arrow[rr, phantom, "C^2\Rrightarrow"] & & \\
 & & & & N \\
 & & M \arrow[urr, "C"'] & &
\end{tikzcd}
=
\begin{tikzcd}
 & & N \arrow[drr, "u1"] \arrow[ddrr, bend right, "1"'] & & \\
M \arrow[urr, "C"] \arrow[dddrr, bend right, "1"'] & & & & NN \arrow[dd, "m"] \arrow[dl, phantom, "\lambda\Rrightarrow"'] \\
 & & & & \\
 & & & & N \\
 & & M \arrow[urr, "C"'] & &
\end{tikzcd}
\tag{OM3}
$$

Remark 2.1.12. Changing the direction of the structure cells in the definition of an opmonoidal arrow and adjusting the axioms to make sense for the compositions with the associators and unitors gives the notion of *monoidal arrow* between right skew monoidales. In the case that both opmonoidal constraints are isomorphisms speak of a *strong monoidal arrow*, and if they are identities we speak of a *strict monoidal arrow*.

Definition 2.1.13. An *opmonoidal cell* between a parallel pair of opmonoidal arrows C and $C' : M \longrightarrow N$ in $\mathcal{M}$ consists of a cell ξ as shown,

$$
\begin{tikzcd}
M \arrow[r, bend left, "C'"] \arrow[r, bend right, "C"'] \arrow[r, phantom, "\xi\Uparrow"] & N
\end{tikzcd}
$$

satisfying two axioms.

$$
\begin{tikzcd}
MM \arrow[r, "C'C'"] \arrow[d, "m"'] \arrow[dr, phantom, "C'^2 \Nearrow"] & NN \arrow[d, "m"] \\
M \arrow[r, bend left, "C'"] \arrow[r, bend right, "C"'] \arrow[r, phantom, "\xi\Uparrow"] & N
\end{tikzcd}
=
\begin{tikzcd}
MM \arrow[r, bend left, "C'C'"] \arrow[r, bend right, "CC"'] \arrow[r, phantom, "\xi\xi\Uparrow"] \arrow[d, "m"'] \arrow[dr, phantom, "C^2 \Nearrow"] & NN \arrow[d, "m"] \\
M \arrow[r, "C"'] & N
\end{tikzcd}
\tag{OM4}
$$

$$
\begin{tikzcd}
 & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "C'^0 \Nearrow"] & \\
M \arrow[rr, bend left, "C'"] \arrow[rr, bend right, "C"'] \arrow[rr, phantom, "\xi\Uparrow"] & & N
\end{tikzcd}
=
\begin{tikzcd}
 & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "C^0 \Nearrow"] & \\
M \arrow[rr, "C"'] & & N
\end{tikzcd}
\tag{OM5}
$$

Remark 2.1.14. The dual concept is that of a *monoidal cell* between a parallel pair of monoidal arrows: this is a cell in $\mathcal{M}$ that satisfies two axioms that correspond to those of an opmonoidal cell. As usual, the process of taking dual categories does the job of switching between these notions, we name all of them as follows:

- $\mathsf{SkOpMon}_{\mathrm{r}}(\mathcal{M}) = \mathsf{SkOpMon}(\mathcal{M})$ is the bicategory of right skew monoidales, opmonoidal arrows, and opmonoidal cells between them. This bicategory is going to be used the most throughout the document, and in order to have a lighter notation we omit the subscript.

- $\mathsf{SkOpMon}_{\mathrm{l}}(\mathcal{M}) = \mathsf{SkOpMon}_{\mathrm{r}}(\mathcal{M}^{\mathrm{rev}})$ is the bicategory of left skew monoidales, opmonoidal arrows and opmonoidal cells between them.

- $\mathsf{SkMon}_{\mathrm{l}}(\mathcal{M}) = \mathsf{SkOpMon}_{\mathrm{r}}(\mathcal{M}^{\mathrm{co}})^{\mathrm{co}}$ is the bicategory of left skew monoidales, monoidal arrows and monoidal cells between them.

- $\mathsf{SkMon}_{\mathrm{r}}(\mathcal{M}) = \mathsf{SkOpMon}_{\mathrm{r}}(\mathcal{M}^{\mathrm{rev\ co}})^{\mathrm{co}}$ is the bicategory of right skew monoidales, monoidal arrows, and monoidal cells between them.

- $\mathsf{OpMon}(\mathcal{M})$ is the bicategory of monoidales, opmonoidal arrows, and opmonoidal cells between them, which can be seen as the full subbicategory of $\mathsf{SkOpMon}_{\mathrm{r}}(\mathcal{M})$ whose objects are monoidales.

By reversing the arrows one gets *(skew) comonoidales*, *comonoidal* and *opcomonoidal arrows* between them, and *comonoidal* and *opcomonoidal cells* between them, but these do not play a role here. When there is no room for ambiguity, we omit the ambient monoidal bicategory $\mathcal{M}$ from the hom categories and write them as $\mathsf{SkOpMon}(M,N) := \mathsf{SkOpMon}(\mathcal{M})(M,N)$, and $\mathsf{OpMon}(M,N) := \mathsf{OpMon}(\mathcal{M})(M,N)$ to save some space.

Skew monoidales in Cat are called *skew monoidal categories*, these appeared in the study of R-bialgebroids in [Szl03, Theorem 9.1] precisely as closed skew monoidal category structures on the categories of one sided R-modules for which the skew unit is R. Skew monoidales appear first in [LS12], where the authors observe that skew monoidales in Span are categories, and skew monoidales in Mod_k are bialgebroids. Here we favour right skew monoidales as in [Szl03] rather than left skew monoidales as in [LS12]. In the following lemma we construct right skew monoidales from adjunctions whose left adjoint has domain I.

Lemma 2.1.15. *For every adjunction in a monoidal bicategory $\mathcal{M}$ as shown below, there is a right skew monoidal structure on R.*

$$\begin{tikzcd} R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i"] \end{tikzcd}$$

Proof.
The structure is given as follows:

$$\begin{array}{llll}
\text{Product} & RR \xrightarrow{i^*1} R & \text{Unit} & I \xrightarrow{i} R \\
\text{Associator} & \begin{tikzcd} RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, "i^*1"] \\ RR \arrow[r, "i^*1"'] & R \end{tikzcd} & \text{Unitors} & \begin{tikzcd} R \arrow[r, "i1"] \arrow[dr, "1"'] \arrow[dr, phantom, bend left, "\eta 1 \Rightarrow"] & RR \arrow[d, "i^*1"] \arrow[dr, phantom, "\cong"] & R \arrow[l, "1i"'] \arrow[d, "i^*"] \arrow[dl, bend left=60, "1"] \\ & R & I \arrow[l, "i"] \arrow[l, phantom, bend right, "\Downarrow\varepsilon"] \end{tikzcd}
\end{array}$$

Note that the associator is an interchange isomorphism, therefore the pentagon (SKM1) holds as an instance of the interchange coherence. The α-λ compatibility (SKM3) holds by naturality of the interchanger. The α-ρ compatibility (SKM4) is also an instance of the naturality of the interchanger, (regardless of the definition of ρ). The α-λ-ρ compatibility (SKM2) and the λ-ρ compatibility (SKM5) are a consequence of the snake equations of the adjunction $i \dashv i^*$.

$$\begin{tikzcd} RR \arrow[r, "i^*1"] \arrow[rr, bend left=40, "1"] \arrow[dr, "1i1"'] \arrow[ddr, bend right=50, "1"'] & I \arrow[r, "i1"] \arrow[u, phantom, "\varepsilon 1 \Uparrow"] \arrow[d, phantom, "\cong"] & RR \arrow[dd, "i^*1"] \\ & RRR \arrow[ur, "i^*11"'] \arrow[d, "1i^*1"] \arrow[dr, phantom, "\cong"] \arrow[l, phantom, "1\eta 1 \Rightarrow"] & \\ & RR \arrow[r, "i^*1"'] & R \end{tikzcd} = \begin{tikzcd} RR \arrow[r, "i^*1"] \arrow[rr, bend left=40, "1"] \arrow[u, phantom, "\varepsilon 1 \Uparrow"] \arrow[ddr, bend right=50, "1"'] & I \arrow[r, "i1"] \arrow[ddr, bend right, "1"'] \arrow[dr, phantom, "\eta 1 \Rightarrow"] & RR \arrow[dd, "i^*1"] \\ & & \\ & RR \arrow[r, "i^*1"'] & R \end{tikzcd} = \begin{tikzcd} RR \arrow[r, bend left, "i^*1"] \arrow[r, bend right, "i^*1"'] & R \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, bend left, "i"] \arrow[r, bend right, "i"'] & R \end{tikzcd} = \begin{tikzcd} & R \arrow[dr, "i^*"] \arrow[ddr, bend left=60, "1"] \arrow[d, phantom, "\eta \Uparrow"] & \\ I \arrow[ur, "i"] \arrow[rr, bend right, "1"'] \arrow[d, "i"'] & & I \arrow[d, "i"] \arrow[r, phantom, "\varepsilon \Rightarrow"] \\ R \arrow[rr, bend right, "1"'] & & R \end{tikzcd} = \begin{tikzcd} & R \arrow[dr, "i^*"] \arrow[ddr, bend left=60, "1"] \arrow[d, "1i"] & \\ I \arrow[ur, "i"] \arrow[d, "i"'] \arrow[r, phantom, "\cong"] & RR \arrow[dr, "i^*1"] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\eta 1 \Uparrow"] & I \arrow[d, "i"] \arrow[r, phantom, "\varepsilon \Rightarrow"] \\ R \arrow[ur, "i1"] \arrow[rr, bend right, "1"'] & & R \end{tikzcd}$$

■

Example 2.1.16. In the case of $\mathcal{M} = \mathsf{Mod}_k$ every finitely generated and projective module P in $\mathsf{Mod}\text{-}R$ induces a right skew monoidal structure $\odot_P$ on $\mathsf{Mod}\text{-}R$. The case where $P = R_R$ is simple yet illuminating; the skew monoidal product on $\mathsf{Mod}\text{-}R$ is given by tensoring over k and forgetting about the right R-module structure of the module on the left. Furthermore, under Szlachányi's equivalence, this skew monoidal category corresponds to the simplest possible R-bialgebroid $B = R^\circ \otimes R$ [Böh09, Example 3.2.3.].

Product	$\begin{aligned} A \underset{R}{\odot} B :=& A \underset{R}{\otimes} R \otimes B \\ \cong& A \otimes B \end{aligned}$	Unit	R
Associator	(invertible)	Unitors	$\begin{aligned} \lambda : B \xrightarrow{\eta\otimes 1} R \otimes B &\cong R \underset{R}{\odot} B \\ b \longmapsto 1 \otimes b& \\ \rho : A \underset{R}{\odot} R \cong A \otimes R &\xrightarrow{A \underset{R}{\otimes} \varepsilon} A \\ a \otimes r \longmapsto ar& \end{aligned}$

In the general case, if P is a finitely generated and projective right R-module and P^* is its dual in $R\text{-}\mathsf{Mod}$, then the skew monoidal structure $\odot_P$ on $\mathsf{Mod}\text{-}R$ obtained by Lemma 2.1.15 is given explicitly below. And if $P \neq R$, then $\odot_P$ does not correspond to bialgebroid under [Szl12, Theorem 9.1] since a necessary condition is that the skew unit is equal to R_R.

Product	$A \underset{P}{\odot} B := A \underset{R}{\otimes} P^* \otimes B$	Unit	P
Associator	(invertible)	Unitors	$\begin{aligned} \lambda &: B \xrightarrow{\eta\otimes 1} P \underset{R}{\otimes} P^* \otimes B \cong P \underset{P}{\odot} B \\ \rho &: A \underset{P}{\odot} P \cong A \underset{R}{\otimes} P^* \otimes P \xrightarrow[A \underset{R}{\otimes} \varepsilon]{} A \end{aligned}$

Example 2.1.17. Two more examples were pointed out by one of the examiners:

- In the case that $\mathcal{M} = \mathsf{Cat}$ a left adjoint $i : 1 \longrightarrow R$ is the same as an initial object i in R. The right skew monoidal structure $\odot$ on R induced by i is given by the second projection $a \odot b = b$, it is strictly associative and left unital. The right unitor is the unique arrow $a \odot i = i \longrightarrow a$ in R.

- When $\mathcal{M}$ is a locally discrete monoidal bicategory, in other words, a monoidal category regarded as a monoidal bicategory, the only example is I itself since adjunctions in $\mathcal{M}$ are the isomorphisms.

Definition 2.1.18. A right bidual of an object R of $\mathcal{M}$ is an object R° equipped with two arrows n and e called unit and counit,

$$I \xrightarrow{n} R^\circ R \qquad\qquad RR^\circ \xrightarrow{e} I$$

and two cells ς_l and ς_r called left and right triangle (or snake) isomorphisms,

$$
\begin{tikzcd}
R^\circ \arrow[r, "n1"] \arrow[dr, "1"'] & R^\circ R R^\circ \arrow[d, "1e"] \arrow[dl, phantom, "\varsigma_l \cong"] \\
& R^\circ
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R R^\circ R \arrow[d, "e1"'] & R \arrow[l, "1n"'] \arrow[dl, "1"] \arrow[dl, phantom, "\cong \varsigma_r"'] \\
R &
\end{tikzcd}
$$

satisfying the swallowtail equations below.

$$
\begin{tikzcd}
R R^\circ \arrow[rr, bend left, "1"] \arrow[dr, "1n1"'] \arrow[ddd, bend right, "1"'] & \varsigma_r 1 \cong & R R^\circ \arrow[dd, "e"] \\
& R R^\circ R R^\circ \arrow[ur, "e11"] \arrow[dd, "11e"] \arrow[dr, phantom, "\cong"] & \\
1\varsigma_l \cong & & I \\
& R R^\circ \arrow[ur, "e"'] &
\end{tikzcd}
=
\begin{tikzcd}
R R^\circ \arrow[r, bend left, "e"] \arrow[r, bend right, "e"'] & I
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \arrow[r, bend left, "n"] \arrow[r, bend right, "n"'] & R^\circ R
\end{tikzcd}
=
\begin{tikzcd}
& R^\circ R \arrow[d, "11n"] \arrow[ddr, bend left, "1"] \arrow[dr, phantom, "1\varsigma_r \cong"] & \\
I \arrow[ur, "n"] \arrow[dd, "n"'] \arrow[r, phantom, "\cong"] & R^\circ R R^\circ R \arrow[ddr, "1e1"] & \\
& & \\
R^\circ R \arrow[uur, "n11"] \arrow[rr, bend right, "1"'] & \varsigma_l 1 \cong & R^\circ R
\end{tikzcd}
$$

This situation is denoted by $R \dashv R^\circ$ and called *a biduality* in $\mathcal{M}$. *Left biduals* are defined as right biduals in $\mathcal{M}^{\mathrm{op\ rev}}$. A monoidal bicategory $\mathcal{M}$ that has right biduals for every object is called *right autonomous* (or *right rigid*); if instead $\mathcal{M}$ has left biduals it is called *left autonomous*, and if it has both left and right biduals $\mathcal{M}$ is called *autonomous*.

Example 2.1.19. This is what bidualities look like in our prototypical monoidal bicategories.

- In Mod_k for a commutative ring k, the bidual of a k-algebra R is the opposite algebra R°, which has the same underlying k-vector space but the reverse multiplication.
- In $\mathsf{Span}^{\mathrm{co}}$ every set is self-bidual, the unit and counit of the biduality are constructed with the unique comonoid structure (duplicate/discard) that every set has.
- In $\mathcal{V}$-Prof for a symmetric monoidal category $\mathcal{V}$, the bidual of a $\mathcal{V}$-category $\mathcal{A}$ is the opposite category $\mathcal{A}^{\mathrm{op}}$.
- In Cat with the cartesian product (and in fact in any cartesian monoidal bicategory) a biduality is far too restrictive, because for a biduality $\mathcal{C} \dashv \mathcal{C}^\circ$ to exist both categories $\mathcal{C}$ and $\mathcal{C}^\circ$ have to be equivalent to the terminal category.

The unit object I is its own two-sided (left and right) bidual in an obvious way, where n and e are the component at I of the right and left unitors for the tensor product on $\mathcal{M}$, respectively. It is worth pointing out that in an autonomous monoidal bicategory $\mathcal{M}$, it is not assumed that right and left biduals for an object are the same, not even equivalent; however, in the case that $\mathcal{M}$ is a braided monoidal bicategory, an object *is* a left bidual if and only if it *is* a right bidual.

Remark 2.1.20. When considering $\mathcal{M}$ as a one object tricategory, the existence of a right bidual for R is the same as requiring that R has a right biadjoint in the tricategory. And in the same way that in a bicategory $\mathcal{B}$ right adjoints are unique up to isomorphism and adjunctions induce a monad; in a tricategory, right biadjoints are unique up to equivalence and biadjunctions induce a pseudomonad. And if we go one dimension down, for a one object bicategory, meaning a monoidal category $\mathcal{V}$, this translates to the fact that a duality $R \dashv R^\circ$ in $\mathcal{V}$ induces a monoid $R^\circ \otimes R$, sometimes called *enveloping* monoid [Sch00]. And for a monoidal bicategory $\mathcal{M}$ this means that right biduals are unique up to equivalence and a biduality $R \dashv R^\circ$ induces a monoidale $R^\circ R$ with the structure below.

Product

$$R^\circ R R^\circ R \xrightarrow{1e1} R^\circ R$$

Unit

$$I \xrightarrow{n} R^\circ R$$

Associator

$$\begin{tikzcd}
R^\circ R R^\circ R R^\circ R \arrow[r, "1e11"] \arrow[d, "11e1"'] \arrow[dr, phantom, "\cong"] & R^\circ R R^\circ R \arrow[d, "1e1"] \\
R^\circ R R^\circ R \arrow[r, "1e1"'] & R^\circ R
\end{tikzcd}$$

Left and right unitors

$$\begin{tikzcd}
R^\circ R \arrow[r, "n1"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\varsigma_l 1 \cong"] & R^\circ R R^\circ R \arrow[d, "1e1"] & R^\circ R \arrow[l, "1n"'] \arrow[dl, "1"] \arrow[dl, phantom, shift right=3, "\cong 1\varsigma_r"] \\
 & R^\circ R &
\end{tikzcd}$$

The pentagon axiom (SKM1) is an instance of the coherence of the interchange law in $\mathcal{M}$, and the triangle axiom (SKM2) is exactly one of the swallowtail equations of the biduality. We call monoidales that arise from bidualities *enveloping monoidales*, as with the case of a monoidal category $\mathcal{V}$.

Likewise, it is possible to generalise the fact that an adjunction $F \dashv G$ in Cat is also given by a pair of functors F, G and a natural isomorphism of sets $\hom(Fx, y) \cong \hom(x, Gy)$ to the case of biadjunctions in a tricategory (see [Ver92, Example 1.1.7] for biadjunctions between bicategories), here we spell it out in terms of bidualities.

Proposition 2.1.21. *In a monoidal bicategory $\mathcal{M}$ the following statements are equivalent:*

(i). There is a biduality $R \dashv R^\circ$.

(ii). There are objects R, R° and for every pair of objects X and Y in $\mathcal{M}$ an adjoint equivalence of categories as shown,

$$\Phi_{X,Y} : \mathcal{M}(RX, Y) \simeq \mathcal{M}(X, R^\circ Y) : \Psi_{X,Y}$$

which is pseudonatural in X and Y. Its unit $N_{X,Y}$ and counit $E_{X,Y}$,

$$
\begin{tikzcd}
 & \mathcal{M}(X,R^\circ Y) \arrow[dr, "\Psi_{X,Y}"] \arrow[d, phantom, "\wr\|N_{X,Y}"] & \\
\mathcal{M}(RX,Y) \arrow[ur, "\Phi_{X,Y}"] \arrow[rr, "1"'] & \, & \mathcal{M}(RX,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}(X,R^\circ Y) \arrow[rr, "1"] \arrow[dr, "\Psi_{X,Y}"'] & \arrow[d, phantom, "\wr\|E_{X,Y}"] & \mathcal{M}(X,R^\circ Y) \\
 & \mathcal{M}(RX,Y) \arrow[ur, "\Phi_{X,Y}"'] &
\end{tikzcd}
$$

are modifications in X and Y satisfying the snake equations. Furthermore, for every Z in $\mathcal{M}$ there are isomorphisms $A^Z_{X,Y}$ and $B^Z_{X,Y}$,

$$
\begin{tikzcd}
\mathcal{M}(RX,Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\_\otimes Z"'] \arrow[dr, phantom, "A^Z_{X,Y}\cong"] & \mathcal{M}(X,R^\circ Y) \arrow[d, "\_\otimes Z"] \\
\mathcal{M}(RXZ,YZ) \arrow[r, "\Phi_{XZ,YZ}"'] & \mathcal{M}(XZ,R^\circ YZ)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}(X,R^\circ Y) \arrow[r, "\Psi_{X,Y}"] \arrow[d, "\_\otimes Z"'] \arrow[dr, phantom, "B^Z_{X,Y}\cong"] & \mathcal{M}(RX,Y) \arrow[d, "\_\otimes Z"] \\
\mathcal{M}(XZ,R^\circ YZ) \arrow[r, "\Psi_{XZ,YZ}"'] & \mathcal{M}(RXZ,YZ)
\end{tikzcd}
$$

which are modifications in X and Y satisfying two axioms, one relating A, B and N, and a similar one relating A, B and E.

$$
\begin{tikzcd}
 & \mathcal{M}(X,R^\circ Y) \arrow[dr, "\Psi_{X,Y}"] \arrow[d, phantom, "\wr\|N_{X,Y}"] & \\
\mathcal{M}(RX,Y) \arrow[ur, "\Phi_{X,Y}"] \arrow[rr, "1"'] \arrow[dd, "\_\otimes Z"'] & \, & \mathcal{M}(RX,Y) \arrow[dd, "\_\otimes Z"] \\
 & & \\
\mathcal{M}(RXZ,YZ) \arrow[rr, "1"'] & & \mathcal{M}(RXZ,YZ)
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{M}(X,R^\circ Y) \arrow[dr, "\Psi_{X,Y}"] \arrow[dd, "\_\otimes Z"] & \\
\mathcal{M}(RX,Y) \arrow[ur, "\Phi_{X,Y}"] \arrow[dd, "\_\otimes Z"'] \arrow[r, phantom, "A^Z_{X,Y}\cong"] & \, \arrow[r, phantom, "B^Z_{X,Y}\cong"] & \mathcal{M}(RX,Y) \arrow[dd, "\_\otimes Z"] \\
 & \mathcal{M}(XZ,R^\circ YZ) \arrow[dr, "\Psi_{XZ,YZ}"] \arrow[d, phantom, "\wr\|N_{XZ,YZ}"] & \\
\mathcal{M}(RXZ,YZ) \arrow[ur, "\Phi_{XZ,YZ}"] \arrow[rr, "1"'] & \, & \mathcal{M}(RXZ,YZ)
\end{tikzcd}
$$

(iii). There are objects R and R°, and for every pair of objects X and Y in $\mathcal{M}$ an adjoint equivalence of categories as shown,

$$\mathcal{M}(XR^\circ,Y)\simeq\mathcal{M}(X,YR)$$

with analogous structure and properties as in (ii).

Proof. [Sketch]

($i \Rightarrow ii$) Define the functor $\Phi_{X,Y}:\mathcal{M}(RX,Y)\longrightarrow\mathcal{M}(X,R^\circ Y)$ on an arrow $f:RX\longrightarrow Y$ to be its *transpose* under the biduality,

$$X\xrightarrow{n1}R^\circ RX\xrightarrow{1f}R^\circ Y$$

and the pseudoinverse $\Psi_{X,Y}:\mathcal{M}(X,R^\circ Y)\longrightarrow\mathcal{M}(RX,Y)$ on an arrow $g:X\longrightarrow R^\circ Y$ to be its *transpose* under the biduality.

$$RX\xrightarrow{1g}RR^\circ Y\xrightarrow{e1}Y$$

These functors are pseudonatural in X and Y and form an equivalence since there are isomorphisms, $E_{X,Y} : \Psi_{X,Y}\Phi_{X,Y}f \cong f$ and $N_{X,Y} : g \cong \Psi_{X,Y}\Phi_{X,Y}g$ given by the isomorphisms below,

$$
\begin{tikzcd}
X \arrow[r, "n1"] \arrow[d, "g"'] \arrow[dr, phantom, "\cong"] & R^\circ RX \arrow[d, "11g"] \\
R^\circ Y \arrow[r, "n11"] \arrow[dr, "1"'] \arrow[r, phantom, shift right=4, "\varsigma_l 1 \cong"] & R^\circ RR^\circ Y \arrow[d, "1e1"] \\
& R^\circ Y
\end{tikzcd}
\qquad
\begin{tikzcd}
RR^\circ Y \arrow[d, "e1"'] \arrow[r, phantom, shift right=6, "\cong"] & RR^\circ RX \arrow[l, "11f"'] \arrow[d, "e11"] & RX \arrow[l, "1n1"'] \arrow[dl, "1"] \arrow[dl, phantom, shift left=3, "\cong \varsigma_r 1"] \\
Y & RX \arrow[l, "f"] &
\end{tikzcd}
$$

which are clearly modifications in X and Y, and satisfy the triangle equations. The isomorphisms $A^Z_{X,Y}$ and $B^Z_{X,Y}$ are given by the structure of $\mathcal{M}$; these are instances of the distributivity of the tensor product along the horizontal composition, and thus are modifications in X and Y. It is routine to prove that they also satisfy the two required axioms.

($ii \Rightarrow i$) Define the unit of the biduality as $n := \Phi_{I,R}(\mathrm{id}_R)$, and the counit as $e := \Psi_{R^\circ,I}(\mathrm{id}_{R^\circ})$, then one can prove that there are two isomorphisms as shown.

$$
\begin{tikzcd}
\mathcal{M}(RX,Y) \arrow[rr, "\Phi_{X,Y}"] \arrow[dr, "R^\circ\otimes -"'] & \cong & \mathcal{M}(X,R^\circ Y) \\
& \mathcal{M}(R^\circ RX, R^\circ Y) \arrow[ur, "\mathcal{M}(nX{,}R^\circ Y)"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}(X,R^\circ Y) \arrow[rr, "\Psi_{X,Y}"] \arrow[dr, "R\otimes -"'] & \cong & \mathcal{M}(RX,Y) \\
& \mathcal{M}(RX, RR^\circ Y) \arrow[ur, "\mathcal{M}(RX{,}eY)"'] &
\end{tikzcd}
$$

Call the bottom composites $\bar{\Phi}_{X,Y}$ and $\bar{\Psi}_{X,Y}$ respectively; these functors are pseudonatural in X and Y and constitute an adjoint equivalence of categories, since $\Phi_{X,Y}$ and $\Psi_{X,Y}$ do. The unit $\bar{N}_{X,Y} : \mathrm{id}_{\mathcal{M}(RX,Y)} \longrightarrow \bar{\Psi}_{X,Y}\bar{\Phi}_{X,Y}$ and the counit $\bar{E}_{X,Y} : \bar{\Phi}_{X,Y}\bar{\Psi}_{X,Y} \longrightarrow \mathrm{id}_{\mathcal{M}(X,R^\circ Y)}$ are modifications in X and Y because $N_{X,Y}$ and $E_{X,Y}$ are. Define the snake isomorphisms as $\varsigma_l := \bar{E}_{R^\circ,I}$ and $\varsigma_r := \bar{N}_{I,R}$, the triangle equations for $\bar{N}_{X,Y}$ and $\bar{E}_{X,Y}$ imply the swallowtail equations for ς_l and ς_r.

($i \Leftrightarrow iii$) This is proven in a similar fashion as ($i \Leftrightarrow ii$). ■

Remark 2.1.22. Every autonomous monoidal bicategory $\mathcal{M}$ is a right closed monoidal bicategory [DS97, Section 2]: the right internal hom is given by $[X,Y] := X^\circ Y$ because of (ii) in the previous lemma. This allows us to think of the enveloping monoidale $R^\circ R$ as the endohom monoidale. An opmonoidal arrow whose source is the monoidal unit I may be called an internal comonoid of the target skew monoidale. Even if the monoidal bicategory is not right closed monoidal or autonomous, for an object R with a bidual R° we may still talk about internal comonoids of the enveloping monoidale $R^\circ R$. These are opmonoidal arrows $I \longrightarrow R^\circ R$, and it is not hard to see that under transposition internal comonoids of $R^\circ R$ correspond to comonads on R, see Remark 2.3.9 below.

Lemma 2.1.23. *For every two bidualities $R \dashv R^\circ$ and $S \dashv S^\circ$ in $\mathcal{M}$ there is an adjoint equivalence of categories*

$$\mathcal{M}(R,S) \simeq \mathcal{M}(S^\circ, R^\circ).$$

More generally,

$$\mathcal{M}(RX, YS) \simeq \mathcal{M}(XS^\circ, R^\circ Y).$$

Proof.

$$\mathcal{M}(RX, YS) \simeq \mathcal{M}(X, R^\circ YS) \simeq \mathcal{M}(XS^\circ, R^\circ Y).$$

■

Remark 2.1.24. If $\mathcal{M}$ is right autonomous, the axiom of choice allows us to choose a bidual R° for every object R, thus the equivalence of Lemma 2.1.23 gives rise to a strong monoidal pseudofunctor in the up to equivalence sense since $(XY)^\circ \simeq Y^\circ X^\circ$.

$$\mathcal{M}^{\mathrm{rev\ op}} \xrightarrow{(\ )^\circ} \mathcal{M}$$

This pseudofunctor is also locally an equivalence in the sense that, for every pair of objects, its action on homs is an equivalence. Furthermore, if $\mathcal{M}$ is autonomous $(\ )^\circ$ is a strong monoidal biequivalence of monoidal bicategories [Str80, 1.33 for definition]. Its pseudoinverse is defined to take an object to its chosen left bidual, thus $(\ )^\circ$ is essentially surjective on objects (in the up to equivalence sense) by the existence of left biduals and the uniqueness up to equivalence of right biduals. This appears first in [DS97, Section 2] but the authors forget to mention left autonomy. When not every object has a right bidual one may restrict the domain of $(\ )^\circ$ to be the full subbicategory on the objects that have right biduals, and the codomain to the full subbicategory on the objects that have left biduals; the same argument proves that these two monoidal subbicategories are monoidally biequivalent. This biequivalence allows us to transpose many structures without losing information, for example, adjunctions.

Lemma 2.1.25. *For every two bidualities $S \dashv S^\circ$ and $R \dashv R^\circ$ in $\mathcal{M}$, adjunctions $f_* \dashv f^* : S \to R$ are in correspondence with adjunctions $f_\circ \dashv f^\circ : S^\circ \to R^\circ$.*

Proof.
Adjunctions are preserved and reflected by biequivalences, thus restricting $(\ )^\circ$ where it is a biequivalence completes the proof. We write this assignation explicitly to fix some notation.

$$\begin{tikzcd} R \arrow[d, bend left, "f^*"] \arrow[d, phantom, "\dashv"] \\ S \arrow[u, bend left, "f_*"] \end{tikzcd} \qquad \begin{aligned} f_\circ &:= (f^*)^\circ : S^\circ \xrightarrow{n1} R^\circ R S^\circ \xrightarrow{1f^*1} R^\circ S S^\circ \xrightarrow{1e} R^\circ \\ f^\circ &:= (f_*)^\circ : R^\circ \xrightarrow{n1} S^\circ S R^\circ \xrightarrow{1f1} S^\circ R R^\circ \xrightarrow{1e} S^\circ \end{aligned} \qquad \begin{tikzcd} R^\circ \arrow[d, bend left, "f^\circ"] \arrow[d, phantom, "\dashv"] \\ S^\circ \arrow[u, bend left, "f_\circ"] \end{tikzcd}$$

The unit $\eta_\circ$ and counit $\varepsilon_\circ$ are defined in a similar way. ■

The adjunction $f_\circ \dashv f^\circ$ is called the *opposite or mate* adjunction of $f_* \dashv f^*$. In what follows adjunctions where $S = I$ and their opposites are constantly used, so we spell out the opposite adjunction to have at hand for future calculations.

$$\begin{tikzcd} R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i"] \end{tikzcd} \qquad \begin{aligned} i_\circ &: I \xrightarrow{n} R^\circ R \xrightarrow{1i^*} R^\circ \\ i^\circ &: R^\circ \xrightarrow{i1} R R^\circ \xrightarrow{e} I \end{aligned} \qquad \begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

$$
\begin{tikzcd}
I \arrow[r, "i_\circ"] \arrow[dr, "1"'] & R^\circ \arrow[d, "i^\circ"] \arrow[dl, phantom, "\eta_\circ \Nearrow"] \\
& I
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
I \arrow[rr, "n"] \arrow[dr, "i"'] \arrow[ddddrrr, bend right=60, "1"'] & & R^\circ R \arrow[r, "1i^*"] \arrow[dd, "i11"] \arrow[dl, phantom, "\cong"] & R^\circ \arrow[dd, "i1"] \arrow[ddl, phantom, "\cong"] \\
& R \arrow[dr, "1n"] \arrow[ddr, bend right=50, "1"'] \arrow[dd, phantom, "\eta \Nearrow"] & & \\
& & RR^\circ R \arrow[r, "11i^*"] \arrow[d, "e1"] \arrow[dl, phantom, "\cong"] & RR^\circ \arrow[dd, "e"] \arrow[dl, phantom, "\cong"] \\
& & R \arrow[dr, "i^*"'] & \\
& & & I
\end{tikzcd}
$$

$$
\begin{tikzcd}
R^\circ \arrow[d, "i^\circ"'] \arrow[dr, "1"] & \\
I \arrow[r, "i_\circ"'] \arrow[ur, phantom, "\varepsilon_\circ \Nearrow"] & R^\circ
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
R^\circ \arrow[dr, "n1"] \arrow[dd, "i1"'] \arrow[ddddrrr, bend left=50, "1"] & & & \\
& R^\circ R R^\circ \arrow[dr, "1i^*1"'] \arrow[ddrr, bend left=40, "1"] \arrow[rr, phantom, "1\cong"] & & \\
RR^\circ \arrow[dd, "e"'] & & R^\circ R^\circ \arrow[dr, "1i1"'] \arrow[ur, phantom, "1\varepsilon 1 \Nearrow"] & \\
& \cong & & R^\circ R R^\circ \arrow[d, "1e"] \\
I \arrow[r, "n"'] & R^\circ R \arrow[rr, "1i^*"'] & & R^\circ
\end{tikzcd}
$$

Note that the associated monad of $i_\circ \dashv i^\circ$ has the same underlying arrow as the monad for $i \dashv i^*$ up to isomorphism, but the multiplication is the opposite one, in the sense that it is reversed.

Another characteristic of biequivalences is that they preserve all lax bilimits and lax bicolimits that exist in their domain; and since Eilenberg-Moore objects are lax bilimits and Kleisli objects are lax bicolimits we conclude the following.

Lemma 2.1.26. *If $\mathcal{M}$ is an autonomous monoidal bicategory then an adjunction in $\mathcal{M}$ is monadic if and only if the opposite adjunction is opmonadic.*

Proof.
By using the biequivalence $(\ )^\circ$, an adjunction $f_* \dashv f^*$ is monadic in $\mathcal{M}$ if and only if the opposite adjunction $f_\circ \dashv f^\circ$ is monadic in $\mathcal{M}^{\mathrm{op}}$. And the latter happens if and only if the adjunction $f_\circ \dashv f^\circ$ is opmonadic in $\mathcal{M}$. ■

In our main examples of monoidal bicategories (Mod_k, $\mathsf{Span}^{\mathrm{co}}$, and Prof) an adjunction is opmonadic if and only if the opposite adjunction is *opmonadic* too! This is due to the fact that lax bilimits and lax bicolimits coincide for bicategories which are locally cocomplete, i.e. all hom categories are cocomplete and composition is a cocontinuous functor, see [Str81, Proposition 1].

We close this section with the example that gave this chapter its name. Recall the definition of an $R|S$-coalgebroid ([Szl05, Definition 1.1] or [Böh09, pp. 185]) which first appeared under the name $R|S$-coring in [Tak87, Definition 3.5].

Definition 2.1.27. Let R and S be k-algebras for a commutative ring k. An *$R|S$-coalgebroid* consists of a module C in RS-Mod-RS, a morphism called comultiplication $\delta : C \longrightarrow C \otimes_S C$ in RS-Mod-RS in which $C \otimes_S C$ uses the two-sided R-module structure given by $r.(c \otimes c').r' = cr' \otimes rc'$, that is (i). $\delta(scs') = \sum sc_{(1)} \otimes c_{(2)}s'$

(ii). $\delta(rcr') = \sum c_{(1)}r' \otimes rc_{(2)}$

and a morphism called counit $\varepsilon : C \longrightarrow S$ in S-Mod-S, that is

(iii). $\varepsilon(scs') = s\varepsilon(c)s'$

subject to the following axioms.

(iv). $\sum rc_{(1)} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)}r$

(v). $\varepsilon(rc) = \varepsilon(cr)$

(vi). (C, ε, δ) forms a comonoid in the monoidal category S-Mod-S

Note that axiom (iv) may be rewritten using the two-sided R-module structure on $C \otimes_S C$ given by $r\cdot(c \otimes c')\cdot r' = (rc) \otimes (c'r')$, which is different than the one used in (i) and (ii).

(iv'). $r\cdot\delta(c) = \delta(c)\cdot r$, the image of the comultiplication δ is in the R-centralizer of $C \otimes_S C$.

According to [Tak87], [Szl05], or [Hai08], conditions (iv) and (v) are logically equivalent. In Example 2.3.13 at the end of Section 2.3, we give another equivalent and simpler definition of a coalgebroid by using the tool developed in that section: oplax actions.

It is immediate from the definition of an $R|S$-coalgebroid that if $R = k$ then conditions (ii), (iv), and (v) are trivial, thus a $k|S$-coalgebroid is nothing but a comonoid in S-Mod-S, i.e. an S-coring. Going up one dimension, since S-Mod-S is a hom category of the monoidal bicategory Mod_k, then an S-coring is a comonad in Mod_k on S. And, as mentioned in Remark 2.1.22, comonads correspond to opmonoidal arrows by transposition, which implies that $k|S$-coalgebroids correspond to opmonoidal arrows $k \longrightarrow S^{\circ}S$.

In fact, all $R|S$-coalgebroids *are* opmonoidal arrows; the following lemma is the behaviour on objects of an isomorphism of bicategories between the full subbicategory $\mathsf{OpMon}^{\mathrm{e}}(\mathsf{Mod}_k)$ of $\mathsf{OpMon}(\mathsf{Mod}_k)$ on the enveloping monoidales in Mod_k, and the bicategory Cgb_k, defined in [Szl05], whose objects are k-algebras and arrows $R \longrightarrow S$ are $R|S$-coalgebroids.

$$\mathsf{OpMon}^{\mathrm{e}}(\mathsf{Mod}_k) \cong \mathsf{Cgb}_k$$

In the proof, there are modules that have more than two actions with respect to the same k-algebra and tensor products of these modules over one or more of these actions. To avoid confusion, we use coloured k-algebras as subscripts for modules and tensor products to distinguish which actions are being used while tensoring. For example, ${}_RM_S$ is a module in R-Mod-S, and with another module ${}_SN_T$ we can form the tensor product ${}_RM_S \underset{S}{\otimes} {}_SN_T$ to get a module ${}_RL_T$.

Lemma 2.1.28. *For a commutative ring k, opmonoidal arrows in the bicategory Mod_k of the form $C : R^{\circ}R \longrightarrow S^{\circ}S$ are $R|S$-coalgebroids.*

Proof.

The isomorphism $R^\circ R$-Mod-$S^\circ S \cong RS$-Mod-RS is used throughout without changing the name of the modules. Let C be an opmonoidal arrow as in the statement. One may rewrite the structure cell C^0 in the language of the category Mod-$S^\circ S$ instead of the language of the monoidal bicategory Mod_k. Both notations are shown below.

$$\begin{tikzcd} & k \arrow[dl, "n"'] \arrow[dr, "n"] \arrow[d, phantom, "C^0 \Nearrow"] & \\ R^\circ R \arrow[rr, "C"'] & & S^\circ S \end{tikzcd} \qquad R_{R^\circ R} \underset{R^\circ R}{\otimes} {}_{R^\circ R}C_{S^\circ S} \xrightarrow{C^0} S_{S^\circ S}$$

And module morphisms C^0 are in bijective correspondence with module morphisms $\varepsilon : C \longrightarrow S$ in S-Mod-S for which the condition (v) $\varepsilon(rc) = \varepsilon(cr)$ is satisfied. Now, one needs to be more careful with the structure cell C^2 as there are several R-actions which may be confusing. Here is where the colours are most helpful; C^2 is a cell in Mod_k as follows.

$$\begin{tikzcd} R^\circ R R^\circ R \arrow[r, "CC"] \arrow[d, "1e1"'] \arrow[dr, phantom, "C^2 \Nearrow"] & S^\circ S S^\circ S \arrow[d, "1e1"] \\ R^\circ R \arrow[r, "C"'] & S^\circ S \end{tikzcd}$$

But now, one may rewrite it in the language of $R^\circ R R^\circ R$-Mod-$S^\circ S$, hence C^2 is a module morphism with source and target as shown below.

$$\left({}_{R^\circ}R_{R^\circ} \otimes {}_{RR^\circ}R \otimes {}_{R}R_{R}\right) \underset{R^\circ R}{\otimes} {}_{R^\circ R}C_{S^\circ S} \xrightarrow{C^2} \left({}_{R^\circ R}C_{S^\circ S} \otimes {}_{R^\circ R}C_{S^\circ S}\right) \underset{S^\circ S S^\circ S}{\otimes} \left({}_{S^\circ}S_{S^\circ} \otimes {}_{SS^\circ}S \otimes {}_{S}S_{S}\right)$$

The source may be simplified as follows,

$$\left({}_{R^\circ}R_{R^\circ} \otimes {}_{RR^\circ}R \otimes {}_{R}R_{R}\right) \underset{R^\circ R}{\otimes} {}_{R^\circ R}C_{S^\circ S} \cong {}_{RR^\circ}R \otimes {}_{R^\circ R}C_{S^\circ S}$$

and the target is simplified as below.

$$\begin{aligned} & \left({}_{R^\circ R}C_{S^\circ S} \otimes {}_{R^\circ R}C_{S^\circ S}\right) \underset{S^\circ S S^\circ S}{\otimes} \left({}_{S^\circ}S_{S^\circ} \otimes {}_{SS^\circ}S \otimes {}_{S}S_{S}\right) \\ \cong\ & \left({}_{R^\circ R}C_{S^\circ S} \otimes {}_{R^\circ R}C_{S^\circ S}\right) \underset{SS^\circ}{\otimes} {}_{SS^\circ}S \\ \cong\ & {}_{R^\circ R}C_{S^\circ S} \underset{S}{\otimes} {}_{R^\circ R}C_{S^\circ S} \end{aligned}$$

Thus in $R^\circ R R^\circ R$-Mod-$S^\circ S$, module morphisms C^2 are in bijection with module morphisms of the following form,

$${}_{RR^\circ}R \otimes {}_{R^\circ R}C_{S^\circ S} \longrightarrow {}_{R^\circ R}C_{S^\circ S} \underset{S}{\otimes} {}_{R^\circ R}C_{S^\circ S}$$

which in turn are in bijection with module morphisms

$$\delta : {}_{R^\circ R}C_{S^\circ S} \longrightarrow {}_{R^\circ}C_{S^\circ S} \underset{S}{\otimes} {}_{R}C_{S^\circ S}$$

in $R^\circ R$-Mod-$S^\circ S$ which satisfy (iv) $\sum r c_{(1)} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)} r$, by using the R-actions. Now that we have translated the data, the three axioms of a comonoid for (C, ε, δ) translate exactly into the those of an opmonoidal arrow for (C, C^0, C^2). ■

2.2 Opmonoidal ⊣ Monoidal Adjunctions and Opmonadicity

In the sequel various opmonadic adjunctions in $\mathcal{M}$ are going to play a central role in various theorems, and we require that these adjunctions behave well with respect to the overall structure of the monoidal bicategory.

Definition 2.2.1. An *opmonadic-friendly monoidal bicategory* $\mathcal{M}$, is a monoidal bicategory such that

- Tensoring with objects on either side preserves opmonadicity.
- Composing with arrows on either side preserves any existing reflexive coequaliser in the hom categories.

A fairly common behaviour of an adjunction in a monoidal bicategory $\mathcal{M}$ between objects that have a (skew) monoidal structure is that the left adjoint is opmonoidal while the right adjoint is monoidal. Surprisingly, these two properties are logically equivalent: for if an opmonoidal arrow has a right adjoint, then the mates of its opmonoidal constraints provides the right adjoint with a monoidal structure and vice versa. Moreover, the right adjoint is strong monoidal if and only if the left adjoint, the unit, and the counit are all opmonoidal, in which case the whole adjunction is in $\mathsf{SkOpMon}(\mathcal{M})$. All of this fits along with a phenomenon called doctrinal adjunction [Kel74].

Definition 2.2.2. An *opmonoidal ⊣ monoidal adjunction* $f \dashv g$ in a monoidal category $\mathcal{M}$, is an adjunction between (skew) monoidales where the left adjoint is opmonoidal and the right adjoint is monoidal.

Examples of opmonoidal ⊣ monoidal adjunctions are presented in what follows.

Lemma 2.2.3. *For every right skew monoidale $(M, m, u, \alpha, \lambda, \rho)$, the unit $u : I \longrightarrow M$ is a (normal) opmonoidal arrow, where I has the trivial monoidal structure. The opmonoidal constraints are given by the diagrams below.*

$$\begin{tikzcd} I \arrow[r, "uu"] \arrow[d, "1"'] \arrow[dr, "u"'] & MM \arrow[d, "m"] \arrow[dl, phantom, "\lambda u \Nearrow"] \\ I \arrow[r, "u"'] & M \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "1"'] \arrow[dr, "u"] & \\ I \arrow[rr, "u"'] & & M \end{tikzcd}$$

■

Remark 2.2.4. As a consequence, every arrow $i : I \longrightarrow R$ that has a right adjoint i^* is automatically opmonoidal, taking the skew monoidal structure on R induced by the adjunction $i \dashv i^*$ in Lemma 2.1.15. In other words, every adjunction such that the source of the left adjoint is I is automatically an "opmonoidal $\dashv$ monoidal adjunction". In general, the unit and counit are neither monoidal nor opmonoidal.

Proposition 2.2.5. *For every biduality $R \dashv R^\circ$ and every adjunction $i \dashv i^*$ the equality between the triangles below holds.*

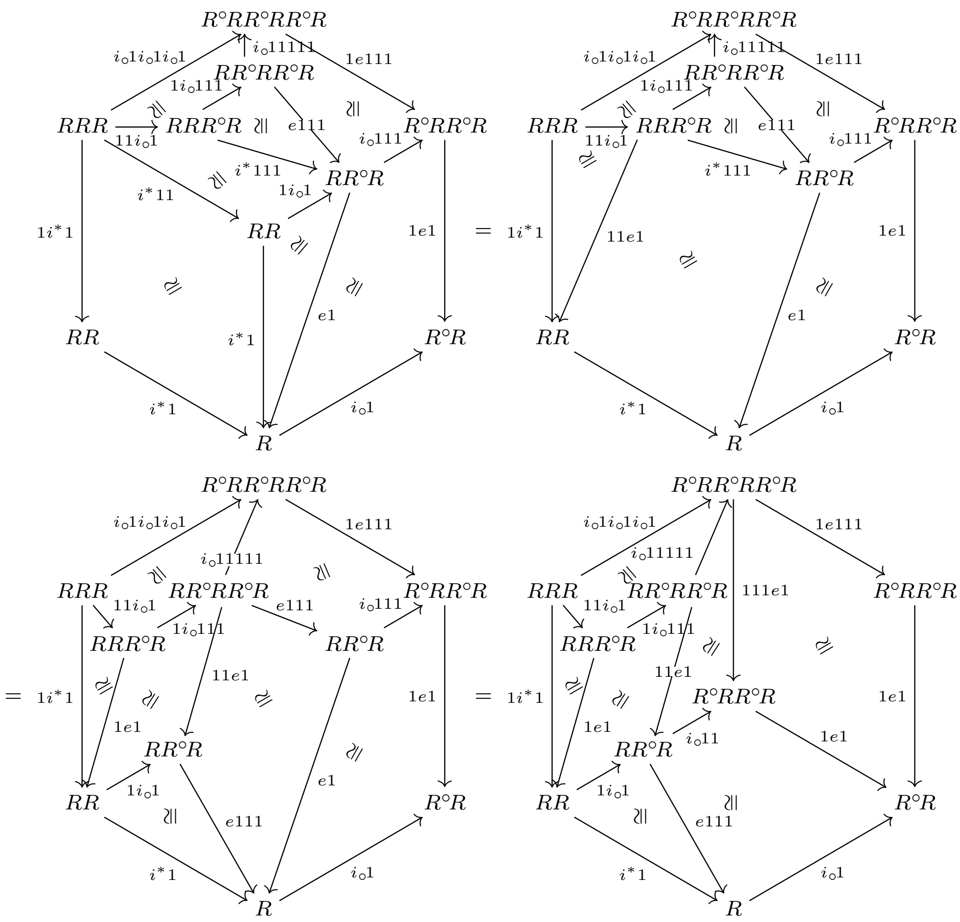
$$
\begin{tikzcd}
 & & I \arrow[dll, "i"'] \arrow[dd, "i_\circ"] \arrow[dr, "n"] & & \\
 & & & R^\circ R \arrow[dl, "1i^*"'] \arrow[dr, "1"] & \\
 & \cong & R^\circ \arrow[r, phantom, "\overset{1\varepsilon}{\Rightarrow}"] \arrow[rr, "1i"] & & R^\circ R \\
R \arrow[rrrr, "i_\circ 1"'] & & & & R^\circ R
\end{tikzcd}
=
\begin{tikzcd}
 & I \arrow[dl, "i"'] \arrow[r, phantom, "\cong"] \arrow[dr, "n"] & \\
R \arrow[rr, "i_\circ 1"'] & \overset{\varepsilon_\circ 1}{\Rrightarrow} & R^\circ R \arrow[ll, "i^\circ 1"'] \arrow[d, "1"] \\
 & & R^\circ R
\end{tikzcd}
$$

Furthermore, taking the skew monoidal structure on R induced by the adjunction $i \dashv i^*$ as in Lemma 2.1.15, and the enveloping monoidale $R^\circ R$ induced by the biduality $R \dashv R^\circ$ as in Remark 2.1.20, the arrow $i_\circ 1 : R \longrightarrow R^\circ R$ is an opmonoidal arrow and its structure cells are the triangle above and the square below.

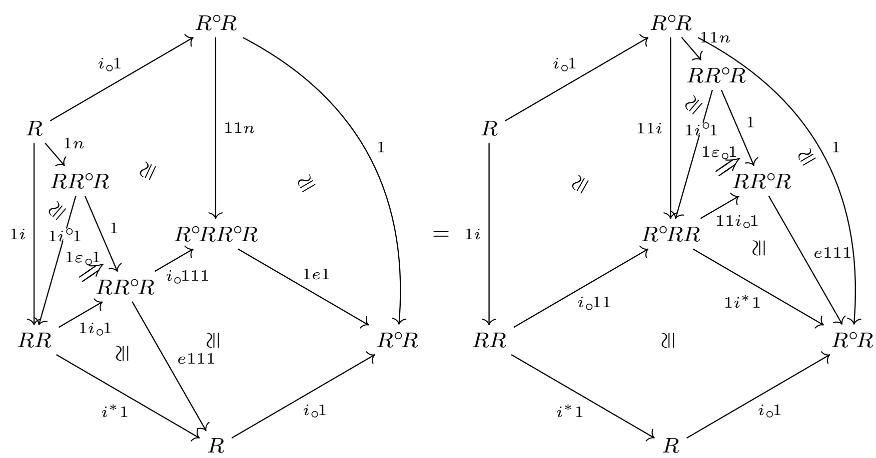
$$
\begin{tikzcd}
RR \arrow[r, "1i_\circ 1"] \arrow[d, "i^*1"'] \arrow[dr, phantom, "\cong"] & RR^\circ R \arrow[r, "i_\circ 111"] \arrow[dl, "e1"] \arrow[dr, phantom, "\cong"] & R^\circ RR^\circ R \arrow[d, "1e1"] \\
R \arrow[rr, "i_\circ 1"'] & & R^\circ R
\end{tikzcd}
$$

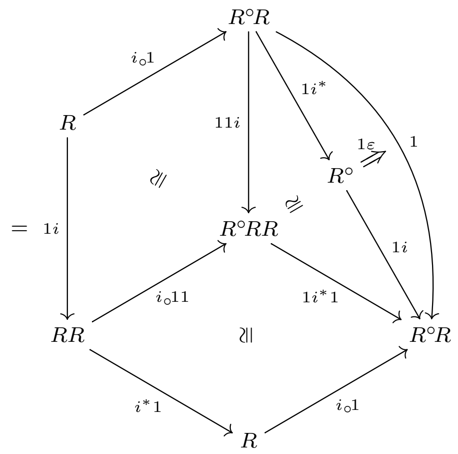
Proof.

The equality between the triangular cells in the statement follows either by direct calculation using the definition of $\varepsilon_\circ$ in terms of ε, or by transposing both triangles along the equivalence $\mathcal{M}(I, R^\circ R) \simeq \mathcal{M}(R, R)$, and noticing that this yields the cell ε in each case. Now we prove that $i_\circ 1$ is opmonoidal; axiom (OM1) follows from the calculation below.

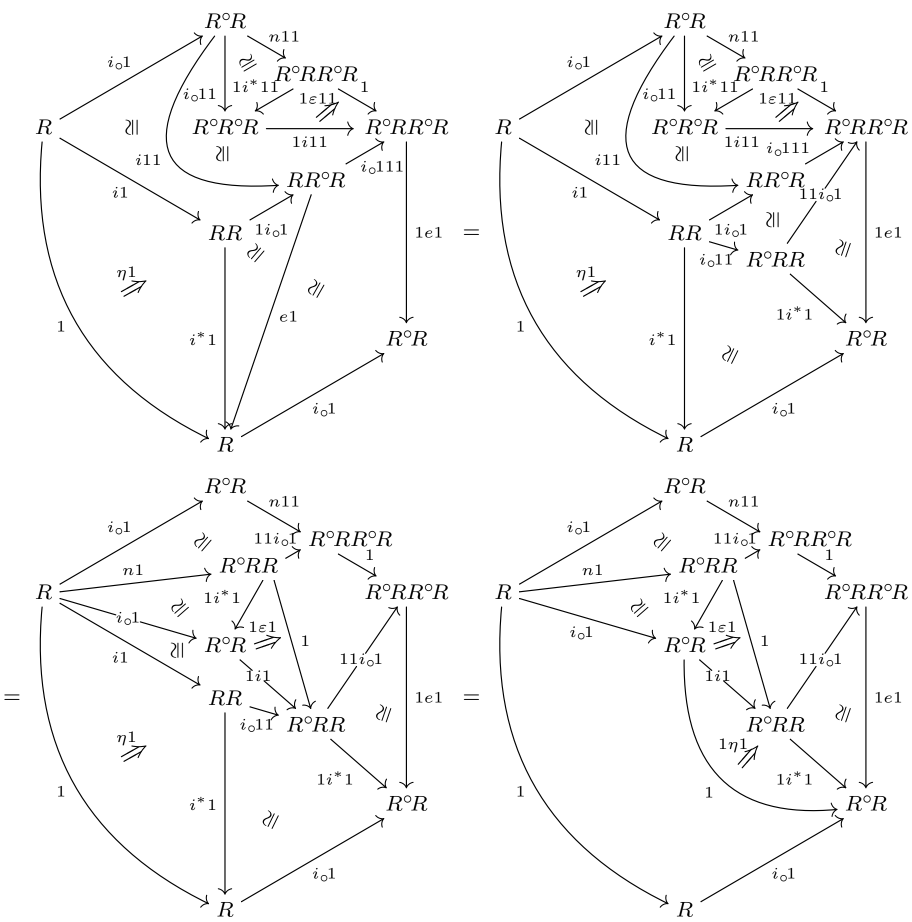


Axiom (OM2) for $i_\circ 1$ is verified as follows.





And axiom (OM3) for $i_\circ 1$ holds by the calculation below.



$$
\begin{tikzcd}
& R^\circ R \arrow[dr, "n11"] \arrow[ddr, bend right, "1"'] & \\
R \arrow[ur, "i_\circ 1"] \arrow[ddr, bend right, "1"'] & & R^\circ R R^\circ R \arrow[d, "1e1"] \arrow[l, phantom, "\cong"] \\
& & R^\circ R \\
& R \arrow[ur, "i_\circ 1"'] &
\end{tikzcd}
$$

■

In Lemma 2.2.3 and Proposition 2.2.5 we exhibit two opmonoidal left adjoints $i : I \longrightarrow R$ and $i_\circ 1 : R \longrightarrow R^\circ R$, which may be composed into a new opmonoidal left adjoint $i_\circ i : I \longrightarrow R^\circ R$. And by a doctrinal adjunction argument, the opmonoidal structures on the left adjoints i, $i_\circ 1$ and $i_\circ i$ induce monoidal structures on the right adjoints i^*, $i^\circ 1$ and $i^\circ i^*$ which in general are not strong monoidal, hence these adjunctions do not belong to $\mathsf{OpMon}(\mathcal{M})$.

2.2.1 A Bicategorical Theorem

We proceed with one of the main results: in an opmonadic-friendly monoidal bicategory $\mathcal{M}$ the functor

$$\mathsf{SkOpMon}(i_\circ 1, N) : \mathsf{OpMon}(R^\circ R, N) \xrightarrow[\simeq]{} \mathsf{SkOpMon}(R, N)$$

is an equivalence of categories, provided that the opmonoidal arrow $i_\circ 1$ in Proposition 2.2.5 is opmonadic, and N is a genuine monoidale (not just a skew one). This is stated formally as Theorem 2.2.8 below. Its proof uses some of the important techniques employed throughout this thesis, and it naturally breaks down into two parts: an isomorphism followed by an equivalence of categories, therefore, to gain some clarity we present these separately in Lemma 2.2.7 and Theorem 2.2.8 below. Taking the middle step and most of the technicalities, there is a category that we denote by $\mathcal{X}(R, N)$. One way to informally interpret the category $\mathcal{X}(R, N)$ is as follows: its objects are opmonoidal arrows $R \longrightarrow N$ equipped with a module structure for the monad induced by the adjunction

$$
\begin{tikzcd}
R^\circ R \arrow[d, bend left, "i^\circ 1"] \arrow[d, phantom, "\dashv"] \\
R \arrow[u, bend left, "i_\circ 1"]
\end{tikzcd}
$$

together with compatibility conditions between the opmonoidal and the module structures which involve the "opmonoidal ⊣ monoidal" structure of the adjunction $i_\circ 1 \dashv i^\circ 1$. What we show in Lemma 2.2.7 is that this extra module structure on the opmonoidal arrows $R \longrightarrow N$ is in fact redundant, hence the isomorphism $\mathcal{X}(R, N) \cong \mathsf{SkOpMon}(R, N)$. And when $i_\circ 1 \dashv$

$i^\circ 1$ is opmonadic the category $\mathcal{X}(R,N)$ of "opmonoidal $\dashv$ monoidal modules" (as we may informally call them) is equivalent to $\mathsf{OpMon}(R^\circ R, N)$, as some sort of "opmonoidal $\dashv$ monoidal opmonadicity".

$$\begin{tikzcd} R^\circ R \arrow[d, bend left, "i^\circ 1"] \arrow[d, phantom, "\dashv"] \\ R \arrow[u, bend left, "i_\circ 1"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{M}(R^\circ R, N) \simeq \mathcal{M}(R,N)^{\mathcal{M}(t1,N)} \arrow[d, bend left, "\mathcal{M}(i_\circ 1,N)"] \arrow[d, phantom, "\dashv"] \\ \mathcal{M}(R,N) \arrow[u, bend left, "\mathcal{M}(i^\circ 1,N)"] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathsf{OpMon}(R^\circ R, N) \simeq \mathcal{X}(R,N) \arrow[d, "\simeq"', "\mathsf{OpMon}(i_\circ 1,N)"] \\ \mathsf{SkOpMon}(R,N) \end{tikzcd}$$

We now make this precise.

Definition 2.2.6. For a right skew monoidale (N,m,u), a biduality $R \dashv R^\circ$, and an adjunction $i_\circ \dashv i^\circ$ in $\mathcal{M}$,

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

the category $\mathcal{X}(R,N)$ has objects pairs (D,φ) where $D : R \longrightarrow N$ is an opmonoidal arrow in $\mathcal{M}$ and φ is a cell

$$\begin{tikzcd} R \arrow[r, "i_\circ 1"'] \arrow[rrr, bend left, "D"] & R^\circ R \arrow[r, "i^\circ 1"'] \arrow[r, phantom, bend left=40, "\varphi\Uparrow"] & R \arrow[r, "D"'] & N \end{tikzcd}$$

satisfying five axioms: two which assert that φ is an action for the monad induced by the adjunction $i_\circ 1 \dashv i^\circ 1$, and three of which express the following compatibility between φ and the opmonoidal constraints of D.

$$\begin{tikzcd} & & I \arrow[dll, "i"'] \arrow[drr, "u"] \arrow[d, phantom, "D^0\Nearrow"] & \\ & & {} & & \\ R \arrow[r, "i_\circ 1"'] \arrow[rrr, bend left, "D"] & R^\circ R \arrow[r, "i^\circ 1"'] \arrow[r, phantom, bend left=40, "\varphi\Uparrow"] & R \arrow[r, "D"'] & N \end{tikzcd} = \begin{tikzcd} & & I \arrow[d, "n"] \arrow[dll, "i"'] \arrow[dd, "i"] \arrow[ddr, "u"] \arrow[dl, phantom, "\cong"] \arrow[ddr, phantom, "D^0\Nearrow"'] & \\ & & R^\circ R \arrow[dll, "i^\circ 1"'] \arrow[dl, "1"] \arrow[d, phantom, "\cong"] \arrow[dl, phantom, "\varepsilon_\circ 1\Nearrow"'] & \\ R \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N \end{tikzcd} \tag{X1}$$

$$
\begin{tikzcd}
RR \arrow[r, "DD"] \arrow[d, "i_\circ 11"'] & NN \arrow[r, "m"] & N \\
R^\circ RR \arrow[d, "i^\circ 11"'] \arrow[ur, phantom, "\varphi D \Uparrow"] & \arrow[r, phantom, "D^2 \Uparrow"] & {} \\
RR \arrow[uur, "DD"] \arrow[r, "i^* 1"'] & R \arrow[uur, "D"'] &
\end{tikzcd}
=
\begin{tikzcd}
RR \arrow[r, "DD"] \arrow[dr, "i^* 1"] \arrow[dd, "i_\circ 11"'] & NN \arrow[r, "m"] \arrow[d, phantom, "D^2 \Uparrow"] & N \\
& R \arrow[ur, "D"] \arrow[d, "i_\circ 1"] \arrow[dr, phantom, "\varphi \Uparrow"] & {} \\
R^\circ RR \arrow[r, "1 i^* 1"] \arrow[ur, phantom, "\cong"] \arrow[d, "i^\circ 11"'] & R^\circ R \arrow[d, "i^\circ 1"] & \\
RR \arrow[r, "i^* 1"'] \arrow[ur, phantom, "\cong"] & R \arrow[uuur, "D"'] &
\end{tikzcd}
\tag{X2}
$$

$$
\begin{tikzcd}
RR \arrow[r, "DD"] \arrow[d, "1 i_\circ 1"'] & NN \arrow[r, "m"] & N \\
R R^\circ R \arrow[d, "1 i^\circ 1"'] \arrow[ur, phantom, "D\varphi \Uparrow"] & \arrow[r, phantom, "D^2 \Uparrow"] & {} \\
RR \arrow[uur, "DD"] \arrow[r, "i^* 1"'] & R \arrow[uur, "D"'] &
\end{tikzcd}
=
\begin{tikzcd}
RR \arrow[r, "DD"] \arrow[d, "1 i_\circ 1"'] \arrow[ddr, bend left, "i^* 1"] & NN \arrow[r, "m"] \arrow[d, phantom, "D^2 \Uparrow"] & N \\
R R^\circ R \arrow[r, "1"] \arrow[d, "1 i^\circ 1"'] \arrow[u, phantom, "\cong"] \arrow[dr, phantom, "1\varepsilon_\circ 1 \Uparrow"] & R R^\circ R \arrow[d, "e1"] & \\
RR \arrow[ur, "1 i_\circ 1"] \arrow[r, "i^* 1"'] & R \arrow[uur, "D"'] &
\end{tikzcd}
\tag{X3}
$$

And an arrow $\gamma : (D, \varphi) \longrightarrow (D', \varphi')$ in $\mathcal{X}(R, N)$ is an opmonoidal cell $\gamma : D \longrightarrow D'$ in $\mathcal{M}$ which preserves the actions φ and φ', in the sense of the equation below.

$$
\begin{tikzcd}
R \arrow[r, "i_\circ 1"'] \arrow[rrr, bend left=60, "D'"] & R^\circ R \arrow[r, "i^\circ 1"'] \arrow[r, phantom, "\varphi' \Uparrow", yshift=30] & R \arrow[r, "D"'] \arrow[r, bend left=60, "D'"] \arrow[r, phantom, "\gamma \Uparrow", yshift=8] & N
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[r, "i_\circ 1"'] \arrow[rrr, bend left=70, "D'"] \arrow[rrr, bend left=30, "D"] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
\tag{X4}
$$

Composition and identities are defined as in $\mathcal{M}(R, N)$.

Lemma 2.2.7. *For every monoidale (N, m, u), every biduality $R \dashv R^\circ$, and every adjunction $i_\circ \dashv i^\circ$ in $\mathcal{M}$*

$$
\begin{tikzcd}
R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\
I \arrow[u, bend left, "i_\circ"]
\end{tikzcd}
$$

the forgetful functor $F : \mathcal{X}(R, N) \longrightarrow \mathsf{SkOpMon}(R, N)$ is an isomorphism of categories.

Proof.

It is clear that F is faithful. To see that F is injective on objects observe that for an object (D,φ) of $\mathcal{X}(R,N)$ the following calculation exhibits φ purely in terms of the opmonoidal constraints D^0 and D^2 of D (note the need for N to be left normal).

$$\begin{tikzcd}
R \arrow[r, "i_\circ 1"'] \arrow[rrr, bend left, "D"] \arrow[rrr, phantom, shift left=4, "\varphi\Uparrow"] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
\overset{\text{(OM3)}}{=}
\begin{tikzcd}
R \arrow[d, "i1"] \arrow[rrr, bend left, "D"] \arrow[rrr, phantom, shift left=4, "D^0D\Uparrow"] \arrow[dd, bend right=60, "1"'] \arrow[dd, phantom, bend right=30, "\overset{\eta 1}{\Rightarrow}"] & & & N \arrow[d, "u1"] \arrow[dd, bend left=60, "1"] \arrow[dd, phantom, bend left=30, "\eqsim"] \\
RR \arrow[d, "i^*1"] \arrow[rrr, bend left, "DD"] \arrow[rrr, phantom, shift left=4, "D^2\Uparrow"] & & & NN \arrow[d, "m"] \\
R \arrow[r, "i_\circ 1"'] \arrow[rrr, bend left, "D"] \arrow[rrr, phantom, shift left=4, "\varphi\Uparrow"] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}$$

$$\overset{\text{(X2)}}{=}
\begin{tikzcd}
R \arrow[d, "i1"] \arrow[rrr, bend left, "D"] \arrow[rrr, phantom, shift left=4, "D^0D\Uparrow"] \arrow[dd, bend right=60, "1"'] \arrow[dd, phantom, bend right=30, "\overset{\eta 1}{\Rightarrow}"] & & & N \arrow[d, "u1"] \arrow[dd, bend left=60, "1"] \arrow[dd, phantom, bend left=30, "\eqsim"] \\
RR \arrow[d, "i^*1"] \arrow[r, "i_\circ 11"'] \arrow[rrr, bend left, "DD"] \arrow[rrr, phantom, shift left=4, "\varphi D\Uparrow"] \arrow[dr, phantom, "\cong"] & R^\circ RR \arrow[d, "1i^*1"] \arrow[r, "i^\circ 1"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, "i^*1"] \arrow[r, "DD"'] \arrow[dr, phantom, "D^2\Uparrow"] & NN \arrow[d, "m"] \\
R \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}$$

$$\overset{\text{(X1)}}{=}
\begin{tikzcd}
R \arrow[d, "i1"] \arrow[r, "n1"] \arrow[rrr, bend left, "D"] \arrow[rr, bend left, "i1"] \arrow[dd, bend right=60, "1"'] \arrow[dd, phantom, bend right=30, "\overset{\eta 1}{\Rightarrow}"] & R^\circ RR \arrow[d, "1"] \arrow[dl, "i^\circ 11"'] \arrow[dr, phantom, "\cong"] \arrow[l, phantom, shift right=4, "\varepsilon_\circ 11 \Uparrow"] & & N \arrow[d, "u1"] \arrow[dd, bend left=60, "1"] \arrow[dd, phantom, bend left=30, "\eqsim"] \arrow[l, phantom, "D^0D\Uparrow"] \\
RR \arrow[d, "i^*1"] \arrow[r, "i_\circ 11"'] \arrow[dr, phantom, "\cong"] & R^\circ RR \arrow[d, "1i^*1"] \arrow[r, "i^\circ 1"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, "i^*1"] \arrow[r, "DD"] \arrow[dr, phantom, "D^2\Uparrow"] & NN \arrow[d, "m"] \\
R \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
=
\begin{tikzcd}
& N \arrow[dr, "u1"] \arrow[dddrr, bend left, "1"] & & \\
& & NN \arrow[ddr, "m"] \arrow[r, phantom, "\cong"] \arrow[dd, phantom, "D^2\Uparrow"] & \\
& RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[u, phantom, "D^0D\Uparrow"] \arrow[d, phantom, "\wr\Vert"] & & \\
R \arrow[uuur, "D"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}$$

To see that F is surjective on objects take an arbitrary opmonoidal arrow D in $\mathsf{OpMon}(R,N)$, then let φ be the cell below.

$$\begin{tikzcd}
R \arrow[r, "i_\circ 1"'] \arrow[rrr, bend left, "D"] \arrow[rrr, phantom, shift left=4, "\varphi\Uparrow"] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
:=
\begin{tikzcd}
& N \arrow[dr, "u1"] \arrow[dddrr, bend left, "1"] & & \\
& & NN \arrow[ddr, "m"] \arrow[r, phantom, "\cong"] \arrow[dd, phantom, "D^2\Uparrow"] & \\
& RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[u, phantom, "D^0D\Uparrow"] \arrow[d, phantom, "\wr\Vert"] & & \\
R \arrow[uuur, "D"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
\tag{2.2.1}$$

This cell φ exhibits D as an object of $\mathcal{X}(R, N)$. We shall prove the five axioms that make it happen, starting with the two that make φ into an action for the monad induced by $i_\circ 1 \dashv i^\circ 1$.

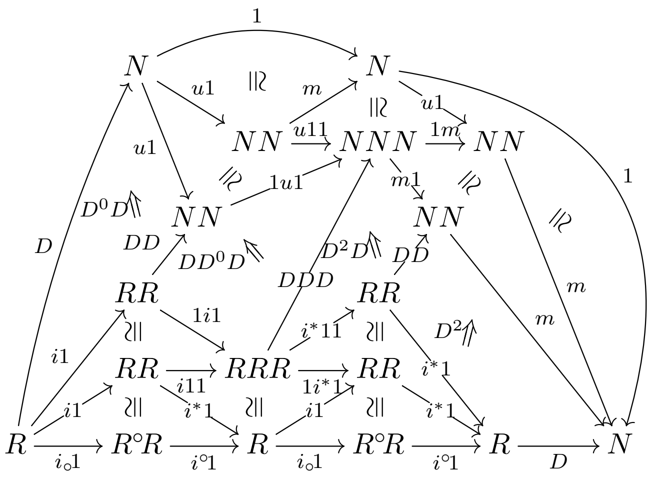
$$
\begin{tikzcd}
& N \arrow[dr, "u1"] \arrow[dddrr, bend left=60, "1"] \arrow[dddrr, phantom, "\cong"] & & \\
& \arrow[r, phantom, "D^0D\Lleftarrow"] & NN \arrow[ddr, "m"] & \\
& RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\cong"] & D^2\Uparrow & \\
R \arrow[uuur, "D"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] \arrow[rr, bend right=60, "1"'] \arrow[rr, phantom, bend right=30, "\eta_\circ 1\Uparrow"] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
=
\begin{tikzcd}
& N \arrow[dr, "u1"] \arrow[dddrr, bend left=60, "1"] \arrow[dddrr, phantom, "\cong"] & & \\
& \arrow[r, phantom, "D^0D\Lleftarrow"] & NN \arrow[ddr, "m"] & \\
& RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] & D^2\Uparrow & \\
R \arrow[uuur, "D"] \arrow[ur, "i1"] \arrow[rr, bend right=60, "1"'] \arrow[r, phantom, "\eta 1\Uparrow"] & & R \arrow[r, "D"'] & N
\end{tikzcd}
$$

$$
\overset{\text{(OM3)}}{=}
\begin{tikzcd}
& N \arrow[dr, "u1"] \arrow[ddrr, bend left=60, "1"] \arrow[ddrr, phantom, "\cong"] \arrow[ddrr, bend right=20, "1"'] \arrow[ddrr, phantom, bend right=10, "\cong"] & & \\
& & NN \arrow[dr, "m"] & \\
R \arrow[uur, "D"] \arrow[rr, "1"'] & & R \arrow[r, "D"'] & N
\end{tikzcd}
= \mathrm{id}_D
$$

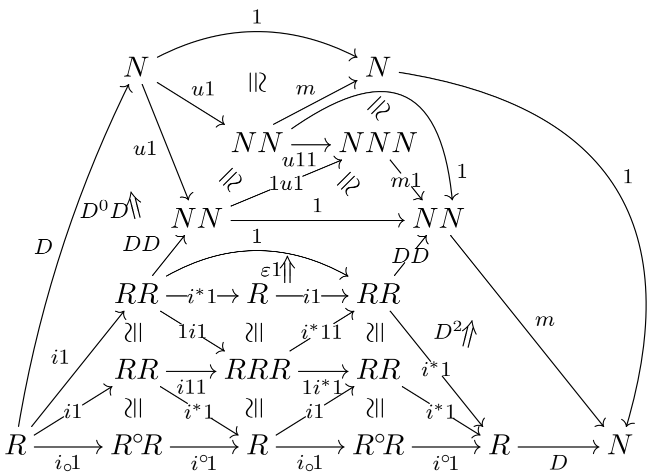
The proof of the second axiom requires N to be a genuine monoidale (not just a skew left normal one).

$$
\begin{tikzcd}
& N \arrow[rr, bend left, "1"] \arrow[dr, "u1"] & \cong & N \arrow[dr, "u1"] \arrow[dddrr, bend left=60, "1"] \arrow[dddrr, phantom, "\cong"] & & \\
& \arrow[r, phantom, "D^0D\Lleftarrow"] & NN \arrow[ur, "m"] & \arrow[r, phantom, "D^0D\Lleftarrow"] & NN \arrow[ddr, "m"] & \\
& RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\cong"] & D^2\Uparrow & RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\cong"] & D^2\Uparrow & \\
R \arrow[uuur, "D"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[uuur, "D"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
$$

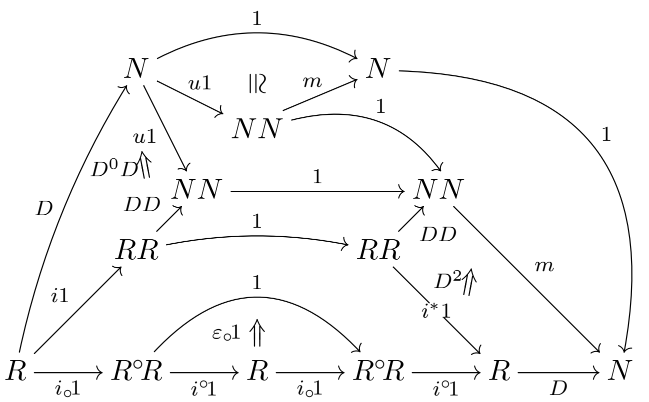
$$
=
\begin{tikzcd}
& N \arrow[rr, bend left, "1"] \arrow[dr, "u1"] & \cong & N \arrow[dr, "u1"] \arrow[d, phantom, "\cong"] \arrow[ddddrr, bend left=60, "1"] \arrow[ddddrr, phantom, "\cong"] & & \\
& & NN \arrow[ur, "m"] \arrow[r, "u11"] & NNN \arrow[r, "1m"] & NN \arrow[dddr, "m"] & \\
& D^0D\Lleftarrow & DD & D^0DD \Rightarrow\ DDD\ DD^2 \Rightarrow & DD & D^2\Uparrow \\
& RR \arrow[uur, "DD"] \arrow[r, "i11"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\cong"] & RRR \arrow[uur, "DDD"] \arrow[r, "1i^*1"] \arrow[d, phantom, "\cong"] & RR \arrow[uur, "DD"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\cong"] & & \\
R \arrow[uuuur, "D"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
$$



(OM1)
$=$



(SKM3)
(OM2)
$=$



(SKM5)
$=$

$$
= \begin{tikzcd}
 & N \arrow[drr, "u1"] \arrow[dddrrrr, bend left, "1"] \arrow[dr, phantom, "D^0D\Uparrow"] & & & & \\
 & & & NN \arrow[ddrr, "m"] \arrow[ur, phantom, "\cong"] & & \\
 & & RR \arrow[ur, "DD"] \arrow[drr, "i^*1"] \arrow[d, phantom, "\wr\|"] \arrow[dr, phantom, "D^2\Uparrow"] & & & \\
R \arrow[uuur, "D"] \arrow[urr, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] \arrow[rr, bend left, "1"] & R \arrow[r, "i_\circ 1"'] \arrow[u, phantom, "\varepsilon_\circ 1\Uparrow"] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
$$

The axiom (X1) holds.

$$
\begin{tikzcd}
I \arrow[r, "u"] \arrow[ddd, "i"'] \arrow[dd, phantom, "\Uparrow D^0"] & N \arrow[dr, "u1"] \arrow[dddrr, bend left, "1"] \arrow[dd, phantom, "D^0D\Uparrow"] & & \\
 & & NN \arrow[ddr, "m"] \arrow[ur, phantom, "\cong"] & \\
 & RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\wr\|"] \arrow[r, phantom, "D^2\Uparrow"] & & \\
R \arrow[uuur, "D"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[r, "u"] \arrow[dr, "u"] \arrow[dd, "i"] \arrow[ddd, "i"'] & N \arrow[dr, "u1"] \arrow[dddrr, bend left, "1"] \arrow[d, phantom, "\cong"] & & \\
 & N \arrow[r, "1u"] & NN \arrow[ddr, "m"] \arrow[ur, phantom, "\cong"] & \\
R \arrow[ur, "D"] \arrow[r, "1i"] \arrow[u, phantom, "\Uparrow D^0"] \arrow[d, phantom, "\wr/\!/"] & RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[u, phantom, "DD^0\Uparrow"] \arrow[d, phantom, "\wr\|"] \arrow[r, phantom, "D^2\Uparrow"] & & \\
R \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
$$

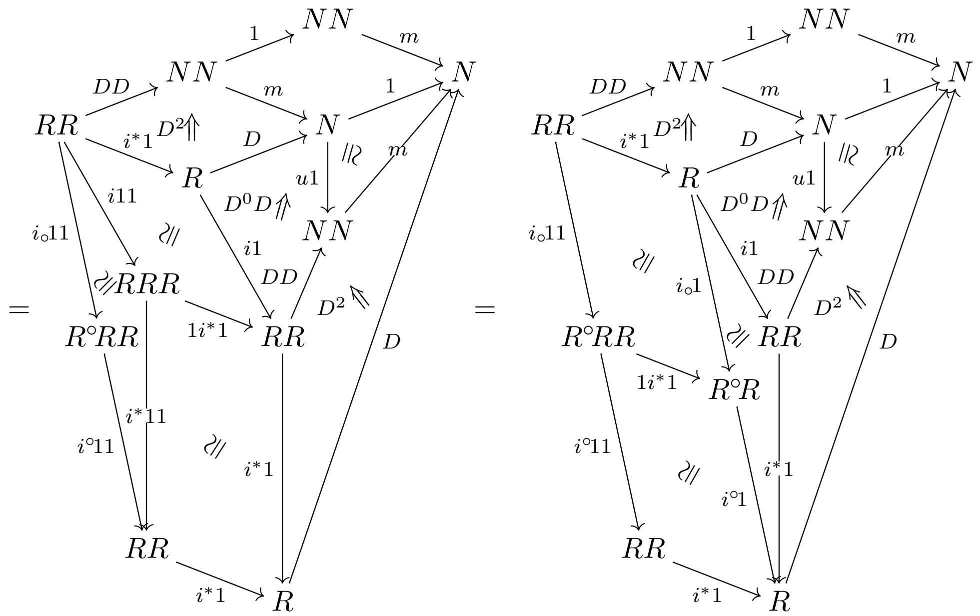
$$
\overset{\text{(SKM5)}}{=}
\begin{tikzcd}
I \arrow[r, "u"] \arrow[dr, "i"] \arrow[ddd, "i"'] & N \arrow[dr, "1u"] \arrow[dddrr, bend left, "1"] \arrow[dl, phantom, "D^0\Uparrow"] & & \\
 & R \arrow[u, "D"] \arrow[d, "1i"] \arrow[r, phantom, "DD^0\Uparrow"] \arrow[dl, phantom, "\wr/\!/"] & NN \arrow[ddr, "m"] \arrow[ur, phantom, "\cong"] & \\
 & RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\wr\|"] \arrow[r, phantom, "D^2\Uparrow"] & & \\
R \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
$$

$$
\overset{\text{(OM2)}}{=}
\begin{tikzcd}
I \arrow[r, "u"] \arrow[dr, "i"] \arrow[ddd, "i"'] & N \arrow[dddrr, bend left, "1"] \arrow[dl, phantom, "D^0\Uparrow"] & & \\
 & R \arrow[u, "D"] \arrow[d, "1i"] \arrow[dr, "i^*"] \arrow[ddr, bend left, "1"] \arrow[dl, phantom, "\wr/\!/"] & & \\
 & RR \arrow[dr, "i^*1"] \arrow[d, phantom, "\wr\|"] & I \arrow[d, "i"] \arrow[u, phantom, "\varepsilon\Nearrow"] & \\
R \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[dd, "i"'] \arrow[dr, "n"] \arrow[ddrr, "i"] \arrow[ddrrr, bend left, "u"] & & N \arrow[d, phantom, "D^0\Nearrow"] & \\
 & R^\circ R \arrow[dl, "i^\circ 1"'] \arrow[d, "1"] \arrow[d, phantom, "\varepsilon_\circ 1\Nearrow"'] & & \\
R \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
$$

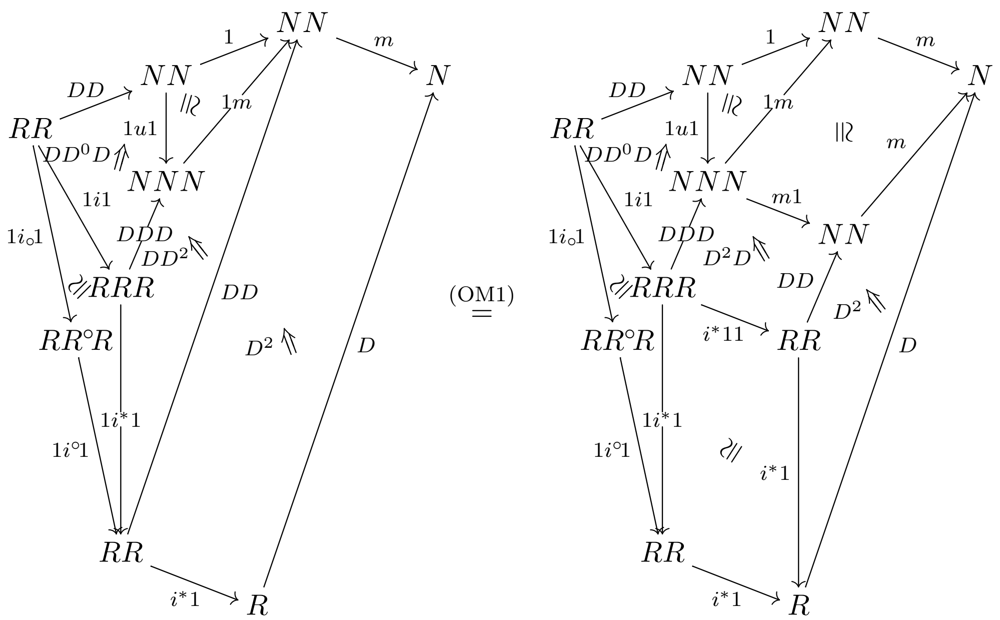
The axiom (X2) holds.

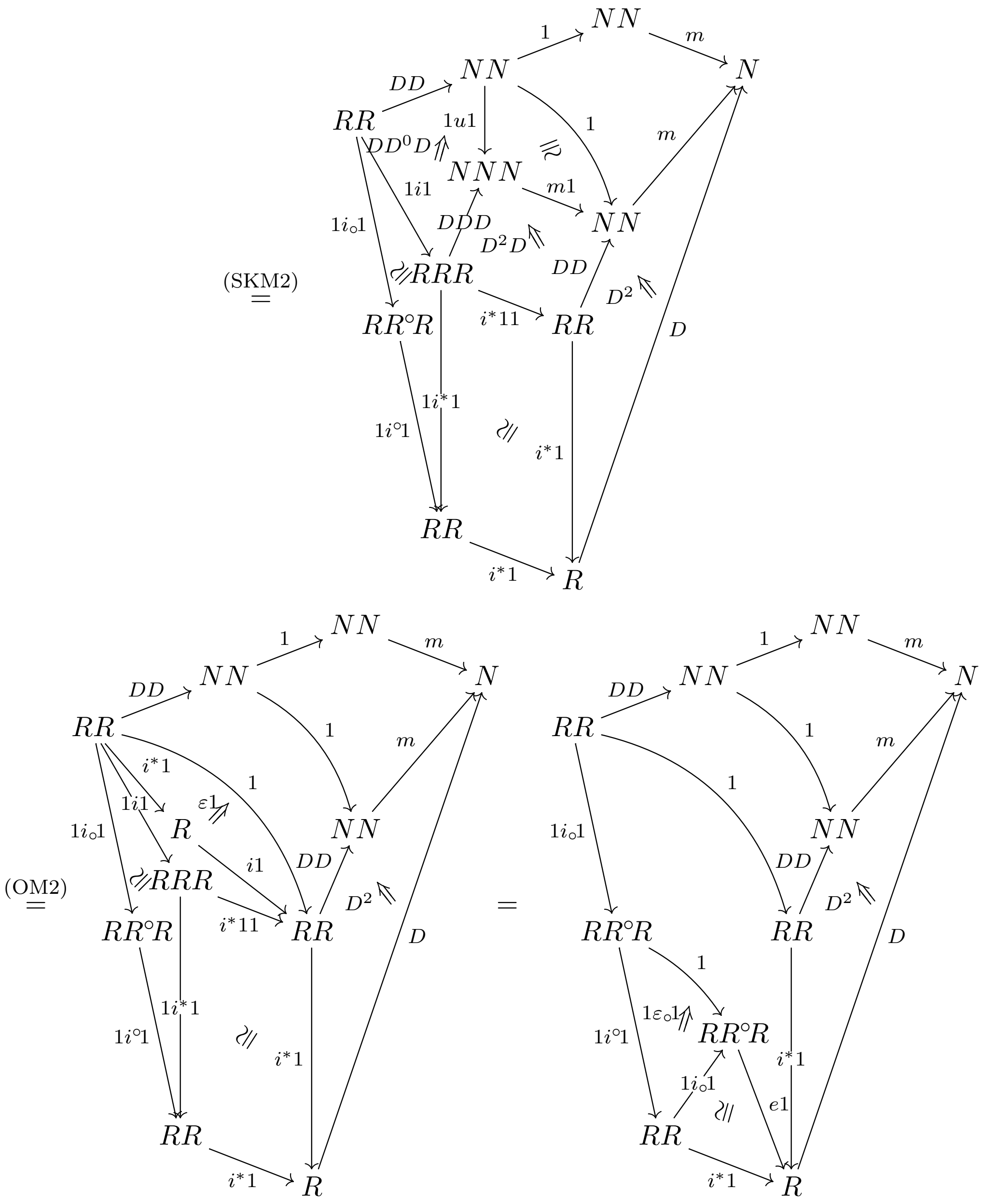
$$
\begin{tikzcd}
 & & NN \arrow[dr, "m"] & \\
 & NN \arrow[ur, "1"] \arrow[dd, "u11"] \arrow[r, phantom, "\cong"] & & N \\
RR \arrow[ur, "DD"] \arrow[ddr, "i11"] \arrow[ddd, "i_\circ 11"'] \arrow[r, phantom, "D^0DD\Uparrow"] & & & \\
 & NNN \arrow[uuur, "m1"'] & & \\
 & RRR \arrow[u, "DDD"] \arrow[dd, two heads, "i^*11"] \arrow[r, phantom, "D^2D\Leftarrow"] & & \\
R^\circ RR \arrow[dr, "i^\circ 11"'] & & \arrow[r, phantom, "D^2\Uparrow"] & \\
 & RR \arrow[uuuuuur, "DD"'] \arrow[dr, "i^*1"'] & & \\
 & & R \arrow[uuuuuur, "D"'] &
\end{tikzcd}
\overset{\text{(OM1)}}{=}
\begin{tikzcd}
 & & NN \arrow[drr, "m"] & & \\
 & NN \arrow[ur, "1"] \arrow[dd, "u11"] \arrow[r, phantom, "\cong"] & & \cong & N \\
RR \arrow[ur, "DD"] \arrow[ddr, "i11"] \arrow[ddd, "i_\circ 11"'] \arrow[r, phantom, "D^0DD\Uparrow"] & & & & \\
 & NNN \arrow[uuur, "m1"'] \arrow[rr, "1m"] & & NN \arrow[uur, "m"'] & \\
 & RRR \arrow[u, "DDD"] \arrow[dd, two heads, "i^*11"] \arrow[drr, "1i^*1"] \arrow[r, phantom, "DD^2\Leftarrow"] & & & \\
R^\circ RR \arrow[dr, "i^\circ 11"'] & & \cong & RR \arrow[uu, "DD"] \arrow[dd, "i^*1"] \arrow[r, phantom, "D^2\Leftarrow"] & \\
 & RR \arrow[drr, "i^*1"'] & & & \\
 & & & R \arrow[uuuuuur, "D"'] &
\end{tikzcd}
$$

$$
\overset{\text{(SKM3)}}{=}
\begin{tikzcd}
 & & & NN \arrow[drr, "m"] & & \\
 & & NN \arrow[ur, "1"] \arrow[dd, "u11"] \arrow[drr, "m"] & & & N \\
 & RR \arrow[ur, "DD"] \arrow[ddr, "i11"] \arrow[ddd, "i_\circ 11"'] \arrow[r, phantom, "D^0DD\Uparrow"] & & \cong & N \arrow[ur, "1"] \arrow[d, "u1"] \arrow[r, phantom, "\cong"] & \\
 & & NNN \arrow[rr, "1m"] & & NN \arrow[uur, "m"'] & \\
 & & RRR \arrow[u, "DDD"] \arrow[dd, two heads, "i^*11"] \arrow[drr, "1i^*1"] \arrow[r, phantom, "DD^2\Leftarrow"] & & & \\
 & R^\circ RR \arrow[dr, "i^\circ 11"'] & & \cong & RR \arrow[uu, "DD"] \arrow[dd, "i^*1"] \arrow[r, phantom, "D^2\Leftarrow"] & \\
 & & RR \arrow[drr, "i^*1"'] & & & \\
 & & & & R \arrow[uuuuuur, "D"'] &
\end{tikzcd}
$$



The axiom (X3) holds.





Hence F is surjective on objects. Now, these actions defined purely in terms of the opmonoidal constraints turn every opmonoidal cell $\gamma : D \longrightarrow D'$ into an arrow in $\mathcal{X}(R, N)$, because as one can see below the axiom (X4) holds.

$$
\begin{tikzcd}
 & N \arrow[dr, "u1"] \arrow[dddrr, bend left=60, "1"] \arrow[dddrr, phantom, bend left=30, "\cong"] & & \\
 & \arrow[r, phantom, "D^0D\Uparrow"] & NN \arrow[ddr, "m"] & \\
 & RR \arrow[ur, "DD"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\wr\|"] & \arrow[r, phantom, "D^2\Uparrow"] & \\
R \arrow[uuur, bend left, "D'"] \arrow[uuur, phantom, bend left=20, "\gamma\Rrightarrow"] \arrow[uuur, "D"'] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
\overset{\text{(OM5)}}{=}
\begin{tikzcd}
 & N \arrow[dr, "u1"] \arrow[dddrr, bend left=60, "1"] \arrow[dddrr, phantom, bend left=30, "\cong"] & & \\
 \arrow[r, phantom, "D'^0D'\Uparrow"] & \arrow[r, phantom, "\gamma\gamma\Uparrow"] & NN \arrow[ddr, "m"] & \\
 & RR \arrow[ur, bend left, "D'D'"] \arrow[ur, "DD"'] \arrow[dr, "i^*1"] \arrow[d, phantom, "\wr\|"] & \arrow[r, phantom, "D^2\Uparrow"] & \\
R \arrow[uuur, bend left, "D'"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
$$

$$
\overset{\text{(OM4)}}{=}
\begin{tikzcd}
 & N \arrow[dr, "u1"] \arrow[dddrr, bend left=60, "1"] \arrow[dddrr, phantom, bend left=30, "\cong"] & & \\
 \arrow[r, phantom, "D'^0D'\Uparrow"] & & NN \arrow[ddr, "m"] & \\
 & RR \arrow[ur, bend left, "D'D'"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\wr\|"] & \arrow[r, phantom, "D'^2\Uparrow"] & \\
R \arrow[uuur, bend left, "D'"] \arrow[ur, "i1"] \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[r, "i^\circ 1"'] & R \arrow[r, bend left, "D'"] \arrow[r, phantom, "\gamma\Uparrow"] \arrow[r, "D"'] & N
\end{tikzcd}
$$

Thus F is full and therefore invertible. ∎

Theorem 2.2.8. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory in the sense of Definition 2.2.1. For every monoidale (N, m, u), every biduality $R \dashv R^\circ$, and every opmonadic adjunction $i_\circ \dashv i^\circ$ in $\mathcal{M}$*

$$
\begin{tikzcd}
R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\
I \arrow[u, bend left, "i_\circ"]
\end{tikzcd}
$$

precomposition with $i_\circ 1 : R \longrightarrow R^\circ R$ defines an equivalence of categories.

$$\mathsf{OpMon}(R^\circ R, N) \simeq \mathsf{SkOpMon}(R, N)$$

Proof.

By Proposition 2.2.5 $i_\circ 1$ is an opmonoidal arrow, thus precomposition along this arrow in $\mathsf{SkOpMon}(\mathcal{M})$ is a well defined functor $\mathsf{SkOpMon}(i_\circ 1, N) : \mathsf{OpMon}(R^\circ R, N) \longrightarrow \mathsf{SkOpMon}(R, N)$. Let G be the composite of $\mathsf{SkOpMon}(i_\circ 1, N)$ followed by the inverse of the isomorphism F in Lemma 2.2.7 which equips an opmonoidal arrow with its canonical module structure (2.2.1) for the monad induced by $i_\circ 1 \dashv i^\circ 1$. We shall now see that the functor G is an equivalence of categories.

$$G : \mathsf{OpMon}(R^\circ R, N) \xrightarrow{\mathsf{SkOpMon}(i_\circ 1, N)} \mathsf{SkOpMon}(R, N) \xrightarrow[\cong]{F^{-1}} \mathcal{X}(R, N)$$

Faithfulness of G follows easily because precomposing with the opmonadic arrow $i^\circ 1$ is faithful in $\mathcal{M}$, and since the forgetful functor

$$\mathsf{SkOpMon}(R^\circ R, N) \longrightarrow \mathcal{M}(R^\circ R, N)$$

is faithful so is precomposing with $i^\circ 1$ in $\mathsf{SkOpMon}(\mathcal{M})$. Now, the functor G is essentially surjective on objects and full, mainly due to the opmonadicity of $i_\circ \dashv i^\circ$. Remember that opmonadicity in $\mathcal{M}$ is preserved by tensoring with objects, so $i_\circ 1 \dashv i^\circ 1$ is opmonadic, and so for an object (D, φ) in $\mathcal{X}(R, N)$, the action cell φ induces an arrow $C : R^\circ R \longrightarrow N$ and an isomorphism

$$\begin{tikzcd}
R \arrow[rr, "D"] \arrow[dr, "i_\circ 1"'] & \arrow[d, phantom, "\cong"] & N \\
& R^\circ R \arrow[ur, "C"'] &
\end{tikzcd} \tag{2.2.2}$$

such that the following equation holds.

$$\begin{tikzcd}
R \arrow[rrrr, "D"] \arrow[dr, "i_\circ 1"'] & & \arrow[d, phantom, "\cong"] & & N \\
& R^\circ R \arrow[rr, "1"] \arrow[dr, "i^\circ 1"'] & \arrow[d, phantom, "\varepsilon_\circ 1 \Uparrow"] & R^\circ R \arrow[ur, "C"'] & \\
& & R \arrow[ur, "i_\circ 1"'] & &
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[rrrr, "D"] \arrow[dr, "i_\circ 1"'] & & \arrow[d, phantom, "\varphi \Uparrow"] & & N \\
& R^\circ R \arrow[dr, "i^\circ 1"'] & & R^\circ R \arrow[ur, "C"'] \arrow[ul, phantom, "\cong"] & \\
& & R \arrow[ur, "i_\circ 1"'] \arrow[uurr, bend left, "D"] & &
\end{tikzcd} \tag{2.2.3}$$

Now, since $i_\circ 1 \dashv i^\circ 1$ is an opmonadic adjunction, its counit $\varepsilon_\circ 1$ is a coequaliser, and then, by hypothesis, the cell

$$\begin{tikzcd}
& I \arrow[dl, "n"'] & \\
R^\circ R \arrow[d, "i^\circ 1"'] \arrow[dr, bend left, "1"] & & \\
R \arrow[r, "i_\circ 1"'] \arrow[ur, phantom, "\varepsilon_\circ 1 \Rrightarrow"] & R^\circ R \arrow[r, "C"'] & N
\end{tikzcd} \tag{2.2.4}$$

is the coequaliser of the parallel pair of cells below.

$$\begin{tikzcd}
& R^\circ R \arrow[dl, "i^\circ 1"'] \arrow[ddl, bend left, "1"] & I \arrow[l, "n"'] & \\
R \arrow[d, "i_\circ 1"'] \arrow[ur, phantom, "\varepsilon_\circ 1 \Rrightarrow"] & & & \\
R^\circ R \arrow[dr, "i^\circ 1"'] & & & N \\
& R \arrow[r, "i_\circ 1"'] & R^\circ R \arrow[ur, "C"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& R^\circ R \arrow[dl, "i^\circ 1"'] & I \arrow[l, "n"'] & \\
R \arrow[d, "i_\circ 1"'] & & & \\
R^\circ R \arrow[dr, "i^\circ 1"'] \arrow[drr, bend left, "1"] & & & N \\
& R \arrow[r, "i_\circ 1"'] \arrow[u, phantom, "\varepsilon_\circ 1 \Uparrow"] & R^\circ R \arrow[ur, "C"'] &
\end{tikzcd} \tag{2.2.5}$$

Taking this into account, one may read axiom (X1) for (D, φ) as saying that precomposing the cell below with each of the parallel cells (2.2.5) gives the same result.

$$
\begin{tikzcd}
 & I \arrow[dl, "n"'] \arrow[ddl, "i"] \arrow[ddr, "u"] & \\
R^\circ R \arrow[d, "i^\circ 1"'] \arrow[dr, phantom, "\cong"] & \arrow[d, phantom, "D^0 \Rrightarrow"] & \\
R \arrow[rr, "D"] \arrow[dr, "i_\circ 1"'] & \cong & N \\
 & R^\circ R \arrow[ur, "C"'] &
\end{tikzcd}
$$

Ergo, by universality of the coequaliser (2.2.4) there exists a cell

$$
\begin{tikzcd}
 & I \arrow[dl, "n"'] \arrow[dr, "u"] \arrow[d, phantom, "C^0 \Rrightarrow"] & \\
R^\circ R \arrow[rr, "C"'] & & N
\end{tikzcd}
$$

such that the following equation holds.

$$
\begin{tikzcd}
 & I \arrow[d, "n"] \arrow[ddl, "i"'] \arrow[ddr, "u"] & \\
 & R^\circ R \arrow[dl, "i^\circ 1"'] \arrow[d, "1"] \arrow[dr, phantom, "C^0 \Rrightarrow"] \arrow[l, phantom, "\cong"] & \\
R \arrow[r, "i_\circ 1"'] \arrow[ur, phantom, "\varepsilon_\circ 1 \Rrightarrow"] & R^\circ R \arrow[r, "C"'] & N
\end{tikzcd}
=
\begin{tikzcd}
 & I \arrow[ddl, "i"'] \arrow[ddr, "u"] \arrow[d, phantom, "D^0 \Rrightarrow"] & \\
 & \cong & \\
R \arrow[r, "i_\circ 1"'] \arrow[rr, bend left, "D"] & R^\circ R \arrow[r, "C"'] & N
\end{tikzcd}
\tag{2.2.6}
$$

The axioms (X2) and (X3) say precisely that D^2 is a morphism of modules for the monads induced by the opmonadic adjunctions

$$
\begin{tikzcd}
R^\circ RR \arrow[d, bend left, "i^\circ 11"] \arrow[d, phantom, "\dashv"] \\
RR \arrow[u, bend left, "i_\circ 11"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
RR^\circ R \arrow[d, bend left, "1i^\circ 1"] \arrow[d, phantom, "\dashv"] \\
RR \arrow[u, bend left, "1i_\circ 1"]
\end{tikzcd}
$$

with the obvious actions on the target of D^2 for each of the monads, and the following actions on the source of D^2.

$$
\begin{tikzcd}
 & R \arrow[dr, "i_\circ 1"] \arrow[rrrd, bend left, "D"] \arrow[d, phantom, "\cong"] & & & \\
 & & R^\circ R \arrow[dr, "i^\circ 1"] \arrow[d, phantom, "\cong"] \arrow[r, phantom, "\varphi \Uparrow"] & & \\
RR \arrow[uur, "i^* 1"] \arrow[r, "i_\circ 11"'] & R^\circ RR \arrow[ur, "1i^* 1"] \arrow[r, "i^\circ 11"'] & RR \arrow[r, "i^* 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \cong & & \\
 & & RR^\circ R \arrow[dr, "e1"] \arrow[d, phantom, "\cong"] & & \\
RR \arrow[r, "1i_\circ 1"'] \arrow[rrr, bend left=60, "i^* 1"] & RR^\circ R \arrow[ur, "1"] \arrow[r, "1i^\circ 1"'] \arrow[u, phantom, "1\varepsilon_\circ 1 \Uparrow"] & RR \arrow[r, "i^* 1"'] \arrow[u, "1i^\circ 1"'] & R \arrow[r, "D"'] & N
\end{tikzcd}
$$

Hence, one may read the axioms (X2) and (X3) for (D, φ) as saying that D^2 is a morphism of modules for the monad induced by the adjunction $i_\circ 1 i_\circ 1 \dashv i^\circ 1 i^\circ 1$ (below left). This adjunction is opmonadic by hypothesis, and so it induces a cell C^2 (as shown on the right)

$$\begin{tikzcd} R^\circ R R^\circ R \arrow[d, bend left, "i^\circ 1 i^\circ 1"] \\ RR \arrow[u, bend left, "i_\circ 1 i_\circ 1"] \arrow[u, phantom, "\dashv"] \end{tikzcd} \qquad\qquad \begin{tikzcd} R^\circ R R^\circ R \arrow[r, "CC"] \arrow[d, "1e1"'] \arrow[dr, phantom, "\Nearrow C^2"] & NN \arrow[d, "m"] \\ R^\circ R \arrow[r, "C"'] & N \end{tikzcd}$$

such that the following equation holds.

$$\begin{tikzcd} RR \arrow[rr, "DD"] \arrow[d, "i^*1"'] \arrow[dr, "i_\circ 1 i_\circ 1"] \arrow[dr, phantom, "\cong", near start, xshift=2em] & & NN \arrow[d, "m"] \\ R \arrow[dr, "i^\circ 1"'] \arrow[r, phantom, "\cong"] & R^\circ R R^\circ R \arrow[ur, "CC"] \arrow[d, "1e1"] \arrow[r, phantom, "\Nearrow C^2"] & N \\ & R^\circ R \arrow[ur, "C"'] & \end{tikzcd} = \begin{tikzcd} RR \arrow[rr, "DD"] \arrow[d, "i^*1"'] \arrow[drr, phantom, "\Nearrow D^2"] & & NN \arrow[d, "m"] \\ R \arrow[rr, "DD"] \arrow[dr, "i^\circ 1"'] \arrow[drr, phantom, "\cong", near end] & & N \\ & R^\circ R \arrow[ur, "C"'] & \end{tikzcd} \qquad (2.2.7)$$

To deduce that the data (C, C^0, C^2) constitute an opmonoidal arrow, first take axiom (OM1) for the data (C, C^0, C^2) and apply the faithful functor given by precomposition with the opmonadic arrow

$$i_\circ 1 i_\circ 1 i_\circ 1 : RRR \longrightarrow R^\circ R R^\circ R R^\circ R$$

to both sides of the axiom; this produces the two sides of axiom (OM1) for D, which are equal. Then, precompose both sides of axiom (OM2) for the data (C, C^0, C^2) with the opmonadic arrow $i_\circ 1 : R \longrightarrow R^\circ R$, and also with the epimorphic cell

$$\begin{tikzcd} & R^\circ R R^\circ R \arrow[dl, "11 i^\circ 1"'] \arrow[dr, "1"] \arrow[d, phantom, "\Nearrow 11\varepsilon_\circ 1"] & \\ R^\circ R R \arrow[rr, "11 i_\circ 1"'] & & R^\circ R R^\circ R \end{tikzcd}$$

at $\mathrm{id}_{R^\circ R R^\circ R}$; this produces the two sides of axiom (OM2) for D, which are equal. And finally, precompose the two sides of axiom (OM3) for the data (C, C^0, C^2) with the opmonadic arrow $i_\circ 1 : R \longrightarrow R^\circ R$, and substitute the cell below for the identity on $i_\circ 1$; this produces both sides of axiom (OM3) for D, which are equal.

$$\begin{tikzcd} R \arrow[r, "i_\circ 1"] \arrow[dr, "1"'] \arrow[r, phantom, "\Nearrow \eta_\circ 1", yshift=-2em] & R^\circ R \arrow[d, "i^\circ 1"] \arrow[dr, "1"] \arrow[dr, phantom, "\Nearrow \varepsilon_\circ 1", xshift=-1em] & \\ & R \arrow[r, "i_\circ 1"'] & R^\circ R \end{tikzcd} = \begin{tikzcd} R \arrow[r, "i_\circ 1"] \arrow[dr, "i1"] \arrow[ddddr, bend right=60, "1"'] \arrow[ddr, phantom, "\Nearrow \eta 1"] & R^\circ R \arrow[dr, "n11"] \arrow[dd, "i11"'] \arrow[ddddr, bend left=60, "1"] & \\ & & R^\circ R R^\circ R \arrow[dl, "i^\circ 111"] \arrow[dd, "1"] \\ & RR \arrow[r, "1 i_\circ 1"] \arrow[dd, "i^*1"'] \arrow[r, phantom, ""] & RR^\circ R \arrow[dl, "e1"] \arrow[dr, "i_\circ 111"'] \arrow[r, phantom, "\Nearrow \varepsilon_\circ 111"] & \\ & & & R^\circ R R^\circ R \arrow[d, "1e1"] \\ & R \arrow[rr, "i_\circ 1"'] & & R^\circ R \end{tikzcd}$$

We have now built an opmonoidal arrow $C : R^{\circ}R \longrightarrow N$ for every D and the isomorphism (2.2.2) reads as $G(C) \cong D$; it is in fact in $\mathcal{X}(R, N)$ by (2.2.3), (2.2.6), and (2.2.7), therefore G is essentially surjective. Now, let $\gamma : D \longrightarrow D'$ be a cell in $\mathcal{X}$; by the opmonadicity of $i_{\circ}1$ axiom (X4) implies the existence of a cell $\xi : C \longrightarrow C'$ in $\mathcal{M}$ such that the following equation holds.

$$
\begin{tikzcd}
R \arrow[rr, "D"] \arrow[rr, bend left=60, "D'"] \arrow[rr, phantom, bend left=30, "\gamma\Uparrow"] \arrow[dr, "i_{\circ}1"'] & \arrow[d, phantom, "\cong"] & N \\
& R^{\circ}R \arrow[ur, "C"'] &
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[rr, bend left=60, "D'"] \arrow[dr, "i_{\circ}1"'] & \cong & N \\
& R^{\circ}R \arrow[ur, "C"'] \arrow[ur, bend left=50, "C'"] \arrow[ur, phantom, bend left=20, "\xi\Uparrow"] &
\end{tikzcd}
$$

To prove that ξ is an opmonoidal arrow, first precompose the two sides of axiom (OM4) for ξ with the opmonadic arrow $i_{\circ}1i_{\circ}1 : RR \longrightarrow R^{\circ}RR^{\circ}R$; this produces each side of axiom (OM5) for γ, which holds true. And then, precompose the two sides of axiom (OM5) for ξ with the epimorphic cell $\varepsilon_{\circ}1$ at $R^{\circ}R$; this produces each side of axiom (OM5) for γ, which holds true. Therefore G is full, and consequently is an equivalence. ∎

Remark 2.2.9. Suppose that $\mathcal{M}$ is a right autonomous opmonadic-friendly monoidal bicategory and that for every object there is an opmonadic adjunction $i_{\circ} \dashv i^{\circ}$, as is the case in the examples $\mathcal{M} = \mathsf{Mod}_k$ and $\mathcal{M} = \mathsf{Span}^{\mathrm{co}}$. Then the equivalence in Theorem 2.2.8 suggests that the assignation $R \longmapsto R^{\circ}R$ behaves as a partial left adjoint to the forgetful functor, or as a strictification of the skew monoidal structure on R into the monoidal structure of $R^{\circ}R$.

$$
\begin{tikzcd}
R^{\circ}R \\
R \arrow[u, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{OpMon}\,\mathcal{M} \arrow[d, bend left, "\text{forget}"] \arrow[d, phantom, "\dashv"] \\
\mathsf{SkOpMon}\,\mathcal{M} \arrow[u, bend left, dotted]
\end{tikzcd}
$$

2.3 Oplax Actions

In this section we introduce a new concept which plays a central role in this thesis, we call it "oplax action". Monoids and actions with respect to a monoid may be defined in a context as general as a monoidal category, but here we work one dimension higher: in the context of a monoidal bicategory. Because of the extra dimension, there are various ways to generalise the concept of a monoid in a monoidal bicategory. Examples of such generalisations are monoidales and skew monoidales which we have used earlier. In these examples the associative and unit laws do not hold strictly as they do with monoids, but only up to a cell satisfying some coherence axioms; for monoidales these cells must be isomorphisms, and for skew monoidales there is no such restriction. Oplax actions are defined with respect to a fixed skew monoidale, they generalise actions with respect to a monoid in a similar way as skew monoidales generalise monoids. That is, the associative and unit laws hold up to a not necessarily invertible cell satisfying some coherence axioms. Syntactically there is no distinction between "actions" and "modules" whatever the context, but their spirit is slightly different, the former focuses on the arrow bit while the latter focuses on the object bit. During this research having the arrow perspective proves to be useful. Oplax actions arise in the following way: we know that a bialgebroid corresponds to an opmonoidal monad on R-Mod-R which in turn corresponds to a skew monoidal structure on Mod-R. We also know that bialgebroids are coalgebroids with some additional structure. One is led to ask what happens when we focus only on the coalgebroid bit of a bialgebroid in the correspondences above. It is known that a coalgebroid corresponds to an opmonoidal functor $R\text{-}\mathsf{Mod}\text{-}R \longrightarrow S\text{-}\mathsf{Mod}\text{-}S$, and in this section we prove that these opmonoidal functors also correspond to oplax action structures on Mod-S with respect to a particular skew monoidal structure on Mod-R.

Definition 2.3.1. Let $(M, u, m, \alpha, \lambda, \rho)$ be a right skew monoidale, and A an object in $\mathcal{M}$. An *oplax right M-action on A* consists of an arrow $a : AM \longrightarrow A$, an associator cell a^2, and a right unitor cell a^0 in $\mathcal{M}$

$$\begin{tikzcd}
AMM \arrow[r, "a1"] \arrow[d, "1m"'] \arrow[dr, phantom, "a^2 \Nearrow"] & AM \arrow[d, "a"] \\
AM \arrow[r, "a"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
AM \arrow[d, "a"'] \arrow[dr, phantom, "\Searrow a^0"] & A \arrow[l, "1u"'] \arrow[dl, "1"] \\
A &
\end{tikzcd}$$

satisfying the following three axioms.

$$\begin{tikzcd}[column sep=small]
 & & AMM \arrow[drr, "a1"] & & \\
AMMM \arrow[urr, "a11"] \arrow[drr, "1m1"] \arrow[dd, "11m"'] & & a^2 1 \Uparrow & & AM \arrow[dd, "a"] \\
 & \alpha \Nearrow & AMM \arrow[urr, "a1"'] \arrow[dd, "1m"] & a^2 \Nearrow & \\
AMM \arrow[drr, "1m"'] & & & & A \\
 & & AM \arrow[urr, "a"'] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & AMM \arrow[drr, "a1"] \arrow[dd, "1m"] & & \\
AMMM \arrow[urr, "a11"] \arrow[dd, "11m"'] & \cong & & a^2 \Nearrow & AM \arrow[dd, "a"] \\
 & & AMM \arrow[drr, "a"] & & \\
AMM \arrow[urr, "a1"] \arrow[drr, "1m"'] & & a^2 \Uparrow & & A \\
 & & AM \arrow[urr, "a"'] & &
\end{tikzcd}
\tag{OLA1}$$

$$\begin{tikzcd}[column sep=small]
 & & A \arrow[ddrr, bend left, "1"] & & \\
AM \arrow[urr, "a"] \arrow[dd, "11u"'] \arrow[ddrr, bend left, "1"] & & & & \\
 & 1\rho \Nearrow & & & \\
AMM \arrow[drr, "1m"'] & & & & A \\
 & & AM \arrow[urr, "a"'] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & A \arrow[dd, "1u"] \arrow[ddrr, bend left, "1"] & & \\
AM \arrow[urr, "a"] \arrow[dd, "11u"'] & \cong & & t^0 \Nearrow & \\
 & & AM \arrow[drr, "a"] & & \\
AMM \arrow[urr, "a1"] \arrow[drr, "1m"'] & & a^2 \Uparrow & & A \\
 & & AM \arrow[urr, "a"'] & &
\end{tikzcd}
\tag{OLA2}$$

$$\begin{tikzcd}[column sep=small]
AM \arrow[rrrr, bend left, "1"] \arrow[drr, "1u1"] \arrow[ddrr, bend right, "1"'] & & a^0 1 \Uparrow & & AM \arrow[d, "a"] \\
 & 1\lambda \Nearrow & AMM \arrow[urr, "a1"'] \arrow[d, "1m"] & a^2 \Nearrow & A \\
 & & AM \arrow[urr, "a"'] & &
\end{tikzcd}
=
\begin{tikzcd}
AM \arrow[r, bend left, "a"] \arrow[r, bend right, "a"'] & A
\end{tikzcd}
\tag{OLA3}$$

Remark 2.3.2. One can similarly define *oplax left actions*, *lax right actions*, and *lax left actions* with respect to a right skew monoidale or with respect to a left skew monoidale. If the associator and the left unitor are isomorphisms we speak of *pseudo right actions*.

For every right skew monoidale $(M, u, m, \alpha, \lambda, \rho)$ there is a *regular* oplax right M-action on M given by its product arrow $m : MM \longrightarrow M$, associator cell α, and right unitor cell ρ; axioms (OLA1), (OLA2), and (OLA3) are respectively axioms (SKM1), (SKM4), and (SKM5) for M. In particular, adjunctions and bidualities induce oplax right actions because, as explained in Lemma 2.1.15, an adjunction

$$\begin{tikzcd} R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i"] \end{tikzcd}$$

induces a right skew monoidal structure on R; and as explained in Remark 2.1.20, a biduality $R \dashv R^\circ$ induces the enveloping monoidale $R^\circ R$. On the other hand a biduality induces another pseudo action with respect to the enveloping monoidale but on the object R given by the arrow $e1 : RR^\circ R \longrightarrow R$. When we think of the enveloping monoidale as an internal endo-hom monoidale the pseudo action $e1$ is the internal evaluation arrow.

Example 2.3.3. In Cat right oplax actions with respect to a skew monoidal category may be called *oplax right actegories*. These are related to right skew monoidal bicategories as defined in [LS14, Section 3] in the following way: a right skew monoidal bicategory $\mathcal{B}$ consists of a set of objects X, Y, and so on; for each object X a right skew monoidal category $\mathcal{B}(X, X)$, and for each pair of objects X and Y a left-$\mathcal{B}(X, X)$ right-$\mathcal{B}(Y, Y)$ oplax actegory $\mathcal{B}(X, Y)$.

Remark 2.3.4. Motivated by the previous example one may have chosen to name oplax actions as skew actions. But one needs to specify if it is a right action or left action, and also if it is right skew or left skew depending on the direction of the cells a^2 and a^0. Thus the full name for right oplax actions with this perspective would be right skew right actions which seems inconveniently long. Furthermore, a monoidale M in $\mathcal{M}$ defines a pseudomonad by tensoring on the right $\_ \otimes M : \mathcal{M} \longrightarrow \mathcal{M}$ whose oplax algebras are our oplax M-actions.

Definition 2.3.5. Let a and a' be oplax right M-actions on A. A *cell of oplax right M-actions on A from a to a'* consists of a cell φ in $\mathcal{M}$

$$\begin{tikzcd} AM \arrow[r, bend left, "a'"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\varphi\Uparrow"] & A \end{tikzcd}$$

satisfying the following two conditions

$$\begin{tikzcd} AMM \arrow[r, "a'1"] \arrow[d, bend right, "1m"'] \arrow[dr, phantom, "a'^2 \Nearrow"] & AM \arrow[d, bend left, "a'"] \\ AM \arrow[r, bend left, "a'"] \arrow[r, bend right, "a"'] \arrow[r, phantom, "\varphi\Uparrow"] & A \end{tikzcd} = \begin{tikzcd} AMM \arrow[r, bend left, "a'1"] \arrow[r, "a1"'] \arrow[r, phantom, shift left=2, "\varphi 1\Uparrow"] \arrow[d, bend right, "1m"'] \arrow[dr, phantom, "a^2 \Nearrow"] & AM \arrow[d, bend right, "a"'] \arrow[d, bend left, "a'"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] \\ AM \arrow[r, bend right, "a"'] & A \end{tikzcd} \tag{OLA4}$$

$$\begin{tikzcd} AM \arrow[d, bend right=40, "a"'] \arrow[d, bend left=40, "a'"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & A \arrow[l, "1u"'] \arrow[dl, "1"] \arrow[dl, phantom, shift right=3, "\Downarrow a'^0"] \\ A & \end{tikzcd} = \begin{tikzcd} AM \arrow[d, bend right=40, "a"'] & A \arrow[l, "1u"'] \arrow[dl, "1"] \arrow[dl, phantom, shift right=3, "\Downarrow a^0"] \\ A & \end{tikzcd} \tag{OLA5}$$

Oplax actions and their cells form a category $\mathsf{OplaxAct}(M; A)$, composition and identities in this category are calculated at the level of their underlying counterparts in $\mathcal{M}(AM, A)$, which means that there is a forgetful functor.

$$\mathsf{OplaxAct}(M; A) \longrightarrow \mathcal{M}(AM, A)$$

Remark 2.3.6. A glance at the axioms reveals that oplax I-actions on an object A are nothing but comonads on A. Another point of view of this phenomenon is that comonads are oplax algebras of the identity pseudomonad, this is considered in [Lac14, Section 9]. In the classical case actions and representations always come hand in hand, and there is no exception with oplax actions. One may as well say that an *oplax representation* of a skew monoidale M with respect to an object A is an opmonoidal arrow to the internal endo-hom monoidale of A.

$$M \longrightarrow [A, A]$$

This means that we require the existence of an object $[A, A]$ in $\mathcal{M}$ with the universal property $\mathcal{M}(AX, A) \simeq \mathcal{M}(X, [A, A])$. One way it might exist is if A has a right bidual, in which case we take the enveloping monoidale induced by the biduality. Another way is if the monoidal bicategory has a right closed monoidal structure as defined in [DS97, Definition 5. and Example 2.]. In any case, right oplax actions and oplax representations are in correspondence by the usual means of transposition. A particular case of this situation was mentioned in Remark 2.1.22 where it is said that comonads on R are oplax R-representations of the unit object I. We make all this very precise in the following theorem which we prove in full detail since with little effort its proof may be adapted to other results: see Corollary 2.3.8.

Theorem 2.3.7. *For every right skew monoidale M and every biduality $S \dashv S^\circ$, there is an equivalence of categories given by transposition along the biduality.*

$$\mathsf{SkOpMon}(M, S^\circ S) \simeq \mathsf{OplaxAct}(M; S)$$

Proof.
The biduality $S \dashv S^\circ$ induces the following equivalences of categories

$$\mathcal{M}(M, S^\circ S) \simeq \mathcal{M}(SM, S) \tag{2.3.1}$$

$$\mathcal{M}(MM, S^\circ S) \simeq \mathcal{M}(SMM, S) \tag{2.3.2}$$

$$\mathcal{M}(I, S^\circ S) \simeq \mathcal{M}(S, S) \tag{2.3.3}$$

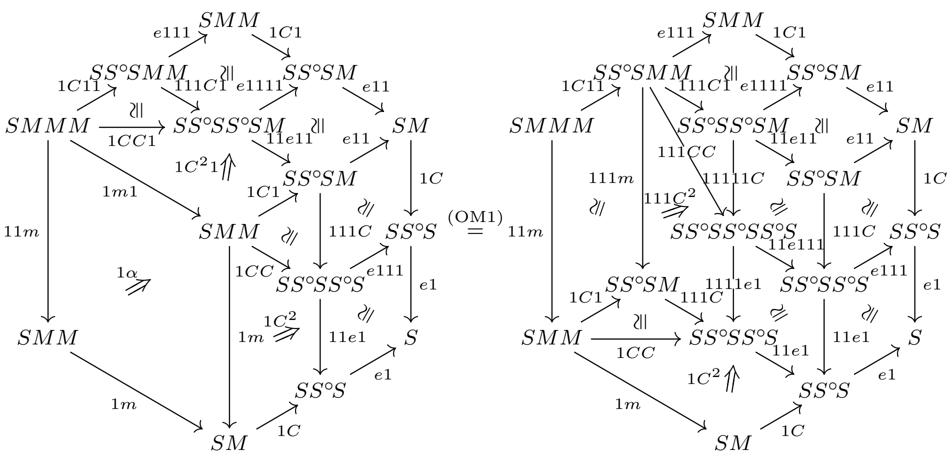
The data of an opmonoidal arrow consists of items in the left hand side of these equivalences: an object in (2.3.1), an arrow in (2.3.2), and an arrow in (2.3.3). These “opmonoidal data” correspond under the equivalences above to the data for an oplax action:

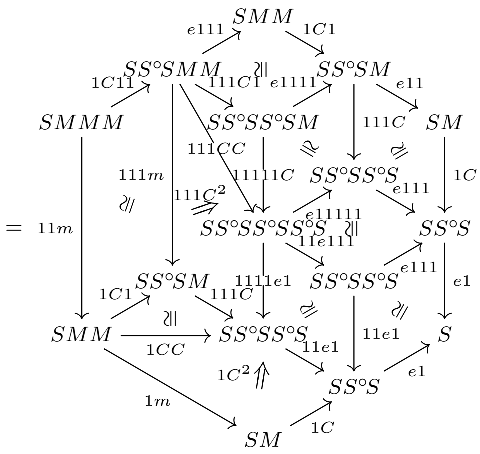
$$\frac{M \xrightarrow{C} S^\circ S}{SM \xrightarrow[s]{} S} \simeq \frac{\begin{tikzcd} MM \arrow[r, "CC"] \arrow[d, "m"'] \arrow[dr, phantom, "C^2 \Nearrow"] & S^\circ S S^\circ S \arrow[d, "m"] \\ M \arrow[r, "C"'] & S^\circ S \end{tikzcd}}{\begin{tikzcd} SMM \arrow[r, "s1"] \arrow[d, "1m"'] \arrow[dr, phantom, "s^2 \Nearrow"] & SM \arrow[d, "s"] \\ SM \arrow[r, "s"'] & S \end{tikzcd}} \cong \frac{\begin{tikzcd} & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "C^0 \Nearrow"] & \\ M \arrow[rr, "C"'] & & S^\circ S \end{tikzcd}}{\begin{tikzcd} SM \arrow[d, "s"'] & S \arrow[l, "1u"'] \arrow[dl, "1"] \arrow[dl, phantom, "\Downarrow s^0"] \\ S & \end{tikzcd}} \cong$$

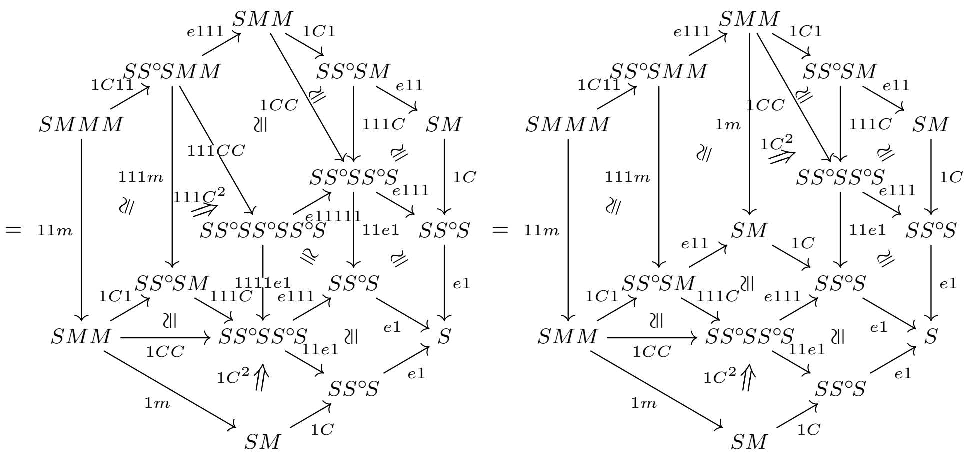
and the data for opmonoidal cells and oplax action cells is in a bijective correspondence.

$$\frac{\begin{tikzcd} M \arrow[r, bend left, "C'"] \arrow[r, bend right, "C"'] \arrow[r, phantom, "\xi \Uparrow"] & S^\circ S \end{tikzcd}}{\begin{tikzcd} SM \arrow[r, bend left, "s'"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\sigma \Uparrow"] & S \end{tikzcd}} \cong$$

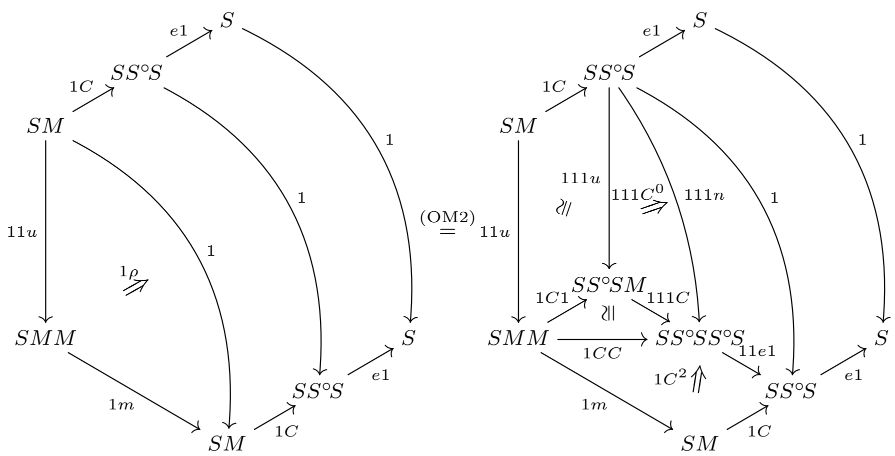
If under this equivalence the property of being opmonoidal corresponds to the property of being an oplax action, then the theorem follows. We prove it by direct calculation: Let $C : M \longrightarrow S^\circ S$ be an opmonoidal arrow, then the axioms (OM1) and (OLA1) are equations that lie in each of the sides of $\mathcal{M}(MMM, S^\circ S) \simeq \mathcal{M}(SMMM, S)$, and the calculation below shows that (OM1) for C corresponds to (OLA1) under the equivalence.

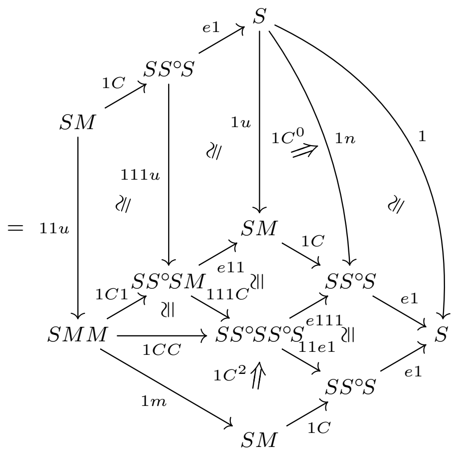




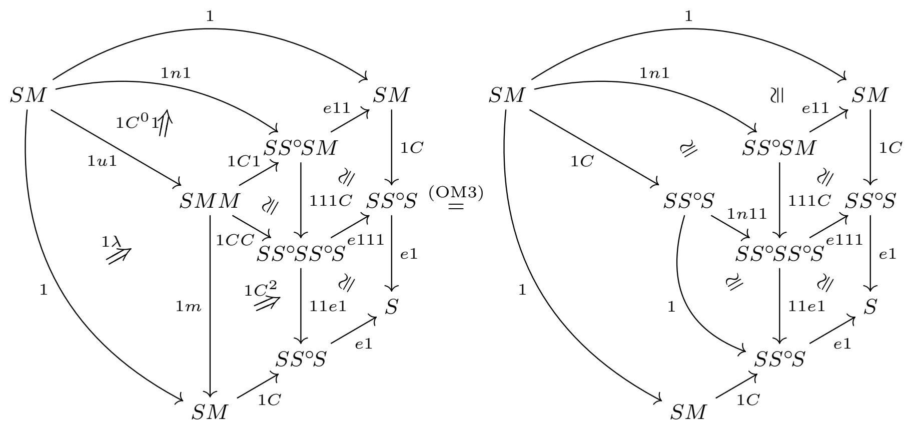


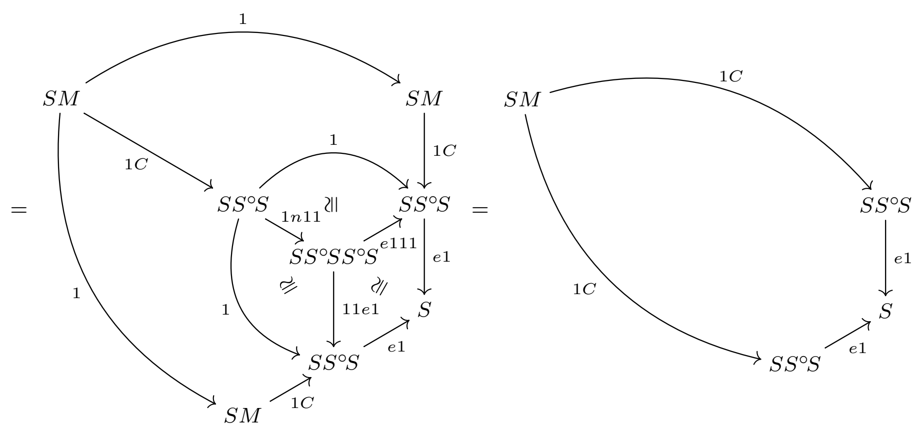
The other axioms (OM2), (OM3), (OLA2), and (OLA3) lie in each of the sides of (2.3.1). The calculation below shows that (OM2) for C corresponds to (OLA2) under the equivalence (2.3.1);





and the calculation below shows that (OM3) for C corresponds to (OLA3) under the equivalence (2.3.1).





Therefore $SM \xrightarrow{1C} SS^\circ S \xrightarrow{e1} S$ is an oplax S-action. Now let $\xi : C \longrightarrow C'$ be an opmonoidal cell, then (OLA4) and (OLA5) hold by the calculations below.

$$
\begin{tikzcd}
SMM \arrow[r, bend left, "1C'1"] \arrow[dd, "1m"'] \arrow[dr, "1C'C'"'] & SS^\circ SM \arrow[r, "e1"] \arrow[d, "111C'"] \arrow[dl, phantom, "\cong"] \arrow[r, phantom, bend right, "\cong"] & SM \arrow[d, bend left, "1C'"] \\
{} \arrow[r, phantom, "1C^2\Rightarrow"] & SS^\circ SS^\circ S \arrow[r, "e111"] \arrow[d, "11e1"] \arrow[dr, phantom, "\cong"] & SS^\circ S \arrow[d, "e1"] \\
SM \arrow[r, bend left, "1C'"] \arrow[r, bend right, "1C"'] \arrow[r, phantom, "1\xi\Uparrow"] & SS^\circ S \arrow[r, "e1"'] & S
\end{tikzcd}
\overset{\text{(OM4)}}{=}
\begin{tikzcd}
SMM \arrow[r, bend left, "1C'1"] \arrow[dd, "1m"'] \arrow[dr, bend left, "1C'C'"] \arrow[dr, bend right, "1CC"'] \arrow[dr, phantom, "1\xi\xi\Rightarrow"] & SS^\circ SM \arrow[r, "e1"] \arrow[d, "111C'"] \arrow[r, phantom, bend right, "\cong"] & SM \arrow[d, bend left, "1C'"] \\
{} \arrow[r, phantom, "1C^2\Rightarrow"] & SS^\circ SS^\circ S \arrow[r, "e111"] \arrow[d, "11e1"] \arrow[dr, phantom, "\cong"] & SS^\circ S \arrow[d, "e1"] \\
SM \arrow[r, bend right, "1C"'] & SS^\circ S \arrow[r, "e1"'] & S
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
SMM \arrow[r, bend left, "1C'1"] \arrow[r, bend right, "1C1"'] \arrow[r, phantom, "1\xi1\Uparrow"] \arrow[dd, "1m"'] \arrow[dr, "1CC"'] & SS^\circ SM \arrow[r, "e1"] \arrow[d, "111C"] \arrow[dl, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & SM \arrow[d, bend right, "1C'"'] \arrow[d, bend left, "1C'"] \arrow[d, phantom, "1\xi\Rightarrow"] \\
{} \arrow[r, phantom, "1C^2\Rightarrow"] & SS^\circ SS^\circ S \arrow[r, "e111"] \arrow[d, "11e1"] \arrow[dr, phantom, "\cong"] & SS^\circ S \arrow[d, "e1"] \\
SM \arrow[r, bend right, "1C"'] & SS^\circ S \arrow[r, "e1"'] & S
\end{tikzcd}
$$

$$
\begin{tikzcd}
SM \arrow[d, bend right, "C'"'] \arrow[d, bend left, "C'"] \arrow[d, phantom, "\xi\Rightarrow"] & S \arrow[l, "1u"'] \arrow[dl, "1n"] \arrow[ddl, "1"] \arrow[l, phantom, shift right=4, "\Downarrow C'^0"] \\
SS^\circ S \arrow[d, "e1"'] \arrow[r, phantom, "\cong"] & {} \\
S &
\end{tikzcd}
=
\begin{tikzcd}
SM \arrow[d, bend right, "C"'] & S \arrow[l, "1u"'] \arrow[dl, "1n"] \arrow[ddl, "1"] \arrow[l, phantom, shift right=4, "\Downarrow C^0"] \\
SS^\circ S \arrow[d, "e1"'] \arrow[r, phantom, "\cong"] & {} \\
S &
\end{tikzcd}
$$

Therefore the cell $\begin{tikzcd} SM \arrow[r, bend left, "1C'"] \arrow[r, bend right, "1C"'] \arrow[r, phantom, "1\xi\Uparrow"] & SS^\circ S \arrow[r, "e1"] & S \end{tikzcd}$ is a cell of oplax right S-actions. ∎

Corollary 2.3.8. *For every object A in a monoidal bicategory $\mathcal{M}$ there is a pseudofunctor*

$$\mathsf{OplaxAct}(\_\,; A) : \mathsf{SkOpMon}^{\mathrm{op}}(\mathcal{M}) \longrightarrow \mathsf{Cat}\,.$$

Proof.
For objects S which are part of a biduality $S \dashv S^\circ$ the previous theorem provides the pseudonatural equivalence given by,

$$\mathsf{SkOpMon}(\_, S^\circ S) \simeq \mathsf{OplaxAct}(\_\,; S)$$

For an arbitrary object A, if $C : M \longrightarrow N$ is an opmonoidal arrow and $a : AN \longrightarrow A$ an oplax right N-action, then the proof that the composite

$$AM \xrightarrow{1C} AN \xrightarrow{a} A$$

is an oplax right M-action is analogous to the big diagram calculation of Theorem 2.3.7 but replacing each instance of $e1 : SS^{\circ}S \longrightarrow S$ with a where appropriate. The same argument goes for morphisms of opmonoidal arrows and cells of oplax actions. ■

Remark 2.3.9. Oplax representations with respect to an object S induce comonads on S. Indeed, let $C : M \longrightarrow S^{\circ}S$ be an oplax representation of a right skew monoidale M; precomposition of C with the unit of M is an opmonoidal arrow

$$I \xrightarrow{u} M \xrightarrow{C} S^{\circ}S$$

since the unit $u : I \longrightarrow M$ is an opmonoidal arrow by Lemma 2.2.3. Then the transposition along the enveloping monoidale $S^{\circ}S$ as in Theorem 2.3.7 is an oplax I-action, in other words a comonad on S, see Remark 2.3.6.

But we may get another perspective on this as a simple application of the previous corollary: any oplax right action $a : AM \longrightarrow A$ induces comonads on A in exactly the same way. This time we may precompose a with the unit of M using Corollary 2.3.8 to get an oplax right I-action on A

$$AI \xrightarrow{1u} AM \xrightarrow{a} A$$

which has comultiplication and counit as shown below.

$$\begin{tikzcd}
 & & A \arrow[dr, "1u"] & & \\
 & AM \arrow[ur, "a"] \arrow[dr, "11u"] \arrow[rr, phantom, "\cong"] & & AM \arrow[dr, "a"] & \\
A \arrow[ur, "1u"] \arrow[dr, "1u"'] \arrow[rr, phantom, "\cong"] & & AMM \arrow[ur, "a1"] \arrow[dr, "1m"] \arrow[rr, phantom, "a^2\Uparrow"] & & A \\
 & AM \arrow[ur, "1u1"] \arrow[rr, bend right, "1"'] \arrow[rr, phantom, "1\lambda\Uparrow"] & & AM \arrow[ur, "a"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "1u"'] \arrow[rr, bend left, "1"] & AM \arrow[r, "a"'] \arrow[u, phantom, "a^0\Uparrow"] & A
\end{tikzcd}$$

A nice case is to take the oplax right R-action $i^*1 : RR \longrightarrow R$ induced by an adjunction $i \dashv i^*$ in $\mathcal{M}$,

$$\begin{tikzcd}
R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\
I \arrow[u, bend left, "i"]
\end{tikzcd}$$

then the process above recovers the comonad on R associated to the adjunction.

The next theorem asserts that the functor

$$\mathsf{OplaxAct}(i_{\circ}1, A) : \mathsf{OplaxAct}(R^{\circ}R; A) \longrightarrow \mathsf{OplaxAct}(R, A)$$

is an equivalence of categories, assuming that the opmonoidal left adjoint $i_\circ 1$ in Proposition 2.2.5 is opmonadic and that we are in an opmonadic-friendly monoidal bicategory. Its proof is entirely analogous to the one of Theorem 2.2.8 so we present only a sketch. Its statement may also be informally interpreted as an "opmonoidal $\dashv$ monoidal opmonadicity", again in analogy with Theorem 2.2.8, but instead of taking the hom functor $\mathsf{OpMon}(\_, N)$, it takes the functor $\mathsf{OplaxAct}(\_; A)$ of Corollary 2.3.8. In the case that A has a right bidual it is possible to make this analogy into a formal statement; it takes the shape of the commutative square of equivalences in Corollary 2.3.11.

Theorem 2.3.10. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every object A, every biduality $R \dashv R^\circ$, and every opmonadic adjunction $i_\circ \dashv i^\circ$ in $\mathcal{M}$,*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

there is an equivalence of categories given by precomposition along $1 i_o 1 : AR \longrightarrow AR^\circ R$.

$$\mathsf{OplaxAct}(R^\circ R; A) \simeq \mathsf{OplaxAct}(R; A)$$

Proof. [Sketch]
The strategy is to consider the category $\mathcal{Y}(R; A)$ of oplax actions $a : AR \longrightarrow A$ together with an action ψ (two axioms) for the monad induced by $1 i_\circ 1 \dashv 1 i^\circ 1$ which is compatible in the appropriate way (three axioms: (Y1), (Y2) and (Y3)) with the oplax action constraints a^2 and a^0,

$$\begin{tikzcd} AR \arrow[r, "1i_\circ 1"'] \arrow[rrr, bend left, "a"] & AR^\circ R \arrow[r, "1i^\circ 1"'] \arrow[r, phantom, shift left=4, "\psi\Uparrow"] & AR \arrow[r, "a"'] & A \end{tikzcd}$$

and then prove the existence of an equivalence and an isomorphism as shown.

$$\mathsf{OplaxAct}(R^\circ R; A) \simeq \mathcal{Y}(R; A) \cong \mathsf{OplaxAct}(R; A)$$

For an object (a, ψ) in $\mathcal{Y}(R; A)$ the action ψ is redundant as it may be written in terms of the oplax action constraints a^0 and a^2,

$$\begin{tikzcd} AR \arrow[r, "1i_\circ 1"'] \arrow[rrr, bend left, "a"] & AR^\circ R \arrow[r, "1i^\circ 1"'] \arrow[r, phantom, shift left=4, "\psi\Uparrow"] & AR \arrow[r, "a"'] & A \end{tikzcd} = \begin{tikzcd} & & AR \arrow[ddr, "a"] & \\ & ARR \arrow[ur, "a1"] \arrow[dr, "1i^*1"] \arrow[d, phantom, "\wr\|"] \arrow[u, phantom, "a^0 1 \Nwarrow"] \arrow[urr, phantom, "a^2 \Nearrow"] & & \\ AR \arrow[r, "1i_\circ 1"'] \arrow[ur, "1i1"] \arrow[uurr, bend left, "1"] & AR^\circ R \arrow[r, "1i^\circ 1"'] & AR \arrow[r, "a"'] & A \end{tikzcd}$$

and for an arbitrary oplax action a the cell ψ written in terms of a^0 and a^2 as above provides a with the structure of an object in $\mathcal{Y}(R; A)$, hence the functor that forgets this structure is an isomorphism $\mathcal{Y}(R; A) \cong \mathsf{OplaxAct}(R; A)$.

Yet with the five axioms that hold for the objects (a, ψ) of $\mathcal{Y}(R; A)$ and the opmonadicity of $i_\circ \dashv i^\circ$, we get the data for an oplax $R^\circ R$-action on A: The first two axioms say that a is a module for the monad induced by $1i_\circ 1 \dashv 1i^\circ 1$, guaranteeing the existence of an arrow $AR^\circ R \longrightarrow A$; axiom (Y1) confirms the existence of the unitor cell; and axioms (Y2) and (Y3) ensure the existence of the associator cell. Finally, this induced data constitute an oplax $R^\circ R$-action with the property that precomposing with $1i_\circ 1 : AR \longrightarrow AR^\circ R$ gives back the original oplax R-action one started with, up to isomorphism. This gives the behaviour on objects of an equivalence of categories $\mathsf{OplaxAct}(R^\circ R; A) \simeq \mathcal{Y}(R; A)$. ■

Corollary 2.3.11. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every two bidualities $R \dashv R^\circ$ and $S \dashv S^\circ$, and every opmonadic adjunction $i_\circ \dashv i^\circ$ in $\mathcal{M}$,*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

there is an equivalence of categories as shown,

$$\mathsf{OpMon}(R^\circ R, S^\circ S) \simeq \mathsf{OplaxAct}(R; S)$$

where R has the skew monoidal structure induced by the adjunction $i \dashv i^*$ opposite to $i_\circ \dashv i^\circ$ as in Lemma 2.1.15. Moreover, the following square of equivalences commutes up to isomorphism,

$$\begin{tikzcd} \mathsf{OpMon}(R^\circ R, S^\circ S) \arrow[r, "\mathsf{SkOpMon}(i_\circ 1, S^\circ S)", "\simeq"'] \arrow[d, "\simeq"] \arrow[dr, phantom, "\cong"] & \mathsf{SkOpMon}(R, S^\circ S) \arrow[d, "\simeq"'] \\ \mathsf{OplaxAct}(R^\circ R; S) \arrow[r, "\simeq", "\mathsf{OplaxAct}(i_\circ 1; S)"'] & \mathsf{OplaxAct}(R; S) \end{tikzcd} \tag{2.3.4}$$

where the vertical functors in the square are given by transposition along $S \dashv S^\circ$ as in Theorem 2.3.7, the functor on the top is an instance of the equivalence in Theorem 2.2.8, and the functor on the bottom is an instance of the equivalence in Theorem 2.3.10.

Proof.
The equivalence in the statement follows from either of the two composites in the square (2.3.4). The commutativity of the square follows strictly in the case that $\mathcal{M}$ is a strict monoidal 2-category, because an opmonoidal arrow C in $\mathsf{OpMon}(R^\circ R, S^\circ S)$ gets sent to the unambiguous composite below,

$$SR \xrightarrow{1i_\circ 1} SR^\circ R \xrightarrow{1C} SS^\circ S \xrightarrow{e1} S$$

which, in the case of an arbitrary monoidal bicategory $\mathcal{M}$, depends on how the parenthesis are placed. The two different ways to do it corresponding to the top path and the bottom path in the square of the statement differ by a coherent isomorphism which consists of instances of the associativity of the composition and instances of the interchanger between the composition and the tensor. ■

Remark 2.3.12. In an opmonadic-friendly monoidal bicategory $\mathcal{M}$, for a duality $R \dashv R^\circ$ and an opmonadic adjunction $i_\circ \dashv i^\circ$ with opposite adjunction $i \dashv i^*$,

$$\begin{tikzcd} R \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

one may take the identity opmonoidal arrow on $R^\circ R$ through all the equivalences in the square (2.3.4) which gives the following interesting items.

$$\begin{array}{ccc} (R^\circ R \xrightarrow{1} R^\circ R) & \longmapsto & (R \xrightarrow{i_\circ 1} R^\circ R) \\ \updownarrow & & \updownarrow \\ (RR^\circ R \xrightarrow[e1]{} R) & \longmapsto & (RR \xrightarrow[i^*1]{} R) \end{array}$$

On the top-right, $i_\circ 1$ has the opmonoidal structure defined in Proposition 2.2.5. On the bottom-left, $e1$ is the evaluation arrow of the internal hom $R^\circ R$ and is canonically a pseudo right $R^\circ R$-action structure on R. And on the bottom-right, i^*1 has the regular oplax R-action structure of the skew monoidale structure on R (as in Remark 2.3.2), which is induced by the adjunction $i \dashv i^*$ as in Lemma 2.1.15.

We close this section by going back to the example of $R|S$-coalgebroids given at the end of Section 2.1, which by means of Corollary 2.3.11 may now be described with less effort.

Example 2.3.13. Let R and S be k-algebras for a commutative ring k. In Lemma 2.1.28 we showed that $R|S$-coalgebroids are opmonoidal arrows between enveloping monoidales in Mod_k. We know that Mod_k is an opmonadic-friendly autonomous monoidal bicategory and every adjunction is monadic and opmonadic. Moreover, the unit of R which might be seen as a ring morphism from k to R induces an adjunction $i \dashv i^*$ and its dual $i_\circ \dashv i^\circ$. Hence, we may use the equivalence in Corollary 2.3.11 to express an $R|S$-coalgebroid via oplax right R-actions on S in Mod_k where R has the right skew monoidal structure induced by the unit $i : k \longrightarrow R$. This definition involves considerably less information than the one in 2.1.27.

An *$R|S$-coalgebroid via oplax right R-actions* consists of an S-coring (C, ε, δ) and a left R-module structure on C compatible with both of its S-module structures such that $\delta(rc) = \sum c_{(1)} \otimes rc_{(2)}$. More explicitly, one has a module C in SR-Mod-S, a morphism $\varepsilon : C \longrightarrow S$ in S-Mod-S, and a morphism $\delta : C \longrightarrow C \otimes_S C$ in SR-Mod-S where the left R-module structure of $C \otimes_S C$ is given by $r.(c \otimes c') = c \otimes rc'$. And together, these constitute a comonoid (C, ε, δ) in the monoidal category S-Mod-S.

2.4 Comodules for Opmonoidal Arrows

In this section we define comodules with respect to an opmonoidal arrow. We saw in Lemma 2.1.28 that $R|S$-coalgebroids are opmonoidal arrows between enveloping monoidales in the bicategory Mod_k. Comodules for $R|S$-coalgebroids are classically defined as the comodules with respect to their underlying comonoid in S-Mod-S. There is no problem in expressing this definition purely in terms of a monoidal bicategory $\mathcal{M}$. Now, by using the same techniques as in the two previous sections, we show in Corollary 2.4.15 that both definitions of a comodule coincide modulo an equivalence of categories. Moreover, we exhibit a monoidal structure on the category of comodules for the underlying opmonoidal arrow of an opmonoidal monad, and this monoidal structure is such that the forgetful functor down to the underlying arrows of the comodules is strong monoidal. This generalises the classical case for $R|S$-coalgebroids in [Hai08].

Definition 2.4.1. Let M and N be two right skew monoidales, A and B two objects, and $a : AM \longrightarrow A$ and $b : BN \longrightarrow B$ two oplax right actions in $\mathcal{M}$. A *right comodule* (Y, C, y) from a to b consists of an arrow $Y : A \longrightarrow B$, an opmonoidal arrow $C : M \longrightarrow N$, and a cell y in $\mathcal{M}$

$$\begin{tikzcd}
AM \arrow[r, "YC"] \arrow[d, "a"'] \arrow[dr, phantom, "y \Nearrow"] & BN \arrow[d, "b"] \\
A \arrow[r, "Y"'] & B
\end{tikzcd}$$

called the C-coaction, satisfying the coassociative and counit laws.

$$\begin{tikzcd}[column sep=small]
& BNN \arrow[dr, "b1"] \arrow[d, phantom, "yC \Uparrow"] & \\
AMM \arrow[ur, "YCC"] \arrow[dr, "a1"] \arrow[d, "1m"'] & |[alias=X]| \; & BN \arrow[dd, "b"] \\
AM \arrow[dr, "a"'] \arrow[r, phantom, "a^2 \Nearrow"] & AM \arrow[ur, "YC"'] \arrow[d, "a"] \arrow[r, phantom, "y \Nearrow"] & \; \\
& A \arrow[r, "Y"'] & B
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& BNN \arrow[dr, "b1"] \arrow[d, "1m"] & \\
AMM \arrow[ur, "YCC"] \arrow[d, "1m"'] \arrow[r, phantom, "YC^2 \Nearrow"] & BN \arrow[dr, "b"] \arrow[r, phantom, "b^2 \Nearrow"] & BN \arrow[d, "b"] \\
AM \arrow[ur, "YC"] \arrow[dr, "a"'] \arrow[rr, phantom, "y \Uparrow"] & & B \\
& A \arrow[ur, "Y"'] &
\end{tikzcd}
\tag{COM1}$$

$$\begin{tikzcd}[column sep=small]
& B \arrow[ddr, bend left, "1"] & \\
A \arrow[ur, "Y"] \arrow[dd, "1u"'] \arrow[ddr, bend left, "1"] \arrow[d, phantom, "a^0 \Nearrow"] & & \\
\; & & \\
AM \arrow[r, "a"'] & A \arrow[r, "Y"'] & B
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& B \arrow[d, "1u"] \arrow[ddr, bend left, "1"] & \\
A \arrow[ur, "Y"] \arrow[d, "1u"'] \arrow[r, phantom, "YC^0 \Nearrow"] & BN \arrow[dr, "b"] \arrow[r, phantom, "b^0 \Nearrow"] & \; \\
AM \arrow[ur, "YC"] \arrow[dr, "a"'] \arrow[rr, phantom, "y \Uparrow"] & & B \\
& A \arrow[ur, "Y"'] &
\end{tikzcd}
\tag{COM2}$$

Remark 2.4.2. For a fixed opmonoidal arrow $C : M \longrightarrow N$ right comodules (Y, C, y) from a to b are also called *right C-comodules from a to b* or *right comodules over C from a to b*, and shall be denoted by (Y, y). Note that the opmonoidal arrow C plays a similar role as a comonoid in the definition of comodules for comonoids. One may similarly define *right modules over* monoidal arrows (instead of opmonoidal ones) between right skew monoidales by changing the direction of y and by modifying the axioms accordingly.

Definition 2.4.3. A morphism $(\gamma, \xi) : (Y, C, y) \longrightarrow (Y', C', y')$ of right comodules from a to b consists of a cell γ and an opmonoidal cell ξ in $\mathcal{M}$,

$$\begin{tikzcd} A \arrow[r, bend left, "Y'"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\gamma\Uparrow"] & B \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, bend left, "C'"] \arrow[r, bend right, "C"'] \arrow[r, phantom, "\xi\Uparrow"] & N \end{tikzcd}$$

satisfying the following equation.

$$\begin{tikzcd} AM \arrow[r, "Y'C'"] \arrow[d, "a"'] \arrow[dr, phantom, "y'\nearrow"] & BN \arrow[d, "b"] \\ A \arrow[r, bend left, "Y'"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\gamma\Uparrow"] & B \end{tikzcd} = \begin{tikzcd} AM \arrow[r, bend left, "Y'C'"] \arrow[r, bend right, "YC"'] \arrow[r, phantom, "\gamma\xi\Uparrow"] \arrow[d, "a"'] \arrow[dr, phantom, "y\nearrow"] & BN \arrow[d, "b"] \\ A \arrow[r, bend right, "Y"'] & B \end{tikzcd} \tag{COM3}$$

Right comodules from a to b and their morphisms constitute a category that we denote by $\mathsf{rComod}((A, M, a), (B, N, b))$, its composition and identities are taken as the ones in $\mathcal{M}(A, B) \times \mathsf{SkOpMon}(M, N)$, hence the forgetful functor below.

$$\begin{aligned} \mathsf{rComod}((A, M, a), (B, N, b)) &\longrightarrow \mathcal{M}(A, B) \times \mathsf{SkOpMon}(M, N) \\ (Y, C, y) &\longmapsto (Y, C) \end{aligned}$$

Remark 2.4.4. There is a horizontal composition functor of right comodules given in the following way,

$$\mathsf{rComod}((A', M', a'), (A'', M'', a'')) \times \mathsf{rComod}((A, M, a), (A', M', a')) \\ \longrightarrow \mathsf{rComod}((A, M, a), (A'', M'', a''))$$

$$((Z, D, z), (Y, C, y)) \longmapsto \begin{tikzcd} AM \arrow[r, "YC"] \arrow[d, "a"'] \arrow[dr, phantom, "y\nearrow"] & A'N \arrow[r, "ZD"] \arrow[d, "a'"'] \arrow[dr, phantom, "z\nearrow"] & A''L \arrow[d, "a''"] \\ A \arrow[r, "Y"'] & A' \arrow[r, "Z"'] & A'' \end{tikzcd}$$

as well as an identity in $\mathsf{rComod}((A, M, a), (A, M, a))$ as shown below.

$$\begin{tikzcd} AM \arrow[r, "1"] \arrow[d, "a"'] & AM \arrow[d, "a"] \\ A \arrow[r, "1"'] & A \end{tikzcd}$$

Together these constitute a bicategory $\mathsf{rComod}(\mathcal{M})$ which comes equipped with a strict functor $\mathsf{rComod}(\mathcal{M}) \longrightarrow \mathcal{M} \times \mathsf{SkOpMon}(\mathcal{M})$. This is the bicategory of oplax right actions in $\mathcal{M}$, right comodules between them, and morphisms of right comodules. The reader should not confuse

$\mathsf{rComod}(\mathcal{M})$ with the bicategory $\mathsf{Comod}(\mathcal{V})$ of comonoids, two sided comodules between them, and their morphisms in suitable a monoidal category $\mathcal{V}$. There is no way to compare, for example, the objects of these bicategories: the data for a comonoid in $\mathcal{V}$ consist of an object V and two arrows $V \longrightarrow VV$ and $V \longrightarrow I$ in $\mathcal{V}$; and the data for an object in $\mathsf{rComod}(\mathcal{M})$ consist of a right skew monoidale M, an object A, and an oplax right M-action $AM \longrightarrow A$ in $\mathcal{M}$.

There are other reasonable names for the objects, arrows, and cells of $\mathsf{rComod}(\mathcal{M})$ which one might be tempted to give. In an action-oriented approach one might say: oplax right actions, oplax morphisms of oplax right actions, and transformations of oplax right actions between them. Although one may feel inclined to reserve the name of *oplax morphisms of oplax actions* for the case of id_M-comodules,

$$\begin{tikzcd} AM \arrow[r, "Y1"] \arrow[d, "a"'] \arrow[dr, phantom, "y \nearrow"] & BM \arrow[d, "b"] \\ A \arrow[r, "Y"'] & B \end{tikzcd}$$

which is certainly the case when we mention them in the introduction. When $\mathcal{M}$ is a locally discrete monoidal bicategory, i.e. it is obtained by adding identity cells to a monoidal category, one has the usual notion of morphism between two actions.

Perhaps for a more module-oriented approach to $\mathsf{rComod}(\mathcal{M})$ one may give the names: oplax right modules, oplax morphisms of oplax right modules, and transformations of oplax right modules between them. We opted for the ones that are conveniently shorter because of how they fit in the forthcoming theorems, particularly in Corollary 2.4.15; where right comodules for an opmonoidal arrow are comodules for a comonad in $\mathcal{M}$.

Remark 2.4.5. For two oplax M-actions a and $a' : AM \longrightarrow A$, right comodules $(\mathrm{id}_A, \mathrm{id}_M, y)$ from a to a' are nothing but cells of oplax actions y from a to a': axiom (COM1) for $(\mathrm{id}_A, \mathrm{id}_M, y)$ is (OLA4) for y, and axiom (COM2) for $(\mathrm{id}_A, \mathrm{id}_M, y)$ is (OLA5) for y. Hence, for a right skew monoidale M and an object A in $\mathcal{M}$, we recover the categories $\mathsf{OplaxAct}(M; A)$ from $\mathsf{rComod}(\mathcal{M})$ by taking the pullback below,

$$\begin{tikzcd} \mathsf{OplaxAct}(M;A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{rComod}(\mathcal{M}) \arrow[d] \\ \mathbb{1} \arrow[r, "(\mathrm{id}_A, \mathrm{id}_M)"'] & \mathcal{M} \times \mathsf{SkOpMon}(\mathcal{M}) \end{tikzcd}$$

which picks those comodules in $\mathsf{rComod}(\mathcal{M})$ of the form $(\mathrm{id}_A, \mathrm{id}_M, y)$ between oplax M-actions on A.

For a fixed opmonoidal arrow $C : M \longrightarrow N$, right C-comodules from a to b also constitute a category which we denote by $\mathsf{rComod}_C((A, a), (B, b))$. This category may be described by a pullback along the forgetful functor from $\mathsf{rComod}((A, M, a), (B, N, b))$ down to

$\mathsf{SkOpMon}(M,N)$ as shown below.

$$\begin{tikzcd}
\mathsf{rComod}_C((A,a),(B,b)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{rComod}((A,M,a),(B,N,b)) \arrow[d] \\
\mathbb{1} \arrow[r, "C"'] & \mathsf{SkOpMon}(M,N)
\end{tikzcd}$$

Example 2.4.6. Now for a biduality $R \dashv R^{\circ}$ and an adjunction $i \dashv i^*$ in an opmonadic-friendly monoidal bicategory the items of Remark 2.3.12 show an even closer relationship. The identity arrow on R comes equipped with a $i_{\circ}1$-comodule structure from i^*1 to $e1$ given by the square below.

$$\begin{tikzcd}
RR \arrow[r, "1i_{\circ}1"] \arrow[d, "i^*1"'] \arrow[dr, phantom, "\cong"] & RR^{\circ}R \arrow[d, "e1"] \\
R \arrow[r, "1"'] & R
\end{tikzcd}$$

Example 2.4.7. Let R and S be two k-algebras and C an $R|S$-coalgebroid. In Lemma 2.1.28 we saw that C is an opmonoidal arrow between enveloping monoidales in Mod_k. In a similar fashion, the objects of $\mathsf{rComod}_C((R,e1),(S,e1))$ may be described in the language of classical ring and module theory. A comodule over the opmonoidal arrow C consists of a module Y in R-Mod-S together with a coaction morphism $\varrho : Y \longrightarrow Y \otimes_S C$ in R-Mod-S in which $Y \otimes_S C$ has the module structure from R to S given by $r(a \otimes c)s = a \otimes rcs$, that is

- $\varrho(as) = \sum a_{(1)} \otimes a_{(2)}s$
- $\varrho(ra) = \sum a_{(1)} \otimes ra_{(2)}$

subject to the following axioms.

(i). $\sum ra_{(1)} \otimes a_{(2)} = \sum a_{(1)} \otimes a_{(2)}r$

(ii). (Y, ϱ) forms a C-comodule in the category R-Mod-S

Note that using the two sided R-module structure on $Y \otimes_S C$ given by $r \cdot (a \otimes c) \cdot r' = ra \otimes cr'$ item (i) may be rewritten as follows.

(i'). The image of the coaction ϱ is in the R-centralizer of $Y \otimes_S C$, that is $r \cdot \varrho(a) = \varrho(a) \cdot r$.

For the rest of this section our goal is to simplify the definition of comodules for opmonoidal arrows between enveloping monoidales in an opmonadic-friendly monoidal bicategory $\mathcal{M}$. And, in the two theorems that follow we apply the same technique used in Corollary 2.3.11 to simplify the definition of coalgebroids in terms of opmonoidal arrows to coalgebroids in terms of oplax actions. This technique consists of two steps: that of Theorem 2.3.7, which is basically the transposition along a biduality; and that of Theorems 2.2.8 or 2.3.10, where the main tool is the universal property of an opmonadic adjunction. Hence the two theorems below: Theorem 2.4.8 is the first step which corresponds to the use of transposition along bidualities, although, in this case it is considerably simpler; and Theorem 2.4.9 below is the second step which is analogous to the one that relies on the opmonadicity of an adjunction. We combine these two results in Corollary 2.4.11 to obtain an equivalence between comodules for opmonoidal arrows between enveloping monoidales and certain "oplax morphisms of oplax actions".

Theorem 2.4.8. *Let $\mathcal{M}$ be a monoidal bicategory. For every right skew monoidale M, every biduality $S \dashv S^\circ$, every oplax right M-action $a : AM \longrightarrow A$, and every opmonoidal arrow $C : M \longrightarrow S^\circ S$ in $\mathcal{M}$ there is an isomorphism between the categories,*

$$\mathsf{rComod}_C((A,a),(S,e1)) \cong \mathsf{rComod}_{\mathrm{id}_M}((A,a),(S,s))$$

$$\begin{tikzcd} AM \arrow[r, "YC"] \arrow[d, "a"'] \arrow[dr, phantom, "\nearrow"] & SS^\circ S \arrow[d, "e1"] \\ A \arrow[r, "Y"'] & S \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} AM \arrow[r, "Y1"] \arrow[d, "a"'] \arrow[dr, phantom, "\nearrow"] & SM \arrow[d, "s"] \\ A \arrow[r, "Y"'] & S \end{tikzcd}$$

where $s : SM \longrightarrow S$ is the oplax right M-action which corresponds to C under Theorem 2.3.7.

Proof.
The objects of these two categories differ only by the isomorphism

$$\begin{tikzcd} SM \arrow[rr, "s"] \arrow[dr, "1C"'] & \cong & S \\ & SS^\circ S \arrow[ur, "e1"'] & \end{tikzcd}$$

induced by the equivalence of Theorem 2.3.7 between the opmonoidal arrow C and the oplax M-action s. ∎

Now, in the following theorem the comodule in Example 2.4.6 plays the role of the opmonoidal left adjoint for the "opmonoidal $\dashv$ monoidal opmonadicity" of Theorem 2.2.8, but in the bicategory $\mathsf{rComod}(\mathcal{M})$ instead of $\mathsf{OpMon}(\mathcal{M})$.

Theorem 2.4.9. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every biduality $R \dashv R^\circ$, every opmonadic adjunction $i_\circ \dashv i^\circ$ in $\mathcal{M}$,*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

every monoidale N, every pseudo right action $b : BN \longrightarrow B$, and every pair of opmonoidal arrows $C : R^\circ R \longrightarrow N$ and $D : R \longrightarrow N$ which correspond to each other under the equivalence of Theorem 2.2.8, there is an isomorphism of categories

$$\mathsf{rComod}_C((R,e1),(B,b)) \cong \mathsf{rComod}_D((R,i^*1),(B,b))$$

$$\begin{tikzcd} RR^\circ R \arrow[r, "YC"] \arrow[d, "e1"'] \arrow[dr, phantom, "\bar{y}\Nearrow"] & BN \arrow[d, "b"] \\ R \arrow[r, "Y"'] & B \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} RR \arrow[r, "YD"] \arrow[d, "i^*1"'] \arrow[dr, phantom, "y\Nearrow"] & BN \arrow[d, "B"] \\ R \arrow[r, "Y"'] & B \end{tikzcd}$$

given by precomposition with the $i_\circ 1$-comodule in Example 2.4.6.

Proof.

Let H be the functor in the statement; its action on objects is given below.

$$H : \mathsf{rComod}_C((R, e1), (B, b)) \longrightarrow \mathsf{rComod}_D((R, i^*1), (B, b))$$

$$\left(\begin{tikzcd} R \arrow[d, "Y"] \\ B \end{tikzcd} , \begin{tikzcd} RR^\circ R \arrow[r, "YC"] \arrow[d, "e1"'] \arrow[dr, phantom, "\bar{y}\Nearrow"] & BN \arrow[d, "b"] \\ R \arrow[r, "Y"'] & B \end{tikzcd} \right) \longmapsto \left(\begin{tikzcd} R \arrow[d, "Y"] \\ B \end{tikzcd} , \begin{tikzcd} RR \arrow[dr, "1i_\circ 1"] \arrow[drr, bend left, "YD"] \arrow[ddr, bend right, "i^*1"'] \arrow[dr, phantom, shift left=4, "\cong"] \arrow[ddr, phantom, shift right=2, "\cong"] & & \\ & RR^\circ R \arrow[r, "YC"] \arrow[d, "e1"'] \arrow[dr, phantom, "\bar{y}\Nearrow"] & BN \arrow[d, "b"] \\ & R \arrow[r, "Y"'] & B \end{tikzcd} \right)$$

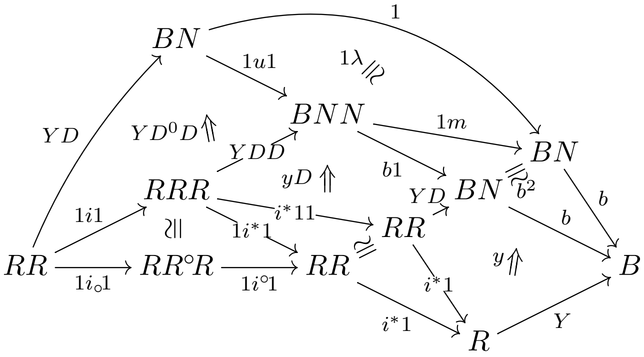
This functor is faithful since it is the identity on the underlying cells $\gamma : Y \longrightarrow Y'$ in $\mathcal{M}$. And it is essentially surjective on objects because for every D-comodule $(Y : R \longrightarrow B, y)$ from $e1$ to b in the codomain of H, the source and target of y have a structure of module for the monad induced by $1i_\circ 1 \dashv 1i^\circ 1$,

$$\begin{tikzcd} & & RR^\circ R \arrow[dr, "e1"] & & \\ RR \arrow[r, "1i_\circ 1"'] & RR^\circ R \arrow[ur, bend left, "1"] \arrow[r, "1i^\circ 1"'] \arrow[r, phantom, shift left=4, "1\varepsilon_\circ 1 \Uparrow"] & RR \arrow[u, "1i_\circ 1"'] \arrow[r, "i^*1"'] & R \arrow[r, "Y"'] & B \end{tikzcd} \qquad \begin{tikzcd} RR \arrow[r, "1i_\circ 1"'] & RR^\circ R \arrow[rr, bend left, "1"] \arrow[r, "1i^\circ 1"'] \arrow[rr, phantom, shift left=4, "1\varphi\Uparrow"] & RR \arrow[r, "YD"'] & BN \arrow[r, "b"'] & B \end{tikzcd}$$

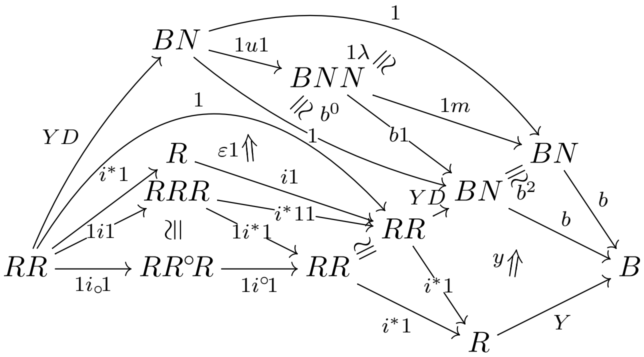
where φ is the action (2.2.1). Furthermore, the axioms for a D-comodule together with how φ is defined in terms of D^2 and D^0 imply that y is a morphism of modules for the monad induced by the adjunction $1i_\circ 1 \dashv 1i^\circ 1$.

$$\begin{tikzcd} & BN \arrow[dr, "1u1"] \arrow[ddrr, bend left, "1"] \arrow[drr, phantom, "1\lambda\cong"] & & & \\ & & BNN \arrow[dr, "1m"] & & \\ & RRR \arrow[ur, "YDD"] \arrow[dr, "1i^*1"] \arrow[d, phantom, "\cong"] \arrow[u, phantom, "YD^0D\Uparrow"] & \arrow[u, phantom, "YD^2\Uparrow"] & BN \arrow[dr, "b"] & \\ RR \arrow[uuur, bend left, "YD"] \arrow[ur, "1i1"] \arrow[r, "1i_\circ 1"'] & RR^\circ R \arrow[r, "1i^\circ 1"'] & RR \arrow[ur, "YD"] \arrow[dr, "i^*1"'] \arrow[rr, phantom, "y\Uparrow"] & & B \\ & & & R \arrow[ur, "Y"'] & \end{tikzcd}$$

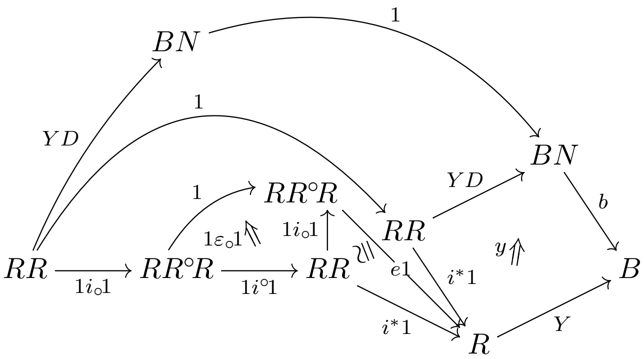
(COM1)
$=$



(COM2)
$=$



(SKM2)
$=$



Therefore by opmonadicity of $1i_\circ 1 \dashv 1i^\circ 1$ there exists a cell $\bar{y}$ as below,

$$\begin{tikzcd}
RR^\circ R \arrow[r, "YC"] \arrow[d, "e1"'] \arrow[dr, phantom, "\bar{y}\nearrow"] & BN \arrow[d, "b"] \\
R \arrow[r, "Y"'] & B
\end{tikzcd}$$

that composed with $1i_\circ 1$ is equal to y. The cell $\bar{y}$ provides the arrow Y with a C-comodule structure; one proves axiom (COM1) for $\bar{y}$ by precomposing both sides with the opmonadic left adjoint $1i_\circ 1 i_\circ 1 : RRR \longrightarrow RR^\circ RR^\circ R$, to get each side of axiom (COM1) for y, which are equal. And to prove axiom (COM2) for $\bar{y}$ precompose both sides of the axiom with the epimorphic cell

$$\begin{tikzcd} & RR^{\circ}R \arrow[dl, "1i^{\circ}1"'] \arrow[dr, "1"] \arrow[d, phantom, "1\varepsilon_{\circ}1 \Nearrow"] & \\ RR \arrow[rr, "1i_{\circ}1"'] & & RR^{\circ}R \end{tikzcd}$$

at $RR^{\circ}R$ to get each side of axiom (COM2) for y, which are equal. Therefore $H(Y, \bar{y}) = (Y, y)$, which means H is surjective on objects. One proves that it is full with a similar calculation for axiom (COM3), so H is an isomorphism. ∎

Remark 2.4.10. In view of Theorem 2.2.8, by varying the opmonoidal arrows C and D we may lift the isomorphisms in the previous theorem to an equivalence

$$\mathsf{rComod}((R, R^{\circ}R, e1), (B, N, b)) \simeq \mathsf{rComod}((R, R, i^{*}1), (B, N, b))$$

between the hom categories of $\mathsf{rComod}(\mathcal{M})$.

Together, the two previous theorems imply the following.

Corollary 2.4.11. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every pair of bidualities $R \dashv R^{\circ}$ and $S \dashv S^{\circ}$, every opmonoidal arrow $C : R^{\circ}R \longrightarrow S^{\circ}S$, and every opmonadic adjunction $i_{\circ} \dashv i^{\circ}$ in $\mathcal{M}$,*

$$\begin{tikzcd} R^{\circ} \arrow[d, bend left, "i^{\circ}"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_{\circ}"] \end{tikzcd}$$

there is an isomorphism of categories,

$$\mathsf{rComod}_{C}((R, e1), (S, e1)) \cong \mathsf{rComod}_{\mathrm{id}_R}((R, i^{*}1), (S, s))$$

where $s : SM \longrightarrow S$ is the oplax right M-action which corresponds to C under Corollary 2.3.11. Moreover, the pentagon below commutes strictly,

$$\begin{tikzcd}[column sep=tiny] \mathsf{rComod}_{C}((R, e1), (S, e1)) \arrow[rr, "\cong"'] \arrow[d, "\cong"] & & \mathsf{rComod}_{D}((R, i^{*}1), (S, e1)) \arrow[d, "\cong"'] \\ \mathsf{rComod}_{\mathrm{id}_{R^{\circ}R}}((R, e1), (S, \widehat{s})) \arrow[dr, "\cong"] & & \mathsf{rComod}_{\mathrm{id}_R}((R, i^{*}1), (S, s)) \\ & \mathsf{rComod}_{i_{\circ}1}((R, i^{*}1), (S, \widehat{s})) \arrow[ur, "\cong"] & \end{tikzcd} \tag{2.4.1}$$

where $\widehat{s} : SR^{\circ}R \longrightarrow S$ is the oplax right action that corresponds to C under the equivalence in Theorem 2.3.7, $D : R \longrightarrow S^{\circ}S$ is the opmonoidal arrow that corresponds to C under the equivalence in Theorem 2.2.8, and the edges of the pentagon are instances of the isomorphisms in Theorems 2.4.9 and 2.4.8. ∎

Remark 2.4.12. Given a duality $R \dashv R^\circ$ and an opmonadic adjunction $i_\circ \dashv i^\circ$ with opposite adjunction $i \dashv i^*$ in an opmonadic-friendly monoidal bicategory $\mathcal{M}$,

$$\begin{tikzcd} R \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

we can take comodules between the actions in Remark 2.3.12 around the pentagon (2.4.1) above. Start with the identity comodule on $e1 : RR^\circ R \longrightarrow R$ that lives in the source category of the pentagon, taking it down the first equivalence does not change it, taking it down-right the second equivalence, as well as to take it from the source to the right, gives the comodule from Example 2.4.6, and to take it all the way to the target gives the identity comodule on $i^*1 : RR \longrightarrow R$.

$$\begin{tikzcd} RR^\circ R \arrow[r, "1"] \arrow[d, "e1"'] & RR^\circ R \arrow[d, "e1"] \\ R \arrow[r, "1"'] & R \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} RR \arrow[r, "1i_\circ 1"] \arrow[d, "i^*1"'] \arrow[dr, phantom, "\cong"] & RR^\circ R \arrow[d, "e1"] \\ R \arrow[r, "1"'] & R \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} RR \arrow[r, "1"] \arrow[d, "i^*1"'] & RR \arrow[d, "i^*1"] \\ R \arrow[r, "1"'] & R \end{tikzcd}$$

Example 2.4.13. Comodules for opmonoidal arrows in Mod_k that live in the source of the pentagon of Corollary 2.4.11 were described in Example 2.4.7. Let us describe what is a comodule in the target of the pentagon, that is, an id_R-comodule from i^*1 to s in Mod_k. Here s is an $R|S$-coalgebroid via oplax actions which is denoted by a module C in SR-Mod-S as in Example 2.3.13. An object in $\mathsf{rComod}_{\mathrm{id}_R}((R, i^*1), (S, s))$ consists of a module Y in R-Mod-S together with a module morphism $\widetilde{\varrho} : Y \longrightarrow Y \otimes_S C$ in Mod-S where $Y \otimes_S C$ takes the right S module structure given by $(y \otimes c).s = y \otimes cs$, hence the condition below,

- $\widetilde{\varrho}(as) = \sum a_{(1)} \otimes a_{(2)} s$

and it is subject to the following axiom.

(i). $(Y, \widetilde{\varrho})$ forms a C-comodule in the category Mod-S

What changed from Example 2.4.7 is that the coaction $\widetilde{\varrho}$ is not necessarily a left R-morphism, condition 2.4.7.(i) vanishes, and $(Y, \widetilde{\varrho})$ is a C-comodule in Mod-S rather than R-Mod-S.

At this point we pause to recall the main results of this and the previous sections. These may be arranged in the chart below using the bar notation for equivalences. So, let two bidualities $R \dashv R^\circ$ and $S \dashv S^\circ$ and an adjunction $i \dashv i^*$ whose opposite is opmonadic all of which are in an opmonadic-friendly monoidal bicategory $\mathcal{M}$. On the left column we have the equivalence of Corollary 2.3.11, and on the right column we have the equivalence of Corollary 2.4.11 for a fixed pair of items in the left column.

<table>
<tr>
<td>

Opmonoidal arrow

$$\frac{R^\circ R \xrightarrow{C} S^\circ S}{SR \xrightarrow[s]{} R}$$

Oplax action

</td>
<td>$\simeq$</td>
<td>

C-comodule

$$\begin{tikzcd} RR^\circ R \arrow[r, "YC"] \arrow[d, "e1"'] \arrow[dr, phantom, "\bar{y}\nearrow"] & S^\circ SS \arrow[d, "e1"] \\ R \arrow[r, "Y"'] & S \end{tikzcd}$$

$$\begin{tikzcd} RR \arrow[r, "Y1"] \arrow[d, "i^*1"'] \arrow[dr, phantom, "y\nearrow"] & SR \arrow[d, "s"] \\ R \arrow[r, "Y"'] & S \end{tikzcd}$$

id_R-comodule to s

</td>
<td>$\cong$</td>
</tr>
</table>

But this table is still incomplete: there is one more equivalence of categories to add at the bottom of the right column and that is precisely what the next theorem is about. This new equivalence is another application of opmonadicity, but this time from the adjunction $i \dashv i^*$. The target category is the category $\mathcal{M}(I,S)_{\mathcal{M}(I,c)}$ of comodules for a comonad $c : S \longrightarrow S$ based at I. What completes the chart is our concluding corollary below, in which the interesting case is when c is the comonad induced by an opmonoidal arrow or by an oplax action as discussed in Remark 2.3.9.

Theorem 2.4.14. *For every biduality $R \dashv R^\circ$, every opmonadic adjunction*

$$\begin{tikzcd} R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i"] \end{tikzcd}$$

and every oplax right R-action $b : BR \longrightarrow B$ with respect to the right skew monoidal structure induced by $i \dashv i^*$ as in Lemma 2.1.15, there is an equivalence of categories

$$\mathsf{rComod}_{\mathrm{id}_R}((R, i^*1), (B, b)) \simeq \mathcal{M}(I,B)_{\mathcal{M}(I,c)}$$

where $c : B \longrightarrow B$ is the comonad induced by b as in Remark 2.3.9; and the equivalence is given as follows.

$$\left(\begin{tikzcd} R \arrow[d, "Y"] \\ B \end{tikzcd} , \begin{tikzcd} RR \arrow[r, "Y1"] \arrow[d, "i^*1"'] \arrow[dr, phantom, "y\nearrow"] & BR \arrow[d, "b"] \\ R \arrow[r, "Y"'] & B \end{tikzcd} \right) \longmapsto \left(\begin{tikzcd} I \arrow[d, "i"] \\ R \arrow[d, "Y"] \\ B \end{tikzcd} , \begin{tikzcd} & & & B \arrow[dr, "1i"] \arrow[dddrr, bend left, "c"] & & \\ & & R \arrow[ur, "Y"] \arrow[dr, "1i"] \arrow[r, phantom, "\cong"] & & BR \arrow[ddr, "b"] & \\ & & & RR \arrow[ur, "Y1"] \arrow[dr, "i^*1"] \arrow[d, phantom, "\Uparrow\eta 1"] \arrow[drr, phantom, "y\nearrow"] & & \\ I \arrow[uurr, "i"] \arrow[rr, "i"'] \arrow[urrr, phantom, "\cong i1"] & & R \arrow[ur] \arrow[r, "1"'] & R \arrow[rr, "Y"'] & & B \end{tikzcd} \right)$$

Proof. [Sketch]

The action on the structure cells y of the proposed functor in the statement may be factorised by first taking the mate of y with respect to the adjunction $i \dashv i^*$ and then by precomposing with i.

For an arrow $Y : R \longrightarrow B$, cells y as in the statement are in bijection with their mates with respect to the adjunction $i \dashv i^*$.

$$\begin{tikzcd}
RR \arrow[r, "Y1"] \arrow[d, "i^*1"'] \arrow[dr, phantom, "y\Nearrow"] & BR \arrow[d, "b"] \\
R \arrow[r, "Y"'] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
RR \arrow[r, "Y1"] \arrow[dr, phantom, "\tilde{y}\Uparrow"] & BR \arrow[d, "b"] \\
R \arrow[u, "i1"] \arrow[r, "Y"'] & B
\end{tikzcd}$$

A cell y satisfies the axioms (COM1) and (COM2) that turn (Y, y) into an id_R-comodule from i^*1 to b if and only if its mate $\tilde{y}$ satisfies two other axioms $(\mathrm{COM1})^{\mathrm{mate}}$ and $(\mathrm{COM2})^{\mathrm{mate}}$ obtained by taking mates of each side of the original ones for y. Call $\mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b))$ the category whose objects consist of an arrow $Y : R \longrightarrow B$ together with a cell $\tilde{y}$ as above, satisfying axioms $(\mathrm{COM1})^{\mathrm{mate}}$ and $(\mathrm{COM2})^{\mathrm{mate}}$. The arrows of $\mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b))$ are cells $\gamma : Y \longrightarrow Y'$ obtained in a similar fashion. Hence the isomorphism of categories below.

$$\mathsf{rComod}_{\mathrm{id}_R}((R, i^*1), (B, b)) \cong \mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b))$$

Now define $\mathcal{Z}$ as the category whose objects are triples (X, x, ζ),

$$I \xrightarrow{X} B \qquad
\begin{tikzcd}
B \arrow[r, "1i"] \arrow[dr, phantom, "x\Uparrow"] & BR \arrow[d, "b"] \\
I \arrow[u, "X"] \arrow[r, "X"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "i"'] \arrow[rrr, bend left, "X"] & R \arrow[r, "i^*"'] \arrow[r, phantom, shift left=4, "\zeta\Uparrow"] & I \arrow[r, "X"'] & B
\end{tikzcd}$$

such that (X, x) is a comodule for the comonad c, and the cell ζ is an action on X with respect to the monad induced by $i \dashv i^*$, satisfying the following compatibility condition,

$$\begin{tikzcd}
B \arrow[rrr, bend left, "1i"] \arrow[rrr, phantom, shift left=3, "x\Uparrow"] & & & BR \arrow[d, "b"] \\
I \arrow[u, "X"] \arrow[r, "i"'] \arrow[rrr, bend left, "X"] & R \arrow[r, "i^*"'] \arrow[r, phantom, shift left=4, "\zeta\Uparrow"] & I \arrow[r, "X"'] & B
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, "1i"] \arrow[dr, phantom, "\cong"] & BR \arrow[r, "1i^*"] \arrow[rr, bend left, "1"] \arrow[r, phantom, shift left=4, "1\varepsilon\Uparrow"] \arrow[dr, phantom, "\cong"] & B \arrow[r, "1i"] \arrow[dr, phantom, "x\Uparrow"] & BR \arrow[d, "b"] \\
I \arrow[u, "X"] \arrow[r, "i"'] & R \arrow[u, "X1"] \arrow[r, "i^*"'] & I \arrow[u, "X"] \arrow[r, "X"'] & B
\end{tikzcd}
\tag{Z1}$$

An arrow $\chi : (X, x, \zeta) \longrightarrow (X', x', \zeta')$ in $\mathcal{Z}$ is a cell $\chi : X \longrightarrow X'$ which is simultaneously a morphism of c-comodules and a morphism of modules for the monad induced by $i \dashv i^*$.

There is an isomorphism of categories $\mathcal{Z} \cong \mathcal{M}(I, B)_{\mathcal{M}(I,c)}$ which is deduced from the redundancy of the action ζ in the objects of $\mathcal{Z}$. Indeed, for every object (X, x, ζ) in $\mathcal{Z}$ one may express ζ in terms of x and b^0 as follows.

$$
\begin{tikzcd}
I \arrow[r, "i"'] \arrow[rrr, bend left, "X"] & R \arrow[r, "i^*"'] \arrow[r, phantom, bend left, "\zeta\Uparrow"] & I \arrow[r, "X"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & B \arrow[dr, "1i"] \arrow[ddr, bend left, "1"] \arrow[dr, phantom, bend left, "b^0\Rightarrow"] & \\
B \arrow[urr, bend left, "1"] \arrow[r, "1i"] & BR \arrow[r, "1i^*"] \arrow[rr, bend left, "1"] \arrow[r, phantom, bend left, "1\varepsilon\Uparrow"] & B \arrow[r, "1i"] & BR \arrow[d, "b"] \\
I \arrow[u, "X"] \arrow[r, "i"'] \arrow[ur, phantom, "\cong"] & R \arrow[u, "X1"] \arrow[r, "i^*"'] \arrow[ur, phantom, "\cong"] & I \arrow[u, "X"] \arrow[r, "X"'] \arrow[ur, phantom, "x\Uparrow"] & B
\end{tikzcd}
$$

And if for an arbitrary c-comodule (X, x) one defines ζ by the equation above, then (X, x, ζ) becomes an object of $\mathcal{Z}$, therefore the functor $\mathcal{Z} \longrightarrow \mathcal{M}(I, B)_{\mathcal{M}(I,c)}$ which forgets the action ζ is an isomorphism of categories.

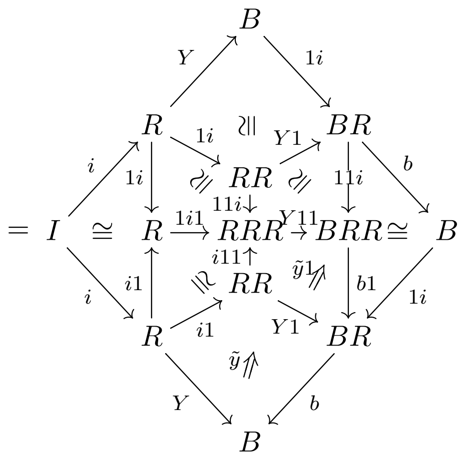
Now, the functor K below induced by precomposition with the opmonadic arrow $i : I \longrightarrow R$ is an equivalence of categories $\mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b)) \simeq \mathcal{Z}$ because of the opmonadicity of $i \dashv i^*$.

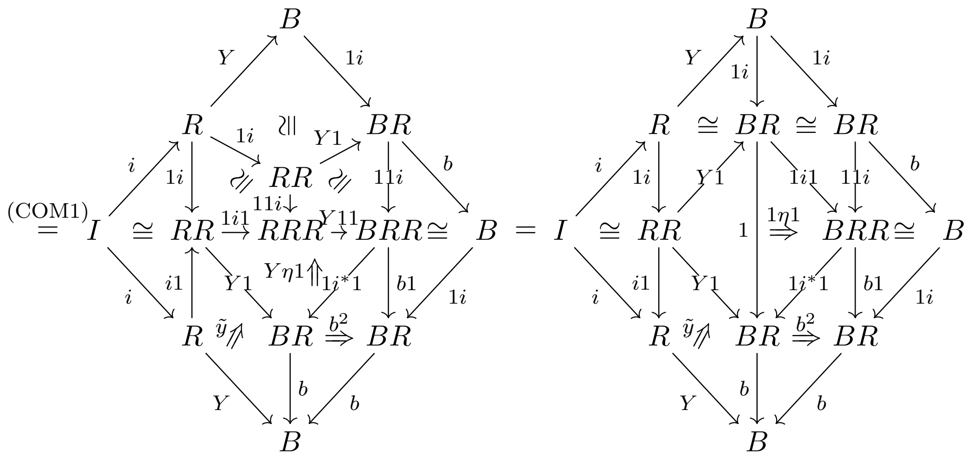
$$K : \mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b)) \longrightarrow \mathcal{M}(I, B)_{\mathcal{M}(I,c)} \cong \mathcal{Z}$$

$$
\left(
\begin{tikzcd}
R \arrow[d, "Y"] \\
B
\end{tikzcd}
,\
\begin{tikzcd}
RR \arrow[r, "Y1"] \arrow[dr, phantom, "\tilde{y}\Uparrow"] & BR \arrow[d, "b"] \\
R \arrow[u, "i1"] \arrow[r, "Y"'] & B
\end{tikzcd}
\right)
\longmapsto
\left(
\begin{tikzcd}
I \arrow[d, "i"] \\
R \arrow[d, "Y"] \\
B
\end{tikzcd}
,\
\begin{tikzcd}
& & B \arrow[dr, "1i"] & & \\
& R \arrow[ur, "Y"] \arrow[dr, "1i"] \arrow[rr, phantom, "\cong"] & & BR \arrow[dr, "b"] & \\
& & RR \arrow[ur, "Y1"] \arrow[dr, phantom, "\tilde{y}\Uparrow"] & & \\
I \arrow[uur, "i"] \arrow[rr, "i"'] \arrow[ur, phantom, "\cong"] & & R \arrow[u, "i1"] \arrow[rr, "Y"'] & & B
\end{tikzcd}
\right)
$$

This assignation K is a well defined functor because the axioms $(\mathrm{COM1})^{\mathrm{mate}}$ and $(\mathrm{COM2})^{\mathrm{mate}}$ translate precisely into the axioms for a c-comodule from I to B. In fact, the unit axiom is literally the same, and the associative axiom follows from the calculation below.

$$
\begin{tikzcd}
& & B \arrow[dr, "1i"] & & \\
& R \arrow[ur, "Y"] \arrow[dr, "1i"] \arrow[rr, phantom, "\cong"] & & BR \arrow[ddr, "b"] & \\
& & RR \arrow[ur, "Y1"] \arrow[r, phantom, "\tilde{y}\Uparrow"] & & \\
I \arrow[uur, "i"] \arrow[ddr, "i"'] \arrow[rr, "i"] \arrow[ur, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & & R \arrow[u, "i1"] \arrow[d, "1i"] \arrow[rr, "Y"] \arrow[dr, phantom, "\cong"] & & B \arrow[ddl, "1i"] \\
& & RR \arrow[dr, "Y1"] & & \\
& R \arrow[ur, "i1"] \arrow[dr, "Y"'] \arrow[rr, phantom, "\tilde{y}\Uparrow"] & & BR \arrow[dl, "b"] & \\
& & B & &
\end{tikzcd}
=
\begin{tikzcd}
& & B \arrow[dr, "1i"] & & \\
& R \arrow[ur, "Y"] \arrow[dr, "1i"] \arrow[rr, phantom, "\cong"] & & BR \arrow[d, "11i"] \arrow[dr, "b"] & \\
& & RR \arrow[d, "11i"] \arrow[r, phantom, "\cong"] & & \\
I \arrow[r, "i"] \arrow[uur, "i"] \arrow[ddr, "i"'] & R \arrow[ur, "i1"] \arrow[dr, "1i"'] \arrow[ul, phantom, "\cong"] \arrow[dl, phantom, "\cong"] & RRR \arrow[r, "Y11"] & BRR \arrow[dd, "b1"] \arrow[r, phantom, "\cong"] & B \arrow[ddl, "1i"] \\
& & RR \arrow[u, "i11"] \arrow[dr, "Y1"] \arrow[ur, phantom, "\tilde{y}1\Uparrow"] & & \\
& R \arrow[ur, "i1"] \arrow[dr, "Y"'] \arrow[rr, phantom, "\tilde{y}\Uparrow"] & & BR \arrow[dl, "b"] & \\
& & B & &
\end{tikzcd}
$$





The functor K is automatically faithful since precomposing with an opmonadic arrow is a faithful process. By the opmonadicity of $i \dashv i^*$ the functor K is essentially surjective on objects and full. Indeed, let (X, x, ζ) be an object in $\mathcal{Z}$, since (X, ζ) is a module for the monad induced by $i \dashv i^*$ there exists an arrow $Y : R \longrightarrow B$ and an isomorphism

$$\begin{tikzcd} I \arrow[rr, "X"] \arrow[dr, "i"'] & \arrow[d, phantom, "\cong"] & B \\ & R \arrow[ur, "Y"'] & \end{tikzcd} \tag{2.4.2}$$

such that the following equation holds.

$$\begin{tikzcd} I \arrow[r, "i"'] \arrow[rrrr, bend left, "X"] & R \arrow[r, "i^*"'] & I \arrow[rr, "X"] \arrow[dr, "i"'] \arrow[u, phantom, "\zeta\Uparrow"] & \arrow[d, phantom, "\cong"] & B \\ & & & R \arrow[ur, "Y"'] & \end{tikzcd} = \begin{tikzcd} I \arrow[r, "i"'] \arrow[rrr, bend left, "X"] & R \arrow[r, bend left, "1"] \arrow[dr, "i^*"'] & \arrow[d, phantom, "\varepsilon\Uparrow"] \arrow[u, phantom, "\cong"] & R \arrow[r, "Y"'] & B \\ & & I \arrow[ur, "i"'] & & \end{tikzcd} \tag{2.4.3}$$

Furthermore, axiom (Z1) may be read as the fact that x is a morphism of modules for the monad induced by $i \dashv i^*$, thus by opmonadicity there exists a cell $\tilde{y}$ such that the following equation holds.

$$
\begin{tikzcd}[column sep=small]
 & B \arrow[dr, "1i"] & \\
 \arrow[r, phantom, "x\Uparrow"] & & BR \arrow[d, "b"] \\
I \arrow[uur, bend left, "X"] \arrow[rr, bend left, "X"] \arrow[r, "i"'] & R \arrow[u, phantom, "\cong"] \arrow[r, "Y"'] & B
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & B \arrow[dr, "1i"] & & \\
 & R \arrow[ur, "Y"] \arrow[dr, "1i"] \arrow[r, phantom, "\cong"] & & BR \arrow[dr, "b"] & \\
 & & RR \arrow[ur, "Y1"] \arrow[r, phantom, "\tilde{y}\Uparrow"] & & \\
I \arrow[uuurr, bend left, "X"] \arrow[uur, "i"] \arrow[rr, "i"'] \arrow[ur, phantom, "\cong"] & & R \arrow[u, "i1"] \arrow[rr, "Y"'] & & B
\end{tikzcd}
\tag{2.4.4}
$$

The data $(Y, \tilde{y})$ constitute an object of $\mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b))$: axiom $(\mathrm{COM1})^{\mathrm{mate}}$ for $(Y, \tilde{y})$ follows by precomposing both sides with the opmonadic arrow $i : I \longrightarrow R$ to obtain each side of the coassociative axiom for the coaction x, which are equal; and axiom $(\mathrm{COM2})^{\mathrm{mate}}$ is equal to the counit axiom for the coaction x. Hence, in light of equations (2.4.3) and (2.4.4) the isomorphism (2.4.2) is in $\mathcal{Z}$ and reads as $K(Y, \tilde{y}) \cong (X, x)$, so K is essentially surjective on objects. Now, let $\chi : (X, x, \zeta) \longrightarrow (X', x', \zeta')$ be a morphism in $\mathcal{Z}$, as χ is a morphism of modules for the monad induced by $i \dashv i^*$ there exists a cell $\gamma : Y \longrightarrow Y'$ in $\mathcal{M}$ such that the following equation holds.

$$
\begin{tikzcd}
I \arrow[rr, bend left=50, "X'"] \arrow[rr, "X"] \arrow[dr, "i"'] & \arrow[u, phantom, "\chi\Uparrow"] \arrow[d, phantom, "\cong"] & B \\
 & R \arrow[ur, "Y"'] &
\end{tikzcd}
=
\begin{tikzcd}
I \arrow[rr, bend left=50, "X'"] \arrow[dr, "i"'] \arrow[r, phantom, "\cong"] & {} & N \\
 & R \arrow[ur, "Y"'] \arrow[ur, bend left=60, "Y'"] \arrow[ur, phantom, "\gamma\Uparrow"] &
\end{tikzcd}
\tag{2.4.5}
$$

To prove that γ is a cell in $\mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b))$ precompose both sides of axiom $(\mathrm{COM3})^{\mathrm{mate}}$ with the opmonadic arrow i; this produces the two sides of the axiom that makes χ into a morphism of c-comodules, which are equal. Thus γ is in $\mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b))$ and the equation (2.4.5) now reads as $K(\gamma) = \chi$, so K is full. The theorem follows by the sequence of equivalences and isomorphisms below.

$$\mathsf{rComod}_{\mathrm{id}_R}((R, i^*1), (B, b)) \cong \mathsf{rComod}^{\mathrm{mate}}_{\mathrm{id}_R}((R, i^*1), (B, b)) \simeq \mathcal{Z} \cong \mathcal{M}(I, B)_{\mathcal{M}(I,c)}$$

$\blacksquare$

Corollary 2.4.15. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every pair of bidualities $R \dashv R^\circ$ and $S \dashv S^\circ$, every opmonoidal arrow $C : R^\circ R \longrightarrow S^\circ S$, and every opmonadic adjunction $i \dashv i^*$ whose dual $i_\circ \dashv i^\circ$ is opmonadic too,*

$$
\begin{tikzcd}
R \arrow[d, bend left, "i^*"] \\
I \arrow[u, bend left, "i"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
\qquad
\begin{tikzcd}
R^\circ \arrow[d, bend left, "i^\circ"] \\
I \arrow[u, bend left, "i_\circ"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

there is an equivalence of categories,

$$\mathsf{rComod}_C((R, e1), (S, e1)) \simeq \mathcal{M}(I, S)_{\mathcal{M}(I,c)}$$

$$\begin{tikzcd} RR^\circ R \arrow[r, "YC"] \arrow[d, "e1"'] \arrow[dr, phantom, "y \nearrow"] & SS^\circ S \arrow[d, "e1"] \\ R \arrow[r, "Y"'] & S \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & S \arrow[dr, "c"] & \\ I \arrow[ur, "X"] \arrow[rr, bend right, "X"'] \arrow[rr, phantom, "x\Uparrow"] & & S \end{tikzcd}$$

where $c : S \longrightarrow S$ is the comonad induced by C as in Remark 2.3.9.

Proof.

Let $s : SR \longrightarrow S$ be the oplax right action that corresponds to the opmonoidal arrow C under the equivalence in Corollary 2.3.11. By Remark 2.3.9 c is the comonad induced both by the opmonoidal arrow C and by the oplax action s. Then there is an equivalence of categories,

$$\mathsf{rComod}_C((R, e1), (S, e1)) \cong \mathsf{rComod}_{\mathrm{id}_R}((R, i^*1), (S, s)) \simeq \mathcal{M}(I, S)_{\mathcal{M}(I,c)}$$

where the isomorphism is an instance of Corollary 2.4.11 and the equivalence is an instance of Theorem 2.4.14. ■

We conclude with a remark about the motivating example $\mathcal{M} = \mathsf{Mod}_k$. In Lemma 2.1.28 we saw that opmonoidal arrows between enveloping monoidales $C : R^\circ R \longrightarrow S^\circ S$ in Mod_k are $R|S$-coalgebroids. There is a standard definition of a comodule for an $R|S$-coalgebroid found for example in [Hai08, 1.4] or [Böh09, 3.6].

Definition 2.4.16. Let R and S be two k-algebras and C an $R|S$-coalgebroid. A C-comodule X is a comodule for the underlying comonoid in S-Mod-S of the coalgebroid C, i.e. a module X in Mod-S together with a module morphism $x : X \otimes_S C \longrightarrow X$ in Mod-S, called the C-coaction, which satisfies coassociative and counit laws.

Remark 2.4.17. If we apply Corollary 2.4.15 to the case $\mathcal{M} = \mathsf{Mod}_k$, we recover [Hai08, Lemma 1.4.1], also found in [Böh09, Lemma 3.17]. This is an equivalence between comodules for coalgebroids, as defined above, and comodules for the opmonoidal arrows in Mod_k that correspond to coalgebroids as described in Example 2.4.7. Moreover, these two versions of comodules for coalgebroids are also equivalent to those defined via oplax actions as in Example 2.4.13. The only difference between the definition of comodule for a coalgebroid via oplax actions and the standard definition is that in the former the underlying module is in R-Mod-S while in the latter is in Mod-S.

Now, a sufficient condition to have a monoidal structure on the category of comodules for a coalgebroid, is that the coalgebroid is in fact a bialgebroid [Hai08, Corollary 1.7.2]. In our language, just as a coalgebroid means an opmonoidal arrow $C : R^\circ R \longrightarrow S^\circ S$, a bialgebroid is an opmonoidal monad $B : R^\circ R \longrightarrow R^\circ R$. This description of bialgebroids in the language of monoidal bicategories is due to [DS04] and it is motivated by the work of [Szl03]. We get a monoidal structure in the category of comodules for opmonoidal monads on an enveloping monoidale $R^\circ R$ in a similar way.

Theorem 2.4.18. *For every biduality $R \dashv R^\circ$ and every opmonoidal monad $B : R^\circ R \longrightarrow R^\circ R$ in $\mathcal{M}$ the category $\mathsf{rComod}_B((R, e1), (R, e1))$ of B-comodules has a monoidal structure such that the forgetful functor*

$$\mathsf{rComod}_B((R, e1), (R, e1)) \longrightarrow \mathcal{M}(R, R)$$

is strong monoidal. The tensor product and unit of B-comodules is calculated are as follows.

$$
\begin{tikzcd}
 & & RR^{\circ}R \arrow[dr, "Y11"] & & \\
 & RR^{\circ}R \arrow[ur, "1B"] \arrow[dr, "Y11"] \arrow[rr, phantom, "\wr\|"] & & RR^{\circ}R \arrow[dr, "Z11"] & \\
RR^{\circ}R \arrow[uurr, bend left, "1B"] \arrow[ur, "1B"] \arrow[rr, "YB"'] \arrow[d, "e1"'] \arrow[u, phantom, "1\mu\Nwarrow"] & & RR^{\circ}R \arrow[ur, "1B"] \arrow[rr, "ZB"'] \arrow[d, "e1"] & & RR^{\circ}R \arrow[d, "e1"] \\
R \arrow[rr, "Y"'] \arrow[urr, phantom, "y\Nearrow"] & & R \arrow[rr, "Z"'] \arrow[urr, phantom, "z\Nearrow"] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
RR^{\circ}R \arrow[r, bend left, "1B"] \arrow[r, "1"'] \arrow[r, phantom, shift left=2, "1\eta\Uparrow"] \arrow[d, "e1"'] & RR^{\circ}R \arrow[d, "e1"] \\
R \arrow[r, "1"'] & R
\end{tikzcd}
$$

Proof.

The associator and left and right unitor isomorphisms are induced by those of the horizontal composition of $\mathsf{rComod}(\mathcal{M})$. And the axioms for a monoidal category follow from the associativity and unitality of the monad structure of B and from coherence axioms for the horizontal composition of B-comodules. ■

3 Oplax Actions

In [LS12, Theorem 5.2] the authors prove that, in a suitable monoidal bicategory $\mathcal{M}$, opmonoidal monads on enveloping monoidales $R^{\circ}R$ are in equivalence with certain right skew monoidales.

$$\frac{R^{\circ}R \longrightarrow R^{\circ}R \text{ Opmonoidal Monads}}{\begin{array}{c} RR \longrightarrow R \text{ Right Skew Monoidales} \\ \text{with fixed unit } (i : I \longrightarrow R) \end{array}}$$

Oplax actions appeared in the previous chapter when we adapted [LS12, Theorem 5.2] to the case of mere opmonoidal arrows on enveloping monoidales; we showed an equivalence between these in Corollary 2.3.11 which holds under the same hypotheses as [LS12, Theorem 5.2].

$$\frac{R^{\circ}R \longrightarrow S^{\circ}S \text{ Opmonoidal Arrows}}{SR \longrightarrow S \text{ Oplax Right Actions}}$$

By requiring the hypotheses of Corollary 2.3.11 to be satisfied globally by $\mathcal{M}$, one may obtain a bicategory of oplax actions by copying the bicategory structure from that of the bicategory $\mathsf{OpMon}(\mathcal{M})$ of monoidales, opmonoidal arrows, and opmonoidal cells in $\mathcal{M}$. And since opmonoidal monads are defined precisely as monads in the bicategory $\mathsf{OpMon}(\mathcal{M})$, monads in the bicategory of oplax actions are right skew monoidales, see Theorem 3.1.1.

In this chapter we generalise this result for an arbitrary monoidal bicategory $\mathcal{M}$. The first thing one notices is that without Corollary 2.3.11 it is not known if there exists a bicategory of oplax actions. So one must rely on a different structure to define monads: here we do that using a simplicial object in Cat.

Simplicial objects in Cat are functors $\Delta^{\mathrm{op}} \longrightarrow \mathsf{Cat}$. There is a notion of weak equivalence between these, it consists of a natural transformation and a pseudonatural transformation going the other way which are pseudoinverse to each other. Each bicategory has an associated simplicial object in Cat that we call the lax-2-nerve: its underlying simplicial set of objects in each dimension is the usual nerve of a bicategory. There is a simplicial set $\mathbb{C}$, called the Catalan simplicial set, that has the property that simplicial morphisms into the nerve of a bicategory $\mathcal{B}$ are in bijection with monads in $\mathcal{B}$.

$$\frac{\mathbb{C}\longrightarrow N\mathcal{B} \text{ Simplicial morphisms}}{\text{Monads in } \mathcal{B}}$$

We show the existence of a simplicial object in Cat of oplax actions $\mathsf{OplaxAct}(\mathcal{M})$, hence one is able to define monads of oplax actions as simplicial maps $\mathbb{C}\longrightarrow\mathsf{OplaxAct}(\mathcal{M})$. We prove that these monads are in bijection with right skew monoidales whose unit has a right adjoint, with no assumptions on $\mathcal{M}$ required. Furthermore, under the hypotheses of Corollary 2.3.11 we show that $\mathsf{OplaxAct}(\mathcal{M})$ is weakly equivalent to the lax-2-nerve of the full subbicategory of $\mathsf{OpMon}(\mathcal{M})$ consisting of enveloping monoidales.

3.1 Preliminaries

As a quick reminder from Section 2.3, *an oplax M-action on A* in a monoidal bicategory $\mathcal{M}$ consists of an arrow $a : AM\longrightarrow A$, together with an associator cell a^2, and a right unitor cell a^0,

$$\begin{tikzcd} AMM \arrow[r, "a1"] \arrow[d, "1m"'] \arrow[dr, phantom, "a^2 \Nearrow"] & AM \arrow[d, "a"] \\ AM \arrow[r, "a"'] & A \end{tikzcd} \qquad \begin{tikzcd} AM \arrow[d, "a"'] \arrow[dr, phantom, "\Searrow a^0"] & A \arrow[l, "1u"'] \arrow[dl, "1"] \\ A & \end{tikzcd}$$

satisfying three axioms (OLA1), (OLA2), and (OLA3). Oplax M-actions on an object A form a category $\mathsf{OplaxAct}(M;A)$ and there is a forgetful functor which takes an oplax action to its underlying arrow $\mathsf{OplaxAct}(M;A)\longrightarrow\mathcal{M}(AM,A)$.

One of the main theorems of the previous chapter is Corollary 2.3.11, it establishes an equivalence of categories between certain categories of oplax actions and hom categories of opmonoidal arrows between enveloping monoidales, provided that $\mathcal{M}$ is an opmonadic-friendly monoidal bicategory in the sense of Definition 2.2.1. The oplax actions involved in this equivalence are on an object S that has a right bidual S°, and with respect to a right skew monoidale R whose unit $i : I\longrightarrow R$ has a right adjoint i^*. In fact, the right skew monoidal structure on R is precisely the one induced by the adjunction $i \dashv i^*$ as in Lemma 2.1.15. Furthermore, R must have a right bidual R° as an object of $\mathcal{M}$, and the opposite adjunction $i_\circ \dashv i^\circ$ of the adjunction $i \dashv i^*$ must be opmonadic.

$$\mathsf{OpMon}(R^\circ R, S^\circ S) \simeq \mathsf{OplaxAct}(R;S)$$

This equivalence exposes an "arrow-like" essence for oplax actions, which justifies the notation for the categories $\mathsf{OplaxAct}(M;A)$. Following this idea we depict oplax actions and cells of oplax actions with arrows and double arrows with a dash in the middle. This also distinguishes them from those of $\mathcal{M}$.

$$\begin{tikzcd} M \arrow[r, bend left, "|"] \arrow[r, bend right, "|"'] \arrow[r, phantom, "\Uparrow"] & A \end{tikzcd}$$

Hence, it is also worthwhile to figure out when the categories of oplax actions are the hom categories of a bicategory. A simple answer is to globally require the hypothesis of Corollary 2.3.11. In which case we call the resulting bicategory $\underline{\mathsf{OplaxAct}}(\mathcal{M})$; the underline is used to distinguish between the bicategory of oplax actions and the simplicial object in Cat of oplax actions that we shall define in the next section.

Theorem 3.1.1. *Let $\mathcal{M}$ be a right autonomous opmonadic-friendly monoidal bicategory such that every object R has a chosen adjunction $i \dashv i^*$ whose opposite adjunction $i_\circ \dashv i^\circ$ is opmonadic.*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

There exists a bicategory of oplax actions $\underline{\mathsf{OplaxAct}}(\mathcal{M})$ *whose objects are those of $\mathcal{M}$ equipped with the skew monoidal structure induced by their chosen adjunction, and with* $\mathsf{OplaxAct}(R;S)$ *as its hom categories. Moreover, there is a biequivalence of bicategories*

$$\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M}) \simeq \underline{\mathsf{OplaxAct}}(\mathcal{M})$$

where $\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M})$ *is the full subbicategory of* $\mathsf{OpMon}(\mathcal{M})$ *whose objects are those enveloping monoidales $R^\circ R$ induced by the chosen right bidual R° for each object R.*

Proof.

Composition and identities are calculated by going back and forth along instances of the equivalence of Corollary 2.3.11. Thus, for an object R in $\mathcal{M}$ the identity functor on R in $\underline{\mathsf{OplaxAct}}(\mathcal{M})$ is given as follows,

$$\mathbb{1} \xrightarrow{\mathrm{id}} \mathsf{OpMon}(R^\circ R, R^\circ R) \simeq \mathsf{OplaxAct}(R; R),$$

its image is the regular oplax action $i^*1 : RR \longrightarrow R$ of the right skew monoidale structure on R induced by the chosen adjunction $i \dashv i^*$, as we mentioned in Remark 2.3.12. For three objects R, S, and T of $\mathcal{M}$, the composition functor is given as follows.

$$\begin{aligned} \mathsf{OplaxAct}(S;T) \times \mathsf{OplaxAct}(R;S) &\simeq \mathsf{OpMon}(S^\circ S, T^\circ T) \times \mathsf{OpMon}(R^\circ R, S^\circ S) \\ &\longrightarrow \mathsf{OpMon}(R^\circ R, T^\circ T) \simeq \mathsf{OplaxAct}(R;T) \end{aligned}$$

Unfortunately, there is no explicit description of the composition functor since the first equivalence relies on the existential part of the universal property of opmonadic adjunctions. ■

Now, at this point we want to draw our attention to some facts regarding monads in $\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M})$, and consequently in $\underline{\mathsf{OplaxAct}}(\mathcal{M})$. First, we know that monads in $\mathsf{OpMon}(\mathcal{M})$ are by definition opmonoidal monads in $\mathcal{M}$. Second, opmonoidal monads on enveloping monoidales are precisely the quantum categories of Day and Street in the case $\mathcal{M} = \mathsf{Mod}(\mathcal{V})$, see [DS04, Proposition 3.3 and Section 12]. And third, Lack and Street characterise opmonoidal monads on enveloping monoidales in [LS12, Theorem 5.2] using the same technique as we did in in Theorem 3.1.1.

Theorem 3.1.2 (Lack-Street)**.** *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every biduality $R \dashv R^\circ$, and every opmonadic adjunction*

$$\begin{tikzcd} R^\circ \arrow[d, bend left=50, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left=50, "i_\circ"] \end{tikzcd}$$

there is an equivalence between

$$\begin{array}{c} R^\circ R \longrightarrow R^\circ R \ \textit{Opmonoidal Monads on } R^\circ R \\ \hline RR \longrightarrow R \ \textit{Right Skew Monoidales} \\ \textit{whose unit } (i : I \longrightarrow R) \textit{ is the opposite of } i^\circ. \end{array}$$

Hence, under the hypotheses of Theorem 3.1.1, we automatically have a description of monads in $\underline{\mathsf{OplaxAct}}(\mathcal{M})$.

Corollary 3.1.3. *Let $\mathcal{M}$ be a right autonomous opmonadic-friendly monoidal bicategory such that every object R has a chosen adjunction $i \dashv i^*$ whose opposite adjunction $i_\circ \dashv i^\circ$ is opmonadic.*

$$\begin{tikzcd} R^\circ \arrow[d, bend left=50, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left=50, "i_\circ"] \end{tikzcd}$$

Monads in $\underline{\mathsf{OplaxAct}}(\mathcal{M})$ are right skew monoidales whose unit has a right adjoint. ■

An explicit horizontal composition that does not require to go back and forth along the equivalence of Theorem 3.1.1 would allow us to give an elementary description of monads in $\underline{\mathsf{OplaxAct}}(\mathcal{M})$. In consequence, it would help us to understand which of the various parts of a right skew monoidale correspond to an "underlying" oplax action and which other parts play the roles of the multiplication and the unit of a monad of oplax actions.

3.2 A simplicial object in Cat of Oplax Actions

3.2.1 Simplicial objects in Cat

The 2-category $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$, where the square brackets denote (2-)functors, (2-)natural transformations, and modifications, is the 2-category of *simplicial objects in* Cat, simplicial morphisms, and simplicial transformations between them. A simplicial object $\mathbb{X}$ in Cat consists of categories $\mathbb{X}_n$ for each natural number, face functors ∂, and degeneracy functors s that satisfy the usual simplicial equations [Mac97, pp. 179].

$$\begin{tikzcd}[column sep=huge] \cdots\ \mathbb{X}_3 \arrow[r, shift left=9, "\partial_0" description] \arrow[r, shift left=3, "\partial_1" description] \arrow[r, shift right=3, "\partial_2" description] \arrow[r, shift right=9, "\partial_3" description] & \mathbb{X}_2 \arrow[l, shift left=6, "s_0" description] \arrow[l, "s_1" description] \arrow[l, shift right=6, "s_2" description] \arrow[r, shift left=6, "\partial_0" description] \arrow[r, "\partial_1" description] \arrow[r, shift right=6, "\partial_2" description] & \mathbb{X}_1 \arrow[l, shift left=3, "s_0" description] \arrow[l, shift right=3, "s_1" description] \arrow[r, shift left=3, "\partial_0" description] \arrow[r, shift right=3, "\partial_1" description] & \mathbb{X}_0 \arrow[l, "s_0" description] \end{tikzcd}$$

The category of sets may be thought as a locally discrete 2-category, and as such, there is a 2-functor $\mathsf{Set} \longrightarrow \mathsf{Cat}$ that sends each set to its corresponding discrete category. It induces a forgetful 2-functor which takes a simplicial object in Cat to the simplicial set that keeps only the set of objects of each category of simplices in each dimension. We refer to it as the *underlying simplicial set* of a simplicial object in Cat.

$$\begin{aligned} [\Delta^{\mathrm{op}}, \mathsf{Cat}] &\longrightarrow [\Delta^{\mathrm{op}}, \mathsf{Set}] \\ \mathbb{X} &\longmapsto \mathbb{X}^{(0)} \end{aligned}$$

The term *simplicial category* is commonly used in the literature to refer to a category enriched in simplicial sets, which in general is different to a simplicial object in Cat. In fact, categories enriched in simplicial sets are in bijection with those simplicial objects in Cat whose underlying simplicial set is a constant functor $\Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$.

Let $\Delta_{\leq n}$ be the standard n-simplex viewed as a category, the inclusion $\Delta_{\leq n} \longrightarrow \Delta$ induces the n-truncation functor

$$\mathsf{tr}_n : [\Delta^{\mathrm{op}}, \mathsf{Cat}] \longrightarrow [\Delta^{\mathrm{op}}_{\leq n}, \mathsf{Cat}]$$

which forgets about the m-simplices for all $m > n$ in a simplicial object in Cat. It has a left adjoint and a right adjoint which are called the *n-skeleton* and *n-coskeleton*, these may be calculated via right and left Kan extensions.

$$\mathsf{sk}_n \dashv \mathsf{tr}_n \dashv \mathsf{cosk}_n$$

One may also describe them inductively; for all $m \leq n$ the categories of m-simplices of both the n-skeleton and n-coskeleton are the equal to $\mathbb{X}_m$. And for $m > n$ the n-skeleton freely adds all degenerate m-simplices for all $(m-1)$-simplices in $\mathsf{sk}_n(\mathbb{X})_{(m-1)}$, and the n-coskeleton freely adds as its m-simplices unique fillers for all $(m-1)$-spheres in $\mathsf{cosk}_n(\mathbb{X})_{(m-1)}$; thus $(\mathsf{sk}_n \mathbb{X})_m \subset (\mathsf{cosk}_n \mathbb{X})_m$. We say that a simplicial object in Cat is *n-(co)skeletal* if it is isomorphic to its own n-(co)skeleton in $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$.

The 2-category of simplicial objects in Cat, apart from having the notions of equality, isomorphism, and equivalence, has a concept of weak equivalence.

Definition 3.2.1. Let be $\mathbb{X}$ and $\mathbb{Y}$ two simplicial objects in Cat. A simplicial morphism $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is a *weak equivalence* if for every n the component of F at n

$$\mathbb{X}_n \xrightarrow[\simeq]{F_n} \mathbb{Y}_n$$

is an equivalence of categories.

To have a weak equivalence $F : \mathbb{X} \longrightarrow \mathbb{Y}$ of simplicial objects in Cat is not the same as having an equivalence in the 2-category $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$, because although it is true that for every n we have pseudoinverses $G_n : \mathbb{Y}_n \longrightarrow \mathbb{X}_n$, in general these do not constitute a simplicial morphism, but only a pseudonatural transformation. So in other words, a simplicial morphism F is a weak equivalence if it is an equivalence when viewed in the bicategory $[\Delta^{\mathrm{op}}, \mathsf{Cat}]_{\mathrm{ps}}$ of 2-functors, pseudonatural transformations, and modifications.

Let nLax be the 2-category of bicategories, normal lax functors, and icons. The following definition may be found in [CCG10, Definition 5.2].

Definition 3.2.2. The *lax-2-nerve* of a bicategory $\mathcal{B}$ is the simplicial object in Cat defined by

$$\begin{aligned} \Delta^{\mathrm{op}} &\longrightarrow \mathsf{Cat} \\ [n] &\longmapsto \mathsf{nLax}([n], \mathcal{B}) \end{aligned}$$

The underlying simplicial set of the lax-2-nerve of a bicategory is the nerve of a bicategory (also called the *Street-nerve* or *1-nerve*), and it is 3-coskeletal. One may see that the lax-2-nerve is also 3-coskeletal by noticing that the category of 3-simplices is the full subcategory of the 3-coskeleton on the 3-simplices of the 1-nerve.

3.2.2 Simplices of Oplax Actions

In an attempt to give a more explicit description of the bicategory $\underline{\mathsf{OplaxAct}}(\mathcal{M})$ found in Corollary 3.1.3 without relying on the structure of $\mathsf{OpMon}(\mathcal{M})$, the first thing one tries is to define a composition of oplax actions, which is a functor

$$\mathsf{OplaxAct}(S;T) \times \mathsf{OplaxAct}(R;S) \xrightarrow{\quad ? \quad} \mathsf{OplaxAct}(R;T)$$

whose argument takes two oplax actions $SR \xrightarrow{s} S$ and $TS \xrightarrow{t} T$. Unfortunately these oplax actions do not seem to compose in any straightforward way. This leaves us unable to access a horizontal composition of oplax actions directly, which exists in the case where $\mathcal{M}$ satisfies the hypotheses of Theorem 3.1.1. To get around this problem, instead of a bicategory of oplax actions, we describe a simplicial object in Cat of oplax actions. We denote it by $\mathsf{OplaxAct}$ (without the underline), or $\mathsf{OplaxAct}(\mathcal{M})$ if one wishes to specify the ambient category. This simplicial object in Cat has two important properties, which solve our initial problem of relating Theorems 3.1.1 and 3.1.2:

(1) There is an appropriate notion of monad in a simplicial set —which for the 1-nerve of a bicategory $\mathcal{B}$ is precisely a monad in $\mathcal{B}$— and a monad in the underlying simplicial set of the simplicial object in Cat of oplax actions $\mathsf{OplaxAct}(\mathcal{M})^{(0)}$ is precisely a right skew monoidale whose unit has a right adjoint.

(2) It is weakly equivalent to the lax-2-nerve of a bicategory of opmonoidal arrows, assuming that $\mathcal{M}$ a right autonomous opmonadic-friendly monoidal bicategory with chosen right biduals R° and chosen left adjoints $i : I \longrightarrow R$ for every object R.

We know that point (1) is true in the case where $\mathcal{M}$ satisfies the hypothesis of Corollary 3.1.3, but point (1) is true in a far more general context: for every monoidal bicategory $\mathcal{M}$ with no extra assumptions whatsoever.

An entirely valid question at this point is: What is the benefit of using simplicial objects in Cat over mere simplicial sets? And while it is true that for point (1) one only needs simplicial sets and the 1-nerve of a bicategory, for point (2) this is not the case. The extra dimension is crucial to express the equivalences (and not mere isomorphisms) such as the one from Corollary 2.3.11, which is basically the case of 1-simplices. To avoid working with this extra categorical dimension on our simplicial objects, one might use a strict monoidal

2-category instead of a monoidal bicategory (or a Gray monoid) and with opmonadicity up-to-isomorphism instead of up-to-equivalence. But we do not intend to restrict in this way, as this will rule out our primary examples $\mathcal{M} = \mathsf{Mod}_k$, $\mathcal{M} = \mathsf{Span}^{\mathrm{co}}$ or more generally $\mathcal{M} = \mathsf{Mod}(\mathcal{V})$ for a symmetric monoidal category $\mathcal{V}$.

The rest of this section is dedicated to the definition of $\mathsf{OplaxAct}(\mathcal{M})$. It is a 3-coskeletal simplicial object in Cat, so it is enough to define the first four categories of simplices and their respective face and degeneracy functors.

$$\begin{tikzcd}
\mathsf{OplaxAct}_3 \arrow[r, shift left=6, "\partial_0"] \arrow[r, shift left=2, "\partial_1"] \arrow[r, shift right=2, "\partial_2"] \arrow[r, shift right=6, "\partial_3"] & \mathsf{OplaxAct}_2 \arrow[l, shift left=4, "s_0"] \arrow[l, "s_1"] \arrow[l, shift right=4, "s_2"] \arrow[r, shift left=4, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=4, "\partial_2"] & \mathsf{OplaxAct}_1 \arrow[l, shift left=2, "s_0"] \arrow[l, shift right=2, "s_1"] \arrow[r, shift left=3, "\partial_0"] \arrow[r, shift right=3, "\partial_1"] & \mathsf{OplaxAct}_0 \arrow[l, "s_0"]
\end{tikzcd}$$

0-simplices

Definition 3.2.3. The category of 0-simplices $\mathsf{OplaxAct}_0$ is the discrete category whose objects are pairs $(R, i \dashv i^*)$ of an object R and an adjunction $i \dashv i^*$ in $\mathcal{M}$.

$$\begin{tikzcd}
R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\
I \arrow[u, bend left, "i"]
\end{tikzcd}$$

1-simplices

Definition 3.2.4. The category of 1-simplices $\mathsf{OplaxAct}_1$ has as objects all oplax actions on the 0-simplices, and as arrows all cells of oplax actions. Explicitly, $\mathsf{OplaxAct}_1$ is the coproduct in Cat over the set of pairs of 0-simplices $((R, i \dashv i^*), (S, j \dashv j^*))$ of the categories of oplax actions $\mathsf{OplaxAct}(R; S)$.

$$\mathsf{OplaxAct}_1 := \coprod_{\substack{(R, i \dashv i^*) \\ (S, j \dashv j^*)}} \mathsf{OplaxAct}(R; S)$$

The picture of oplax actions with a dashed arrow $r : R \nrightarrow S$ was done on purpose to resemble the geometrical shape of a standard 1-simplex, this shows easily what the *face functors* are: $\partial_0(r) = (S, j \dashv j^*)$ and $\partial_1(r) = (R, i \dashv i^*)$. The *degeneracy functor* is defined for a 0-simplex $(R, i \dashv i^*)$ as the regular oplax R-action on R of the right skew monoidal structure on R induced by the adjunction $i \dashv i^*$, see Lemma 2.1.15 and Remark 2.3.2. In short, $s_0(R)$ is $i^*1 : RR \longrightarrow R$ with structure cells as shown below.

$$\begin{tikzcd}
RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, "i^*1"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
RR \arrow[dd, "i^*1"'] \arrow[dr, phantom, "\cong"] & & R \arrow[ll, "1i"'] \arrow[dl, "i^*"] \arrow[ddll, bend left=60, "1"] \\
& I \arrow[dl, "i"] \arrow[dr, phantom, "\Downarrow\varepsilon"] & \\
R & &
\end{tikzcd}$$

2-simplices

For pedagogical purposes relevant in Section 3.4 we first define categories of 2-simplices with fixed 0-faces.

Definition 3.2.5. Given three 0-simplices $(R, i \dashv i^*)$, $(S, j \dashv j^*)$, and $(T, k \dashv k^*)$, the category $\mathsf{OplaxAct}(R; S; T)$ has objects given by quadruples (s, t, v, α), where the first three entries are oplax actions $s : SR \longrightarrow S$, $t : TS \longrightarrow T$, and $v : TR \longrightarrow T$; and α is a square in $\mathcal{M}$ that we depict as a double arrow with a dash inside a triangle of dashed arrows,

$$
\begin{tikzcd}
R \arrow[rr, bend left, "v"] \arrow[dr, "s"'] & \alpha\Uparrow & T \\
& S \arrow[ur, "t"'] &
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
TSR \arrow[r, "t1"] \arrow[d, "1s"'] \arrow[dr, phantom, "\alpha\Nearrow"] & TR \arrow[d, "v"] \\
TS \arrow[r, "t"'] & T
\end{tikzcd}
$$

satisfying the following three axioms.

$$
\begin{tikzcd}
& TSR \arrow[dr, "t1"] & \\
TSSR \arrow[ur, "t11"] \arrow[dr, "1j^*11"] \arrow[dd, "11s"'] \arrow[rr, phantom, "t^21\Uparrow"] & & TR \arrow[dd, "v"] \\
& TSR \arrow[ur, "t1"'] \arrow[dd, "1s"] \arrow[dl, phantom, "\cong"] \arrow[dr, phantom, "\alpha\Nearrow"] & \\
TSS \arrow[dr, "1j^*1"'] & & T \\
& TS \arrow[ur, "t"'] &
\end{tikzcd}
=
\begin{tikzcd}
& TSR \arrow[dr, "t1"] \arrow[dd, "1s"] & \\
TSSR \arrow[ur, "t11"] \arrow[dd, "11s"'] \arrow[dr, phantom, "\cong"] & & TR \arrow[dd, "v"] \arrow[dl, phantom, "\alpha\Nearrow"] \\
& TS \arrow[dr, "t"] & \\
TSS \arrow[ur, "t1"] \arrow[dr, "1j^*1"'] \arrow[rr, phantom, "t^2\Uparrow"] & & T \\
& TS \arrow[ur, "t"'] &
\end{tikzcd}
\tag{2SIM1}
$$

$$
\begin{tikzcd}
& TRR \arrow[dr, "v1"] & \\
TSRR \arrow[ur, "t11"] \arrow[dr, "1s1"] \arrow[dd, "11i^*1"'] \arrow[rr, phantom, "\alpha1\Uparrow"] & & TR \arrow[dd, "v"] \\
& TSR \arrow[ur, "t1"'] \arrow[dd, "1s"] \arrow[dl, phantom, "1s^2\Nearrow"] \arrow[dr, phantom, "\alpha\Nearrow"] & \\
TSR \arrow[dr, "1s"'] & & T \\
& TS \arrow[ur, "t"'] &
\end{tikzcd}
=
\begin{tikzcd}
& TRR \arrow[dr, "v1"] \arrow[dd, "1i^*1"] & \\
TSRR \arrow[ur, "t11"] \arrow[dd, "11i^*1"'] \arrow[dr, phantom, "\cong"] & & TR \arrow[dd, "v"] \arrow[dl, phantom, "v^2\Nearrow"] \\
& TR \arrow[dr, "v"] & \\
TSR \arrow[ur, "t1"] \arrow[dr, "1s"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & T \\
& TS \arrow[ur, "t"'] &
\end{tikzcd}
\tag{2SIM2}
$$

$$
\begin{tikzcd}
& T \arrow[dddr, bend left, "1"] & \\
TS \arrow[ur, "t"] \arrow[dd, "11i"'] \arrow[dddr, bend left, "1"] & & \\
& & \\
TSR \arrow[dr, "1s"'] \arrow[r, phantom, "1s^0\Nearrow"] & & T \\
& TS \arrow[ur, "t"'] &
\end{tikzcd}
=
\begin{tikzcd}
& T \arrow[dd, "1i"] \arrow[dddr, bend left, "1"] & \\
TS \arrow[ur, "t"] \arrow[dd, "11i"'] \arrow[dr, phantom, "\cong"] & & \\
& TR \arrow[dr, "v"] \arrow[ur, phantom, "v^0\Nearrow"] & \\
TSR \arrow[ur, "t1"] \arrow[dr, "1s"'] \arrow[rr, phantom, "\alpha\Uparrow"] & & T \\
& TS \arrow[ur, "t"'] &
\end{tikzcd}
\tag{2SIM3}
$$

The arrows $(\sigma, \tau, \nu) : (s, t, v, \alpha) \longrightarrow (s', t', v', \alpha')$ of $\mathsf{OplaxAct}(R; S; T)$ are triples of oplax action cells

$$
\begin{tikzcd}
R \arrow[r, bend left, "s'"] \arrow[r, bend right, "s"'] \arrow[r, phantom, "\sigma\Uparrow"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, bend left, "t'"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\tau\Uparrow"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, bend left, "v'"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\nu\Uparrow"] & T
\end{tikzcd}
$$

satisfying the following axiom.

$$
\begin{tikzcd}
TSR \arrow[r, bend left, "t'1"] \arrow[d, bend right, "1s"'] \arrow[d, bend left, "1s'"] \arrow[d, phantom, "\overset{1\sigma}{\Rightarrow}"] \arrow[dr, phantom, "\alpha'\Nearrow"] & TR \arrow[d, bend left, "v'"] \\
TS \arrow[r, bend left, "t'"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\tau\Uparrow"] & T
\end{tikzcd}
=
\begin{tikzcd}
TSR \arrow[r, bend left, "t'1"] \arrow[r, bend right, "t1"'] \arrow[r, phantom, "\tau1\Uparrow"] \arrow[d, bend right, "1s"'] \arrow[dr, phantom, "\alpha\Nearrow"] & TR \arrow[d, bend right, "v"'] \arrow[d, bend left, "v'"] \arrow[d, phantom, "\overset{\nu}{\Rightarrow}"] \\
TS \arrow[r, bend right, "t"'] & T
\end{tikzcd}
\tag{2SIM4}
$$

Composition and identities are calculated as in the category below.

$$\mathsf{OplaxAct}(R;S) \times \mathsf{OplaxAct}(S;T) \times \mathsf{OplaxAct}(R;T)$$

Definition 3.2.6. The category of 2-simplices $\mathsf{OplaxAct}_2$ is the coproduct in Cat over the set of triples of 0-simplices $((R, i \dashv i^*), (S, j \dashv j^*), (T, k \dashv k^*))$ of the categories $\mathsf{OplaxAct}(R;S;T)$.

$$\mathsf{OplaxAct}_2 := \coprod_{\substack{(R,i\dashv i^*)\\(S,j\dashv j^*)\\(T,k\dashv k^*)}} \mathsf{OplaxAct}(R;S;T)$$

Again, the picture with dashes of a 2-simplex is set to resemble the geometrical shape of a standard 2-simplex, and remind us that the *face functors* are given by $\partial_0(\alpha) = t$, $\partial_1(\alpha) = v$ and $\partial_2(\alpha) = s$. For an oplax action $t : S \relbar\joinrel\mapstochar\joinrel\rightarrow T$ the *degeneracy functors* are defined as follows,

$$
\begin{tikzcd}
S \arrow[rr, bend left, "t"] \arrow[dr, "\mathfrak{s}_0(S)"'] \arrow[rr, phantom, "\mathfrak{s}_0(t)\Uparrow"] & & T \\
& S \arrow[ur, "t"'] &
\end{tikzcd}
=
\begin{tikzcd}
TSS \arrow[r, "t1"] \arrow[d, "1j^*1"'] \arrow[dr, phantom, "t^2\Nearrow"] & TS \arrow[d, "t"] \\
TS \arrow[r, "t"'] & T
\end{tikzcd}
$$

$$
\begin{tikzcd}
S \arrow[rr, bend left, "t"] \arrow[dr, "t"'] \arrow[rr, phantom, "\mathfrak{s}_1(t)\Uparrow"] & & T \\
& T \arrow[ur, "\mathfrak{s}_0(T)"'] &
\end{tikzcd}
=
\begin{tikzcd}
TTS \arrow[r, "k^*11"] \arrow[d, "1t"'] \arrow[dr, phantom, "\cong"] & TS \arrow[d, "t"] \\
TT \arrow[r, "k^*1"'] & T
\end{tikzcd}
$$

and the axioms for a 2-simplex are trivially satisfied. Similarly for a cell of oplax actions $\tau : t \Longrightarrow t'$ the degeneracies are $\mathfrak{s}_0(\tau) = (\mathrm{id}_{\mathfrak{s}_0(S)}, \tau, \tau)$ and $\mathfrak{s}_1(\tau) = (\tau, \mathrm{id}_{\mathfrak{s}_0(T)}, \tau)$.

3-simplices

In the same way as for 2-simplices, we first define categories of 3-simplices with fixed 0-faces.

Definition 3.2.7. Given four 0-simplices $(R, i \dashv i^*)$, $(S, j \dashv j^*)$, $(T, k \dashv k^*)$, and $(U, l \dashv l^*)$, define the category $\mathsf{OplaxAct}(R; S; T; U)$ as the full subcategory of

$$\mathsf{OplaxAct}(R; S; T) \times \mathsf{OplaxAct}(R; S; U) \times \mathsf{OplaxAct}(R; T; U) \times \mathsf{OplaxAct}(S; T; U)$$

consisting of the quadruples $\Gamma = (\alpha, \beta, \gamma, \zeta)$ of 2-simplices of oplax actions whose boundaries match appropriately to be arranged in a tetrahedral configuration,

$$\begin{tikzcd}
R \arrow[rr, bend left, "w"] \arrow[dr, "s"'] \arrow[drr, "v"] & \arrow[d, phantom, "\alpha\Uparrow"] & \arrow[d, phantom, "\gamma\Uparrow"] U \\
& S \arrow[r, "t"'] & T \arrow[u, "u"']
\end{tikzcd}
\overset{\Gamma}{\Rrightarrow}
\begin{tikzcd}
R \arrow[rr, bend left, "w"] \arrow[dr, "s"'] & \arrow[d, phantom, "\beta\Uparrow"] & U \\
& S \arrow[ur, bend left, "x"] \arrow[r, "t"'] & T \arrow[u, "u"'] \arrow[ul, phantom, "\zeta\Uparrow"]
\end{tikzcd}$$

and that satisfy the following "composing" condition.

$$\begin{tikzcd}
& & USR \arrow[drr, "x1"] \arrow[dd, phantom, "\zeta 1\Uparrow"] & & \\
UTSR \arrow[urr, "u11"] \arrow[dd, "11s"'] \arrow[drr, "1t1"] & & & & UR \arrow[dd, "w"] \\
& & UTR \arrow[urr, "u1"] \arrow[dd, "1v"] & & \\
UTS \arrow[drr, "1t"'] \arrow[r, phantom, "1\alpha"] & & & & U \arrow[l, phantom, "\gamma"] \\
& & UT \arrow[urr, "u"'] & &
\end{tikzcd}
=
\begin{tikzcd}
& & USR \arrow[drr, "x1"] \arrow[dd, "1s"] & & \\
UTSR \arrow[urr, "u11"] \arrow[dd, "11s"'] \arrow[r, phantom, "\cong"] & & & & UR \arrow[dd, "w"] \arrow[l, phantom, "\beta"] \\
& & US \arrow[drr, "x"] \arrow[dd, phantom, "\zeta\Uparrow"] & & \\
UTS \arrow[urr, "u1"] \arrow[drr, "1t"'] & & & & U \\
& & UT \arrow[urr, "u"'] & &
\end{tikzcd}
\tag{3SIM}$$

Definition 3.2.8. The category of 3-simplices $\mathsf{OplaxAct}_3$ is the coproduct in Cat over the set of quadruples of 0-simplices $((R, i \dashv i^*), (S, j \dashv j^*), (T, k \dashv k^*), (U, l \dashv l^*))$ of the categories $\mathsf{OplaxAct}(R; S; T; U)$.

$$\mathsf{OplaxAct}_3 := \coprod_{\substack{(R, i \dashv i^*) \\ (S, j \dashv j^*) \\ (T, k \dashv k^*) \\ (U, l \dashv l^*)}} \mathsf{OplaxAct}(R; S; T; U)$$

Remark 3.2.9. The condition "to match in a tetrahedral configuration" in the definition of the categories $\mathsf{OplaxAct}(R; S; T; U)$ means that $\mathsf{OplaxAct}_3$ is not only a full subcategory of quadruples of objects in $\mathsf{OplaxAct}_2$ but a full subcategory of the 3-coskeleton of the 2-truncated simplicial object in Cat of the simplices of oplax actions defined so far.

$$\begin{tikzcd}
\mathsf{OplaxAct}_2 \arrow[r, shift left=4, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=4, "\partial_2"] & \mathsf{OplaxAct}_1 \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "s_1"] \arrow[r, shift left=2, "\partial_0"] \arrow[r, shift right=2, "\partial_1"'] & \mathsf{OplaxAct}_0 \arrow[l, "s_0"]
\end{tikzcd}$$

Hence, a quadruple $(\alpha, \beta, \gamma, \zeta)$ of 2-simplices in this 2-coskeleton is in $\mathsf{OplaxAct}(\mathcal{M})$ if and only if it satisfies condition (3SIM), so in practice we say that a 3-simplex is a quadruple $(\alpha, \beta, \gamma, \zeta)$ satisfying the property (3SIM).

Once more, for a 3-simplex Γ as above, the tetrahedral picture with dashes is set up to resemble a standard 3-simplex and to remind us of the fact that the *face functors* are defined on objects as $\partial_0(\Gamma) = \alpha$, $\partial_1(\Gamma) = \beta$, $\partial_2(\Gamma) = \gamma$ and $\partial_3(\Gamma) = \zeta$, and in a similar way on arrows. For a 2-simplex (s, t, v, α) the *degeneracy functors* are defined on objects as: $\mathscr{s}_0(\alpha)$ is axiom (2SIM2) for α,

$$\begin{tikzcd} R \arrow[rrr, bend left, "v"] \arrow[rr, bend left=15, "s"] \arrow[dr, "\mathscr{s}_0(R)"'] \arrow[dr, phantom, shift left=3, "\mathscr{s}_0(s)\Uparrow"] & & \arrow[r, phantom, "\alpha\Uparrow"] & T \\ & R \arrow[r, "s"'] & S \arrow[ur, "t"'] & \end{tikzcd} \overset{\mathscr{s}_0(\alpha)}{\Rrightarrow} \begin{tikzcd} R \arrow[rrr, bend left, "v"] \arrow[dr, "\mathscr{s}_0(R)"'] & \arrow[r, phantom, "\mathscr{s}_0(v)\Uparrow"] & & T \\ & R \arrow[r, "s"'] \arrow[urr, bend left, "v"] \arrow[ur, phantom, shift left=3, "\alpha\Uparrow"] & S \arrow[ur, "t"'] & \end{tikzcd}$$

$\mathscr{s}_1(\alpha)$ is axiom (2SIM1) for α,

$$\begin{tikzcd} R \arrow[rrr, bend left, "v"] \arrow[rr, bend left=15, "s"] \arrow[dr, "s"'] \arrow[dr, phantom, shift left=3, "\mathscr{s}_1(s)\Uparrow"] & & \arrow[r, phantom, "\alpha\Uparrow"] & T \\ & S \arrow[r, "\mathscr{s}_0(s)"'] & S \arrow[ur, "t"'] & \end{tikzcd} \overset{\mathscr{s}_1(\alpha)}{\Rrightarrow} \begin{tikzcd} R \arrow[rrr, bend left, "v"] \arrow[dr, "s"'] & \arrow[r, phantom, "\alpha\Uparrow"] & & T \\ & S \arrow[r, "\mathscr{s}_0(S)"'] \arrow[urr, bend left, "t"] \arrow[ur, phantom, shift left=3, "\mathscr{s}_0(t)\Uparrow"] & S \arrow[ur, "t"'] & \end{tikzcd}$$

$\mathscr{s}_2(\alpha)$ is an instance of the coherence law for the interchanger isomorphisms.

$$\begin{tikzcd} R \arrow[rrr, bend left, "v"] \arrow[rr, bend left=15, "v"] \arrow[dr, "s"'] \arrow[dr, phantom, shift left=3, "\alpha\Uparrow"] & & \arrow[r, phantom, "\mathscr{s}_1(v)\Uparrow"] & T \\ & S \arrow[r, "t"'] & T \arrow[ur, "\mathscr{s}_0(T)"'] & \end{tikzcd} \overset{\mathscr{s}_2(\alpha)}{\Rrightarrow} \begin{tikzcd} R \arrow[rrr, bend left, "v"] \arrow[dr, "s"'] & \arrow[r, phantom, "\alpha\Uparrow"] & & T \\ & S \arrow[r, "t"'] \arrow[urr, bend left, "t"] \arrow[ur, phantom, shift left=3, "\mathscr{s}_1(t)\Uparrow"] & T \arrow[ur, "\mathscr{s}_0(T)"'] & \end{tikzcd}$$

This finishes the definition of $\mathsf{OplaxAct}(\mathcal{M})$, now we may address points (1) and (2) made at the beginning of this section. This is done in the Sections 3.3 and 3.4.

3.3 Monads of Oplax Actions and Skew Monoidales

In [BGLS14], [Buc16], [Gre15], and [BL17] the authors classify various monoidal-like notions as simplicial morphisms out of a particular simplicial set $\mathbb{C}$, whose name is *the Catalan simplicial set*. One might think of the Catalan simplicial set as the "free living monoidal-like structure" in the sense that one decides which kind of monoidal-like structure to get by choosing different kinds of nerves. For example, simplicial morphisms from $\mathbb{C}$ into the *1-nerve* of a bicategory are in bijection with monads in the bicategory. By using other kinds of nerves as the target of a simplicial morphism one may get monoids in a monoidal category, or monoidal categories in Cat, or skew monoidales in a monoidal bicategory, and so on. The Catalan simplicial set owes its name is to the fact that the number of n-simplices is the nth-Catalan number.

Definition 3.3.1. The *Catalan simplicial set* $\mathbb{C}$ is the simplicial set that has a unique 0-simplex $\star$, two 1-simplices; one degenerate $s_0(\star) = e : \star \longrightarrow \star$, and a unique non-degenerate $c : \star \longrightarrow \star$; and the rest is built by coskeletality.

The 1-nerve of a bicategory is 3-coskeletal, and so it is reasonable to say:

Definition 3.3.2. A *monad* in a 3-coskeletal simplicial set $\mathbb{X}$ is a simplicial morphism as shown.

$$\mathbb{C} \longrightarrow \mathbb{X}$$

3.3.1 Monads of oplax actions

In the previous section we built a simplicial object in Cat using oplax actions in $\mathcal{M}$. But in this section we are interested in its underlying simplicial set $\mathsf{OplaxAct}(\mathcal{M})^{(0)}$ obtained by taking the set of objects in each dimension. Particularly, we are interested in monads in $\mathsf{OplaxAct}(\mathcal{M})^{(0)}$ because as we shall see in Theorem 3.3.9 these are right skew monoidales in $\mathcal{M}$ whose unit has a right adjoint.

Definition 3.3.3. A *monad of oplax actions* $(R, i \dashv i^*, r, \mu_0, \mu_2)$ in a monoidal bicategory $\mathcal{M}$ is a monad in the simplicial set $\mathsf{OplaxAct}(\mathcal{M})^{(0)}$. More explicitly, it consists of the following items.

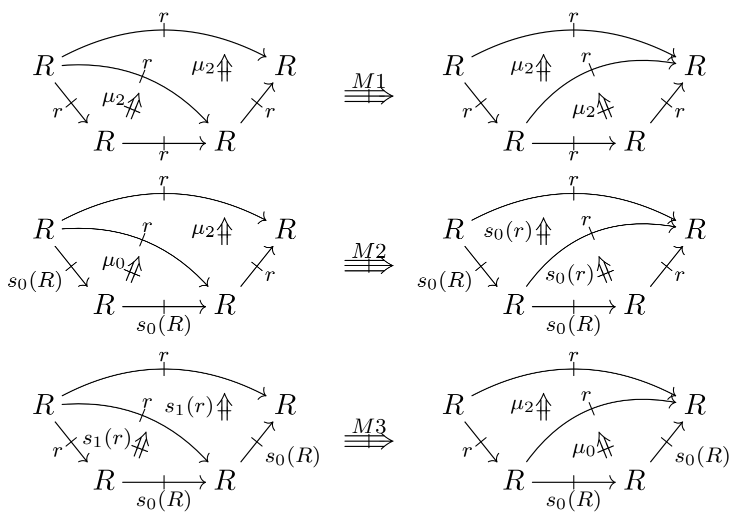
(i). One 0-simplex $(R, i \dashv i^*)$

(ii). One 1-simplex $r : R \nrightarrow R$

(iii). Two 2-simplices

$$\begin{tikzcd} R \arrow[rr, bend left, "r"] \arrow[dr, "i^*1"'] & \mu_0\Uparrow & R \\ & R \arrow[ur, "i^*1"'] & \end{tikzcd} \qquad \begin{tikzcd} R \arrow[rr, bend left, "r"] \arrow[dr, "r"'] & \mu_2\Uparrow & R \\ & R \arrow[ur, "r"'] & \end{tikzcd}$$

(iv). Three 3-simplices



But if we fully unpack each item and enumerate with the same order, a monad of oplax actions amounts to the items below.

(i). An object R, together with the right skew monoidale structure induced by the adjunction $i \dashv i^*$ as in Lemma 2.1.15.

(ii). An oplax right R-action on R with respect to the right skew monoidal structure in (i); that is, an arrow $r : RR \longrightarrow R$ with structure cells r^2 and r^0 that satisfy axioms (OLA1), (OLA2), and (OLA3).

$$
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "r^2 \Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
RR \arrow[d, "r"'] & S \arrow[l, "1i"'] \arrow[dl, "1"] \arrow[dl, phantom, near start, "\Swarrow r^0"'] \\
R &
\end{tikzcd}
$$

(iii). Two quadrangular cells μ_2 and μ_0 each satisfying instances of the three axioms (2SIM1), (2SIM2), and (2SIM3).

$$
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1r"'] \arrow[dr, phantom, "\mu_2 \Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
\qquad
\begin{tikzcd}
RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "\mu_0 \Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
$$

(iv). Three instances of the axiom (3SIM) as follows,

$$
\begin{tikzcd}[column sep=small]
 & RRR \arrow[dr, "r1"] \arrow[d, phantom, "\mu_2 1 \Uparrow"] & \\
RRRR \arrow[ur, "r11"] \arrow[dr, "1r1"] \arrow[dd, "11r"'] & {} & RR \arrow[dd, "r"] \\
 & RRR \arrow[ur, "r1"] \arrow[dd, "1r"] \arrow[dl, phantom, "1\mu_2 \Nearrow"] \arrow[dr, phantom, "\mu_2 \Nearrow"] & \\
RRR \arrow[dr, "1r"'] & & R \\
 & RR \arrow[ur, "r"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & RRR \arrow[dr, "r1"] \arrow[dd, "1r"] & \\
RRRR \arrow[ur, "r11"] \arrow[dd, "11r"'] \arrow[r, phantom, "\cong"] & {} & RR \arrow[dd, "r"] \arrow[dl, phantom, "\mu_2 \Nearrow"] \\
 & RR \arrow[dr, "r"] \arrow[d, phantom, "\mu_2 \Uparrow"] & \\
RRR \arrow[ur, "r1"] \arrow[dr, "1r"'] & {} & R \\
 & RR \arrow[ur, "r"'] &
\end{tikzcd}
\tag{M1}
$$

$$
\begin{tikzcd}[column sep=small]
 & RRR \arrow[dr, "r1"] \arrow[d, phantom, "r^2 1 \Uparrow"] & \\
RRRR \arrow[ur, "r11"] \arrow[dr, "1i^*11"] \arrow[dd, "11i^*1"'] & {} & RR \arrow[dd, "r"] \\
 & RRR \arrow[ur, "r1"] \arrow[dd, "1r"] \arrow[dl, phantom, "1\mu_0 \Nearrow"] \arrow[dr, phantom, "\mu_2 \Nearrow"] & \\
RRR \arrow[dr, "1i^*1"'] & & R \\
 & RR \arrow[ur, "r"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & RRR \arrow[dr, "r1"] \arrow[dd, "1i^*1"] & \\
RRRR \arrow[ur, "r11"] \arrow[dd, "11i^*1"'] \arrow[r, phantom, "\cong"] & {} & RR \arrow[dd, "r"] \arrow[dl, phantom, "r^2 \Nearrow"] \\
 & RR \arrow[dr, "r"] \arrow[d, phantom, "r^2 \Uparrow"] & \\
RRR \arrow[ur, "r1"] \arrow[dr, "1i^*1"'] & {} & R \\
 & RR \arrow[ur, "r"'] &
\end{tikzcd}
\tag{M2}
$$

$$
\begin{tikzcd}[column sep=small]
 & & RRR \arrow[drr, "r1"] & & \\
RRRR \arrow[urr, "i^*111"] \arrow[dr, "1i^*11"'] \arrow[dd, "11r"'] & & \arrow[u, phantom, "\mu_0 1 \Uparrow"] & & RR \arrow[dd, "r"] \\
 & \arrow[d, phantom, "\cong"] & RRR \arrow[urr, "i^*11"'] \arrow[dd, "1r"] & \arrow[d, phantom, "\cong"] & \\
RRR \arrow[drr, "1i^*1"'] & & & & R \\
 & & RR \arrow[urr, "i^*1"'] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & RRR \arrow[drr, "r1"] \arrow[dd, "1r"] & & \\
RRRR \arrow[urr, "i^*111"] \arrow[dd, "11r"'] & \arrow[d, phantom, "\cong"] & & \arrow[d, phantom, "\mu_2"] & RR \arrow[dd, "r"] \\
 & & RR \arrow[drr, "r"] & & \\
RRR \arrow[urr, "i^*11"] \arrow[drr, "1i^*1"'] & & \arrow[u, phantom, "\mu_0 \Uparrow"] & & R \\
 & & RR \arrow[urr, "i^*1"'] & &
\end{tikzcd}
\tag{M3}
$$

This gives a total of twelve axioms.

As one can see, a monad of oplax actions amounts to a lot of information. Fortunately, some of it is redundant. We shall see that the cell r^2 may be written in terms of the rest of the structure and that all but five of the axioms are not actually needed. In fact, a monad of oplax actions is a right skew monoidale whose unit has a right adjoint, and this does not require any extra assumptions on the monoidal bicategory $\mathcal{M}$.

Lemma 3.3.4. *For every monad of oplax actions $(R, i \dashv i^*, r, \mu_0, \mu_2)$ in a monoidal bicategory $\mathcal{M}$ the following equality holds.*

$$
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "r^2"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
=
\begin{tikzcd}
RRR \arrow[r, "1i^*1"] \arrow[rr, bend left=50, "1"] \arrow[dr, "11i1"'] \arrow[ddr, bend right=50, "1"'] & RR \arrow[r, "1i1"] \arrow[u, phantom, "1\varepsilon 1 \Uparrow"] \arrow[d, phantom, "\cong"] & RRR \arrow[r, "r1"] \arrow[d, "1r"] \arrow[dr, phantom, "\mu_2"] & RR \arrow[d, "r"] \\
 \arrow[r, phantom, "11\eta 1"] & RRRR \arrow[ur, "1i^*11"'] \arrow[d, "11i^*1"] \arrow[r, phantom, "1\mu_0"] & RR \arrow[r, "r"'] & R \\
 & RRR \arrow[ur, "1i^*1"'] & &
\end{tikzcd}
$$

Proof.
Starting with the right hand side, one uses the following calculation.

$$
\begin{tikzcd}[column sep=small]
 & RR \arrow[rr, bend left, "1"] & & \\
RRR \arrow[ur, "r1"] \arrow[r, "1i^*1"] \arrow[drr, bend left=40, "1"] \arrow[dr, "11i1"'] \arrow[ddd, bend right=50, "1"'] & RR \arrow[dr, "1i1"] \arrow[u, phantom, "1\varepsilon 1"] \arrow[d, phantom, "\cong"] & & RR \arrow[dd, "r"] \\
 \arrow[r, phantom, "11\eta 1"] & RRRR \arrow[r, "1i^*11"] \arrow[dd, "11i^*1"'] \arrow[d, phantom, "1\mu_0"] & RRR \arrow[ur, "r1"] \arrow[ddd, "1r"] \arrow[r, phantom, "\mu_2"] & \\
 & & & R \\
 & RRR \arrow[dr, "1i^*1"'] & & \\
 & & RR \arrow[uur, "r"'] &
\end{tikzcd}
\overset{\text{(OLA2)}}{=}
\begin{tikzcd}[column sep=small]
 & RR \arrow[dr, "1i1"] \arrow[rr, bend left, "1"] \arrow[d, phantom, "\cong"] & \arrow[u, phantom, "r^0 1"] & \\
RRR \arrow[ur, "r1"] \arrow[dr, "11i1"'] \arrow[ddd, bend right=50, "1"'] & & RRR \arrow[r, "r1"] \arrow[d, phantom, "r^2 1 \Uparrow"] & RR \arrow[dd, "r"] \\
 \arrow[r, phantom, "11\eta 1"] & RRRR \arrow[ur, "r11"] \arrow[r, "1i^*11"] \arrow[dd, "11i^*1"'] \arrow[d, phantom, "1\mu_0"] & RRR \arrow[ur, "r1"] \arrow[ddd, "1r"] \arrow[r, phantom, "\mu_2"] & \\
 & & & R \\
 & RRR \arrow[dr, "1i^*1"'] & & \\
 & & RR \arrow[uur, "r"'] &
\end{tikzcd}
$$

$$
\overset{(M2)}{=}
\begin{tikzcd}
 & RR \arrow[dr, "1i1"'] \arrow[ddrr, bend left, "1"] \arrow[dr, phantom, shift left=3, "r^0 1 \Nearrow"] & & \\
RRR \arrow[ur, "r1"] \arrow[dr, "11i1"] \arrow[dddr, bend right, "1"'] \arrow[r, phantom, "\cong"] \arrow[ddr, phantom, "11\eta 1 \Nearrow"] & & RRR \arrow[dr, "r1"] \arrow[dd, "1i^*1"] & \\
 & RRRR \arrow[ur, "r11"] \arrow[dd, "11i^*1"'] \arrow[dr, phantom, "\cong"] & & RR \arrow[dd, "r"] \arrow[dl, phantom, "r^2 \Nearrow"] \\
 & & RR \arrow[dr, "r"] & \\
 & RRR \arrow[ur, "r1"] \arrow[dr, "1i^*1"'] \arrow[rr, phantom, "r^2 \Uparrow"] & & R \\
 & & RR \arrow[ur, "r"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & RR \arrow[dr, "1i1"'] \arrow[ddrr, bend left, "1"] \arrow[ddr, bend right, "1"'] \arrow[dr, phantom, shift left=3, "r^0 1 \Nearrow"] \arrow[ddr, phantom, "1\eta 1 \Nearrow"] & & \\
RRR \arrow[ur, "r1"] \arrow[dddr, bend right, "1"'] & & RRR \arrow[dr, "r1"] \arrow[dd, "1i^*1"] & \\
 & & & RR \arrow[dd, "r"] \arrow[dl, phantom, "r^2 \Nearrow"] \\
 & & RR \arrow[dr, "r"] & \\
 & RRR \arrow[ur] \arrow[dr, "1i^*1"'] \arrow[rr, phantom, "r^2 \Uparrow"] & & R \\
 & & RR \arrow[ur, "r"'] &
\end{tikzcd}
$$

$$
\overset{(OLA3)}{=}
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "r^2 \Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
$$

■

Note that for a monad of oplax actions $(R, i \dashv i^*, r, \mu_0, \mu_2)$ axiom (2SIM1) for μ_0 is precisely the axiom (2SIM1μ_0) from Lemma 3.3.7. Hence, we get a pair of mates λ and κ each of which amounts to the same information as the cell μ_0 satisfying (2SIM1). This simplifies the formula from Lemma 3.3.4.

$$
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "r^2 \Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
:=
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, bend right, "1i^*1"'] \arrow[d, bend left, "1r"] \arrow[d, phantom, "\overset{1\kappa}{\Rightarrow}"] \arrow[dr, phantom, "\mu_2 \Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
\qquad (3.3.1)
$$

With this information we describe a right skew monoidale induced by a monad of oplax actions.

Proposition 3.3.5. *For every monad of oplax actions $(R, i \dashv i^*, r, \mu_0, \mu_2)$ in a monoidal bicategory $\mathcal{M}$ there is a right skew monoidale on the object R with structure given as follows,*

Product $RR \xrightarrow{r} R$ *Unit* $I \xrightarrow{i} R$

Associator
$$
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1r"'] \arrow[dr, phantom, "\mu_2 \Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
$$

Unitors
$$
\begin{tikzcd}
R \arrow[r, "i1"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\lambda \Nearrow"] & RR \arrow[d, "r"] & R \arrow[l, "1i"'] \arrow[dl, "1"] \arrow[dl, phantom, shift right=3, "\Downarrow r^0"] \\
 & R &
\end{tikzcd}
$$

where the left unitor λ is obtained from μ_0 as in Lemma 3.3.7.

Proof.

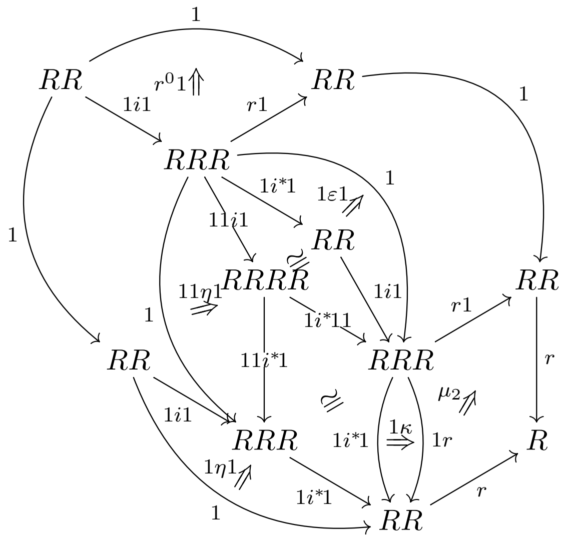
Let λ and κ as in Lemma 3.3.7, then we have the following formulas.

$$
\mu_0 = \begin{tikzcd}
RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, bend right=40, "i^*1"'] \arrow[d, bend left=40, "r"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
\qquad
\kappa = \begin{tikzcd}
RR \arrow[r, "i^*1"] \arrow[rr, bend left=40, "1"] \arrow[rr, phantom, bend left=20, "\varepsilon 1\Uparrow"] \arrow[dr, "1i1"'] \arrow[dddr, bend right=40, "1"'] \arrow[ddr, phantom, "1\eta 1 \Rightarrow"] & R \arrow[r, "i1"] \arrow[d, phantom, "\wr\|"] & RR \arrow[dd, "r"] \\
& RRR \arrow[ur, "i^*11"'] \arrow[dd, "1i^*1"] \arrow[r, phantom, "\mu_0 \Rightarrow"] & \\
& & R \\
& RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
$$

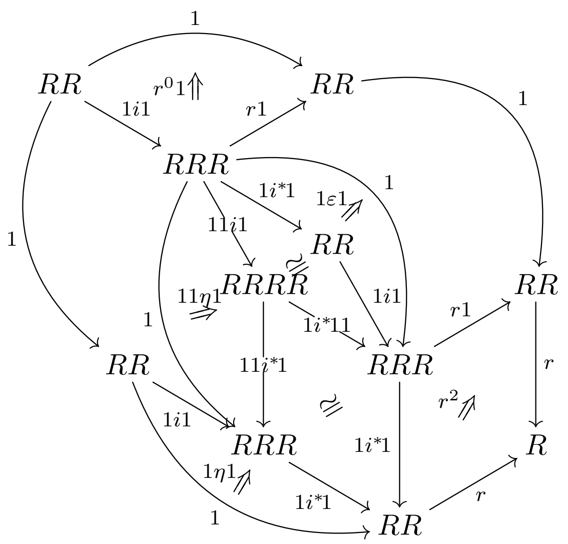
Now, axiom (SKM1) is literally axiom (M1) for the monad of oplax actions and axiom (SKM4) is literally axiom (2SIM3) for μ_2. Since the proposed unit for the right skew monoidale has a right adjoint, for the rest of the axioms one may prove their alternative versions written in terms of κ as in Remark 3.3.6. The following calculation proves axiom (SKM2'), where the first equality substitutes the definition of κ above, the second replaces the instance of μ_0 in terms of κ, and the third is an instance of the equation (3.3.1) mentioned before the proposition.

$$
\begin{tikzcd}
RR \arrow[rr, bend left=40, "1"] \arrow[dr, "1i1"] \arrow[ddr, bend right=50, "1"'] \arrow[ddr, phantom, "1\eta 1\Rightarrow"] & r^0 1 \Uparrow & RR \arrow[dd, "r"] \\
& RRR \arrow[ur, "r1"'] \arrow[d, bend right=30, "1i^*1"'] \arrow[d, bend left=30, "1r"] \arrow[d, phantom, "\overset{1\kappa}{\Rightarrow}"] \arrow[dr, phantom, "\mu_2 \Rightarrow"] & \\
& RR \arrow[r, "r"'] & R
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
RR \arrow[rrrr, bend left=30, "1"] \arrow[drr, "1i1"] \arrow[ddddr, bend right=30, "1"'] & & & r^0 1\Uparrow & RR \arrow[dddrr, bend left=30, "1"] & & \\
& & RRR \arrow[urr, "r1"'] \arrow[drr, "1i^*1"] \arrow[ddr, "11i1"'] \arrow[dddrrr, bend left=30, "1"] \arrow[ddddr, bend right=30, "1"'] & & 1\varepsilon 1\Rightarrow & & \\
& & & & RR \arrow[ddr, "1i1"] \arrow[dl, phantom, "\cong"] & & \\
& & 11\eta 1\Rightarrow & RRRR \arrow[drr, "1i^*11"'] \arrow[dd, "11i^*1"] & & & RR \arrow[dd, "r"] \\
& RR \arrow[drr, "1i1"] \arrow[ddrrrr, bend right=30, "1"'] & & 1\mu_0\Rightarrow & & RRR \arrow[ur, "r1"] \arrow[dd, "1r"] & \mu_2\Rightarrow \\
& & 1\eta 1\Rightarrow & RRR \arrow[drr, "1i^*1"] & & & R \\
& & & & & RR \arrow[ur, "r"'] &
\end{tikzcd}
$$

=


RR
RR
RRR
RR
RRRR
RR
RR
RRR
RRR
R
RR
1
1
1
1
1
1
1i1
r1
r^0 1
1i*1
1ε1
11i1
11η1
1i1
r1
r
1i*11
11i*1
μ_2
1i1
1i*1
1κ
1r
1η1
1i*1
r

$\overset{(3.3.1)}{=}$


RR
RR
RRR
RR
RRRR
RR
RR
RRR
RRR
R
RR
1
1
1
1
1
1
1i1
r1
r^0 1
1i*1
1ε1
11i1
11η1
1i1
r1
r
1i*11
11i*1
r^2
1i1
1i*1
1η1
1i*1
r

$$
= \quad
\begin{tikzcd}
RR \arrow[rr, bend left, "1"] \arrow[dr, "1i1"'] \arrow[dddd, bend right, "1"'] \arrow[r, phantom, "r^0 1 \Uparrow"] & & RR \arrow[dddrr, bend left, "1"] & & \\
& RRR \arrow[ur, "r1"'] \arrow[dr, "1i^*1"'] \arrow[dddrr, bend left, "1"] \arrow[drr, phantom, "1\varepsilon 1 \Rrightarrow"] & & & \\
& & RR \arrow[ddr, "1i1"] \arrow[ddddr, bend right, "1"'] \arrow[ddr, phantom, "1\eta 1 \Rrightarrow"'] & & \\
& & & & RR \arrow[dd, "r"] \\
RR \arrow[dr, "1i1"'] \arrow[ddrrr, bend right, "1"'] & & & RRR \arrow[ur, "r1"] \arrow[dd, "1i^*1"'] \arrow[r, phantom, "r^2 \Rrightarrow"] & \\
& RRR \arrow[drr, "1i^*1"'] \arrow[d, phantom, "1\eta 1 \Rrightarrow"] & & & R \\
& & & RR \arrow[ur, "r"'] &
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
RR \arrow[rr, bend left, "1"] \arrow[dr, "1i1"'] \arrow[dddr, bend right, "1"'] \arrow[r, phantom, "r^0 1 \Uparrow"] & & RR \arrow[dd, "r"] \\
& RRR \arrow[ur, "r1"] \arrow[dd, "1i^*1"'] \arrow[l, phantom, "1\eta 1 \Rrightarrow"'] \arrow[r, phantom, "r^2 \Rrightarrow"] & \\
& & R \\
& RR \arrow[ur, "r"'] &
\end{tikzcd}
\overset{\text{(OLA3)}}{=}
\begin{tikzcd}
RR \arrow[r, bend left, "r"] \arrow[r, bend right, "r"'] & R
\end{tikzcd}
$$

Axiom (SKM3') follows from axiom (M3) for the monad of oplax actions.

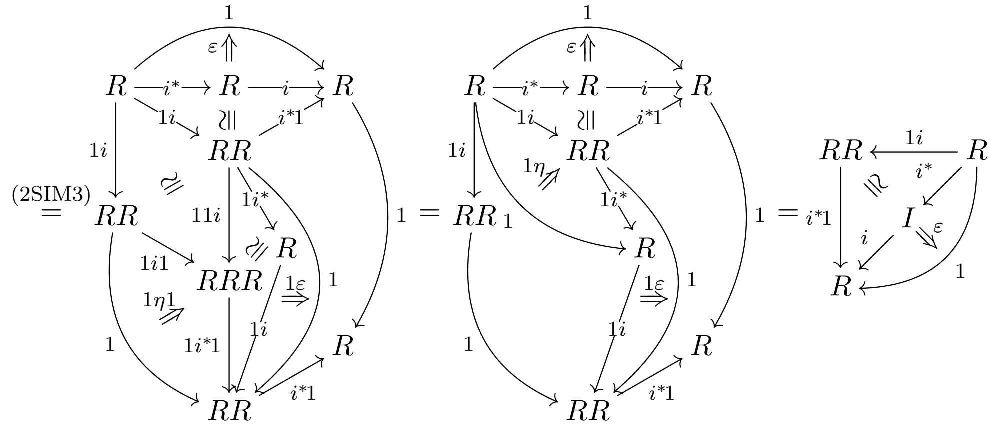
$$
\begin{tikzcd}
RRR \arrow[r, bend left, "r1"] \arrow[r, bend right, "i^*11"'] \arrow[r, phantom, "\kappa 1 \Uparrow"] \arrow[d, "1r"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, "r"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
=
\begin{tikzcd}
& & RRR \arrow[ddr, "r1"] & \\
& RR \arrow[ur, "i11"] \arrow[ur, phantom, "\varepsilon 11 \Uparrow"'] & & \\
RRR \arrow[uurr, bend left, "1"] \arrow[ur, "i^*11"] \arrow[r, "1i11"'] \arrow[drr, bend right, "1"'] \arrow[dd, "1r"'] & RRRR \arrow[uur, "i^*111"] \arrow[dr, "1i^*11"] \arrow[d, phantom, "1\eta 11 \Uparrow"] \arrow[r, phantom, "\mu_0 1 \Uparrow"] & & RR \arrow[dd, "r"] \\
& & RRR \arrow[ur, "i^*11"] \arrow[dd, "1r"] \arrow[dr, phantom, "\cong"] & \\
RR \arrow[drr, bend right, "1"'] & & & R \\
& & RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
=
$$

$$
\begin{tikzcd}
& & RRR \arrow[ddr, "r1"] & \\
& RR \arrow[ur, "i11"] & & \\
RRR \arrow[uurr, bend left, "1"] \arrow[ur, "i^*11"] \arrow[r, "1i11"'] \arrow[dd, "1r"'] & RRRR \arrow[uur, "i^*111"'] \arrow[dr, "1i^*11"] \arrow[dd, "11r"] \arrow[ul, phantom, "\varepsilon 11 \Uparrow"] \arrow[ur, phantom, "\mu_0 1 \Uparrow"] & & RR \arrow[dd, "r"] \\
& & RRR \arrow[ur, "i^*11"'] \arrow[dd, "1r"] \arrow[dl, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & \\
RR \arrow[r, "1i1"'] \arrow[drr, bend right, "1"'] \arrow[ur, phantom, "\cong"] & RRR \arrow[dr, "1i^*1"] \arrow[d, phantom, "1\eta 1 \Uparrow"] & & R \\
& {} & RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
\overset{(M3)}{=}
\begin{tikzcd}
& & RRR \arrow[ddr, "r1"] \arrow[dd, "1r"'] & \\
& RR \arrow[ur, "i11"] & & \\
RRR \arrow[uurr, bend left, "1"] \arrow[ur, "i^*11"] \arrow[r, "1i11"'] \arrow[dd, "1r"'] & RRRR \arrow[uur, "i^*111"'] \arrow[dd, "11r"] \arrow[ul, phantom, "\varepsilon 11 \Uparrow"] \arrow[dr, phantom, "\cong"] & & RR \arrow[dd, "r"] \arrow[dl, phantom, "\mu_2 \Nearrow"] \\
& & RR \arrow[dr, "r"] & \\
RR \arrow[r, "1i1"'] \arrow[drr, bend right, "1"'] \arrow[ur, phantom, "\cong"] & RRR \arrow[ur, "i^*11"] \arrow[dr, "1i^*1"] \arrow[d, phantom, "1\eta 1 \Uparrow"] & \arrow[u, phantom, "\mu_0 \Uparrow"] & R \\
& {} & RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
& & RRR \arrow[dr, "r1"] \arrow[dd, "1r"'] & \\
RRR \arrow[urr, bend left, "1"] \arrow[ddd, "1r"'] & & & RR \arrow[ddd, "r"] \arrow[dl, phantom, "\mu_2 \Nearrow"] \\
& & RR \arrow[ddr, "r"] & \\
& R \arrow[ur, "i1"] & & \\
RR \arrow[uurr, bend left, "1"] \arrow[ur, "i^*1"] \arrow[r, "1i1"'] \arrow[drr, bend right, "1"'] \arrow[u, phantom, "\varepsilon 1 \Uparrow"] & RRR \arrow[uur, "i^*11"'] \arrow[dr, "1i^*1"] \arrow[d, phantom, "1\eta 1 \Uparrow"] & \arrow[u, phantom, "\mu_0 \Uparrow"] & R \\
& {} & RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
=
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1r"'] \arrow[dr, phantom, "\mu_2 \Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, bend left, "r"] \arrow[r, bend right, "i^*1"'] \arrow[r, phantom, "\kappa \Uparrow"] & R
\end{tikzcd}
$$

And axiom (SKM5') follows from axiom (2SIM3) for μ_0.

$$
\begin{tikzcd}
RR \arrow[d, bend right, "i^*1"'] \arrow[d, bend left, "r"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] & R \arrow[l, "1i"'] \arrow[dl, bend left, "1"] \arrow[dl, phantom, "r^0 \Swarrow"] \\
R &
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[rr, bend left, "1"] \arrow[d, "1i"'] & & R \arrow[d, "1i"] \arrow[ddd, bend left=60, "1"] \\
RR \arrow[r, "i^*1"] \arrow[rr, bend left, "1"] \arrow[dr, "1i1"'] \arrow[ddr, bend right, "1"'] & R \arrow[r, "i1"] \arrow[u, phantom, "\varepsilon 1 \Uparrow"] \arrow[d, phantom, "\wr\Vert"] & RR \arrow[dd, "r"] \arrow[dd, phantom, bend left=30, "\overset{r^0}{\Rightarrow}"] \\
& RRR \arrow[ur, "i^*11"'] \arrow[d, "1i^*1"] \arrow[dl, phantom, "1\eta 1 \Nearrow"] \arrow[dr, phantom, "\mu_0 \Nearrow"] & \\
& RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
=
\begin{tikzcd}
R \arrow[r, "i^*"] \arrow[rr, bend left, "1"] \arrow[dr, "1i"'] \arrow[dd, "1i"'] & R \arrow[r, "i"] \arrow[u, phantom, "\varepsilon \Uparrow"] \arrow[d, phantom, "\wr\Vert"] & R \arrow[dd, "1i"] \arrow[dddd, bend left=60, "1"] \\
& RR \arrow[ur, "i^*1"'] \arrow[dd, "11i"] \arrow[dl, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & \\
RR \arrow[dr, "1i1"'] \arrow[ddr, bend right, "1"'] & & RR \arrow[dd, "r"] \arrow[dd, phantom, bend left=30, "\overset{r^0}{\Rightarrow}"] \\
& RRR \arrow[ur, "i^*11"'] \arrow[d, "1i^*1"] \arrow[dl, phantom, "1\eta 1 \Nearrow"] \arrow[dr, phantom, "\mu_0 \Nearrow"] & \\
& RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
$$



■

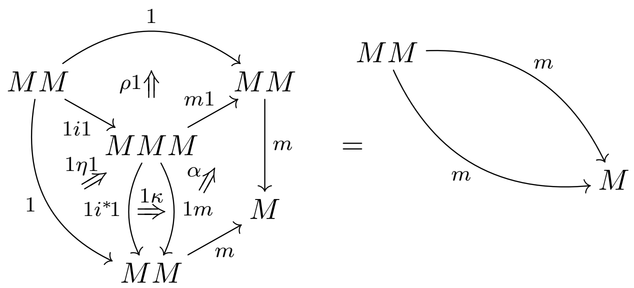
3.3.2 Skew monoidales whose unit has a right adjoint

We shall now investigate skew monoidales whose unit has a right adjoint. An example of such right skew monoidales is given by Lemma 2.1.15. It says that adjunctions $i \dashv i^*$ in $\mathcal{M}$

$$\begin{tikzcd} R \arrow[d, bend left, "i^*"] \\ I \arrow[u, bend left, "i"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

induce right skew monoidal structures on R, with product arrow $i^*1 : RR \longrightarrow R$ and unit i. We also know that if $\mathcal{M}$ satisfies the hypothesis of Corollary 3.1.3 all monads of oplax actions are skew monoidales whose unit has a right adjoint. But, as we mentioned above, this holds true for an arbitrary monoidal bicategory $\mathcal{M}$.

Remark 3.3.6. A right skew monoidale $(M, i, m, \alpha, \lambda, \rho)$ whose the unit has a right adjoint $i \dashv i^*$ may be expressed in simpler terms using the mate κ of the cell λ under the adjunction. This involves the data $(M, i, m, \alpha, \kappa, \rho)$ satisfying five axioms; two of which remain unchanged since they do not involve the left unitor cell λ: the pentagon (SKM1) and the α-ρ compatibility (SKM4). But the remaining three have modified versions as follows.



(SKM2')

$$
\begin{tikzcd}
RRR \arrow[r, bend left, "m1"] \arrow[r, bend right, "i^*11"'] \arrow[r, phantom, "\kappa 1 \Uparrow"] \arrow[d, "1m"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, "m"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
=
\begin{tikzcd}
RRR \arrow[r, "m1"] \arrow[d, "1m"'] \arrow[dr, phantom, "\alpha \Nearrow"] & RR \arrow[d, "m"] \\
RR \arrow[r, bend left, "m"] \arrow[r, bend right, "i^*1"'] \arrow[r, phantom, "\kappa \Uparrow"] & R
\end{tikzcd}
\tag{SKM3'}
$$

$$
\begin{tikzcd}
RR \arrow[d, bend right, "i^*1"'] \arrow[d, bend left, "m"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] & R \arrow[l, "1i"'] \arrow[dl, bend left, "1"] \arrow[dl, phantom, "\Searrow \rho"] \\
R &
\end{tikzcd}
=
\begin{tikzcd}
RR \arrow[dd, "i^*1"'] \arrow[dr, phantom, "\cong"] & & R \arrow[ll, "1i"'] \arrow[dl, "i^*"'] \arrow[ddll, bend left, "1"] \\
& I \arrow[dl, "i"] \arrow[dr, phantom, "\Searrow \varepsilon"] & \\
R & &
\end{tikzcd}
\tag{SKM5'}
$$

The following is a technical lemma involving a pair of mates λ and κ as above.

Lemma 3.3.7. *For every object R, every arrow $m : RR \longrightarrow R$, and every adjunction $i \dashv i^*$ in $\mathcal{M}$*

$$
\begin{tikzcd}
R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\
I \arrow[u, bend left, "i"]
\end{tikzcd}
$$

there are bijections between cells μ_0 that satisfy the equation (2SIM1μ_0) *below, cells κ, and cells λ with sources and targets as shown.*

$$
\left\{
\begin{tikzcd}
RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "\mu_0 \Nearrow"] & RR \arrow[d, "m"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
\;\middle|\; (2\mathrm{SIM}1\mu_0)
\right\}
\cong
\left\{
\begin{tikzcd}
RR \arrow[d, bend right, "i^*1"'] \arrow[d, bend left, "m"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] \\
R
\end{tikzcd}
\right\}
\cong
\left\{
\begin{tikzcd}
R \arrow[r, "i1"] \arrow[dr, "1"'] & RR \arrow[d, "m"] \arrow[dl, phantom, "\lambda \Nearrow"] \\
& R
\end{tikzcd}
\right\}
$$

$$
\begin{tikzcd}[column sep=small]
& RRR \arrow[dr, "i^*11"] \arrow[d, phantom, "\cong"] & \\
RRRR \arrow[ur, "i^*111"] \arrow[r, "1i^*11"'] \arrow[d, "11i^*1"'] \arrow[dr, phantom, "\cong"] & RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"] \arrow[dr, phantom, "\mu_0 \Nearrow"] & RR \arrow[d, "m"] \\
RRR \arrow[r, "1i^*1"'] & RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& RRR \arrow[dr, "i^*11"] \arrow[d, "1i^*1"] \arrow[dr, phantom, "\mu_0 \Nearrow"] & \\
RRRR \arrow[ur, "i^*111"] \arrow[d, "11i^*1"'] \arrow[dr, phantom, "\cong"] & RR \arrow[dr, "i^*1"] & RR \arrow[d, "m"] \\
RRR \arrow[ur, "i^*11"] \arrow[r, "1i^*1"'] \arrow[ur, phantom, "\cong"] & RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
\tag{2SIM1μ_0}
$$

Proof.

The bijection between cells λ and cells κ is done by the calculus of mates of the adjunction $i1 \dashv i^*1$. The interesting part is to prove the bijection between cells κ and cells μ_0 satisfying

(2SIM1μ_0), for which one defines the functions below.

$$
\begin{tikzcd}
RR \arrow[d, bend right, "i^*1"'] \arrow[d, bend left, "m"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] \\
R
\end{tikzcd}
\longmapsto
\widetilde{\kappa} :=
\begin{tikzcd}
RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, bend right, "i^*1"'] \arrow[d, bend left, "m"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
$$

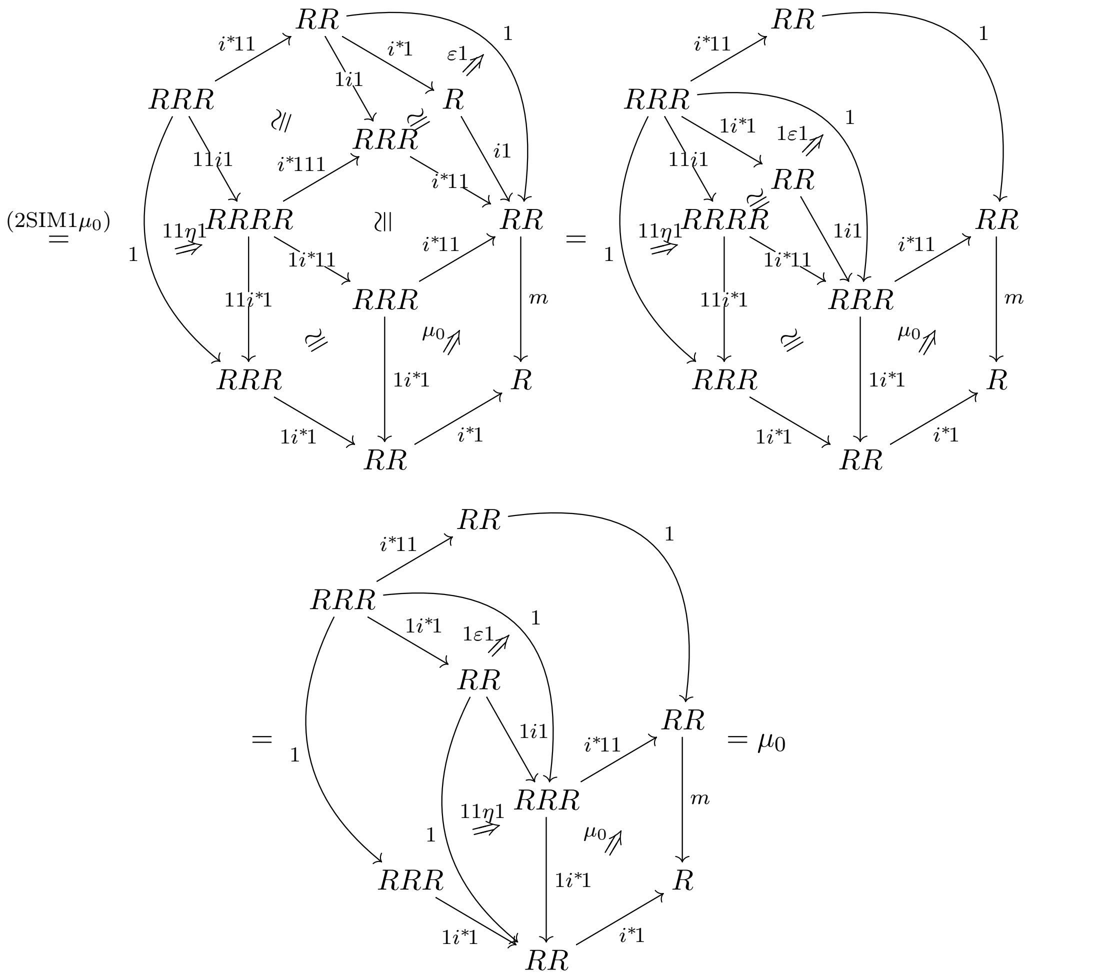
$$
\overline{\mu}_0 :=
\begin{tikzcd}
RR \arrow[r, "i^*1"] \arrow[rr, bend left=50, "1"] \arrow[dr, "1i1"'] \arrow[ddr, bend right=50, "1"'] & R \arrow[r, "i1"] \arrow[u, phantom, "\varepsilon 1 \Uparrow"] \arrow[d, phantom, "\cong"] & RR \arrow[d, "m"] \\
& RRR \arrow[ur, "i^*11"'] \arrow[d, "1i^*1"] \arrow[r, phantom, "\mu_0 \Nearrow"] \arrow[l, phantom, "1\eta 1 \Nearrow"] & R \\
& RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
\longleftarrow\!\!\shortmid
\begin{tikzcd}
RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "\mu_0 \Nearrow"] & RR \arrow[d, "m"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
$$

It is easy to see that given a cell κ the cell $\widetilde{\kappa}$ satisfies axiom (2SIM1μ_0) as a consequence of the coherence of the interchange law. Now, the following calculations show that these functions are inverse to each other. First, let κ as in the statement and argue as follows,

$$
\overline{\widetilde{\kappa}} =
\begin{tikzcd}
RR \arrow[r, "i^*1"] \arrow[rr, bend left=50, "1"] \arrow[dr, "1i1"'] \arrow[ddr, bend right=50, "1"'] & R \arrow[r, "i1"] \arrow[u, phantom, "\varepsilon 1 \Uparrow"] \arrow[d, phantom, "\cong"] & RR \arrow[d, bend right, "i^*1"'] \arrow[d, bend left, "m"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] \\
& RRR \arrow[ur, "i^*11"'] \arrow[d, "1i^*1"] \arrow[r, phantom, "\cong"] \arrow[l, phantom, "1\eta 1 \Nearrow"] & R \\
& RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
=
\begin{tikzcd}
RR \arrow[r, "i^*1"] \arrow[rr, bend left=50, "1"] \arrow[ddr, bend right=50, "1"'] & R \arrow[r, "i1"] \arrow[u, phantom, "\varepsilon 1 \Uparrow"] \arrow[dr, bend right, "1"'] \arrow[d, phantom, "1\eta 1 \Nearrow"] & RR \arrow[d, bend right, "i^*1"'] \arrow[d, bend left, "m"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] \\
& & R \\
& RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
= \kappa
$$

where the first equality holds by the definition of $\overline{\widetilde{\kappa}}$, the second by Gray monoid axioms, and the third by the snake equation of the adjunction. Now let μ_0 be a cell satisfying axiom (2SIM1μ_0), then the other composite is the identity.

$$
\widetilde{\overline{\mu}}_0 =
\begin{tikzcd}
& & RR \arrow[dr, "i^*1"] \arrow[dd, "1i1"'] \arrow[dddrr, bend left=60, "1"] \arrow[dddd, bend right=50, "1"'] & & \\
RRR \arrow[urr, "i^*11"] \arrow[ddddr, bend right=50, "1"'] & & & R \arrow[ddr, "i1"] \arrow[dl, phantom, "\cong"] \arrow[r, phantom, "\varepsilon 1 \Nearrow"] & \\
& & RRR \arrow[drr, "i^*11"] \arrow[dd, "1i^*1"] \arrow[l, phantom, "1\eta 1 \Nearrow"] & & \\
& & & & RR \arrow[dd, "m"] \\
& & RR \arrow[drr, "i^*1"] \arrow[ur, phantom, "\mu_0 \Nearrow"] & & \\
& RRR \arrow[ur, "i^*11"] \arrow[dr, "1i^*1"'] & \cong & & R \\
& & RR \arrow[urr, "1i^*1"'] & &
\end{tikzcd}
=
\begin{tikzcd}
& & RR \arrow[drr, "i^*1"] \arrow[ddr, "1i1"'] \arrow[dddrrr, bend left=60, "1"] & & & \\
RRR \arrow[urr, "i^*11"] \arrow[ddr, "11i1"'] \arrow[ddddr, bend right=50, "1"'] & & \cong & & R \arrow[ddr, "i1"] \arrow[dl, phantom, "\cong"] \arrow[r, phantom, "\varepsilon 1 \Nearrow"] & \\
& & & RRR \arrow[drr, "i^*11"] \arrow[dd, "1i^*1"] & & \\
& RRRR \arrow[urr, "i^*111"] \arrow[dd, "11i^*1"] \arrow[l, phantom, "11\eta 1 \Nearrow"] & \cong & & \mu_0 \Nearrow & RR \arrow[dd, "m"] \\
& & & RR \arrow[drr, "i^*1"] & & \\
& RRR \arrow[urr, "i^*11"] \arrow[drr, "1i^*1"'] & & \cong & & R \\
& & & RR \arrow[urr, "i^*1"'] & &
\end{tikzcd}
$$



■

Now, analogous to how a monad in a bicategory has an underlying arrow, we show that a right skew monoidale whose unit has a right adjoint has an "underlying oplax action". More precisely, there is a monad of oplax actions that carries the same information as the right skew monoidale, and the "underlying oplax action" is the 1-simplex part of such a monad of oplax actions. The existence of such monad of oplax actions is shown in Theorem 3.3.9 below. First, we construct the "underlying oplax action" of a right skew monoidale. To do so, we look at what we know about the oplax actions which are the 1-simplex part of a monad of oplax actions; formula (3.3.1) tells us that the associator r^2 is written in terms of μ_2 and a cell κ induced by μ_0. Now, note that the associator α of a right skew monoidale on an object R is of the same type as the cell μ_2 for a monad of oplax actions on a 0-simplex $(R, i \dashv i^*)$. Thus, one may use formula (3.3.1) as a guide to define the associator r^2 for the "underlying oplax action" of a right skew monoidale by replacing the instance of μ_2 for α and using the mate κ of λ.

Proposition 3.3.8. *Every right skew monoidale $(R, i, m, \alpha, \lambda, \rho)$ for which the unit has a right adjoint $i \dashv i^*$ induces an oplax right R-action (R, r) with respect to the right skew monoidal* *structure on R induced by $i \dashv i^*$ as in Lemma 2.1.15. The new oplax action structure is given by $r := m$, $r^0 := \rho$, and*

$$
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "r^2 \Rrightarrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, bend right, "1i^*1"'] \arrow[d, bend left, "1r"] \arrow[d, phantom, "\overset{1\kappa}{\Rightarrow}"] \arrow[dr, phantom, "\alpha \Rrightarrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
$$

where κ is the mate of λ under the adjunction $i \dashv i^*$.

Proof.

Using the alternative versions of the axioms for the skew monoidale $(R, i, m, \alpha, \kappa, \rho)$ as described in Remark 3.3.6, axiom (OLA1) for (R, r) follows by using axiom (SKM3').

$$
\begin{tikzcd}
& RRR \arrow[dr, "r1"] & \\
RRRR \arrow[ur, "r11"] \arrow[dr, bend left, "1r1"] \arrow[dr, bend right, "1i^*11"'] \arrow[dr, phantom, "1\kappa1 \Rrightarrow"] \arrow[dd, "11i^*1"'] & \alpha1 \Uparrow & RR \arrow[dd, "r"] \\
& RRR \arrow[ur, "r1"'] \arrow[dd, bend right, "1i^*1"'] \arrow[dd, bend left, "1r"] \arrow[dd, phantom, "\overset{1\kappa}{\Rightarrow}"] & \\
RRR \arrow[dr, "1i^*1"'] \arrow[r, phantom, "\cong"] & & R \arrow[ul, phantom, "\alpha \Rrightarrow"] \\
& RR \arrow[ur, "r"'] &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
& RRR \arrow[dr, "r1"] & \\
RRRR \arrow[ur, "r11"] \arrow[dr, "1r1"] \arrow[dr, bend right, "1i^*11"'] \arrow[dr, phantom, "1\kappa1 \Rrightarrow"] \arrow[dd, bend right, "11i^*1"'] \arrow[dd, bend left, "11r"] \arrow[dd, phantom, "\overset{11\kappa}{\Rightarrow}"] & \alpha1 \Uparrow & RR \arrow[dd, "r"] \\
\cong & RRR \arrow[ur, "r1"'] \arrow[dd, "1r"] & \alpha \Rrightarrow \\
RRR \arrow[dr, "1i^*1"'] & & R \\
& RR \arrow[ur, "r"'] &
\end{tikzcd}
\overset{\text{(SKM3)}}{=}
\begin{tikzcd}
& RRR \arrow[dr, "r1"] & \\
RRRR \arrow[ur, "r11"] \arrow[dr, "1r1"] \arrow[dd, bend right, "11i^*1"'] \arrow[dd, bend left, "11r"] \arrow[dd, phantom, "\overset{11\kappa}{\Rightarrow}"] & \alpha1 \Uparrow & RR \arrow[dd, "r"] \\
1\alpha \Rrightarrow & RRR \arrow[ur, "r1"'] \arrow[dd, "1r"] & \alpha \Rrightarrow \\
RRR \arrow[dr, bend left, "1r"] \arrow[dr, bend right, "1i^*1"'] \arrow[dr, phantom, "1\kappa \Rrightarrow"] & & R \\
& RR \arrow[ur, "r"'] &
\end{tikzcd}
$$

$$
\overset{\text{(SKM1)}}{=}
\begin{tikzcd}
& RRR \arrow[dr, "r1"] \arrow[dd, "1r"] & \\
RRRR \arrow[ur, "r11"] \arrow[dd, bend right, "11i^*1"'] \arrow[dd, bend left, "11r"] \arrow[dd, phantom, "\overset{11\kappa}{\Rightarrow}"] & \cong \quad \alpha \Rrightarrow & RR \arrow[dd, "r"] \\
& RR \arrow[dr, "r"] & \\
RRR \arrow[ur, "r1"] \arrow[dr, bend left, "1r"] \arrow[dr, bend right, "1i^*1"'] \arrow[dr, phantom, "1\kappa \Rrightarrow"] & \alpha \Uparrow & R \\
& RR \arrow[ur, "r"'] &
\end{tikzcd}
=
\begin{tikzcd}
& RRR \arrow[dr, "r1"] \arrow[dd, bend right, "1i^*1"'] \arrow[dd, bend left, "1r"] \arrow[dd, phantom, "\overset{1\kappa}{\Rightarrow}"] & \\
RRRR \arrow[ur, "r11"] \arrow[dd, "11i^*1"'] & \cong \quad \alpha \Rrightarrow & RR \arrow[dd, "r"] \\
& RR \arrow[dr, "r"] & \\
RRR \arrow[ur, "r1"] \arrow[dr, bend left, "1r"] \arrow[dr, bend right, "1i^*1"'] \arrow[dr, phantom, "1\kappa \Rrightarrow"] & \alpha \Uparrow & R \\
& RR \arrow[ur, "r"'] &
\end{tikzcd}
$$

To prove axiom (OLA2) one makes use of axiom (SKM5').

$$
\begin{tikzcd}[column sep=small]
 & R \arrow[dddr, bend left, "1"] & \\
RR \arrow[ur, "r"] \arrow[dd, "11i"'] \arrow[dr, "1i^*"] \arrow[dddr, bend left, "1"] & & \\
 & R \arrow[dd, "1i"] \arrow[ur, phantom, "1\varepsilon\Nearrow"] & \\
RRR \arrow[dr, "1i^*1"'] \arrow[ur, phantom, "\cong"] & & R \\
 & RR \arrow[ur, "r"'] &
\end{tikzcd}
\overset{\text{(SKM5')}}{=}
\begin{tikzcd}[column sep=small]
 & R \arrow[ddr, bend left, "1"] & \\
RR \arrow[ur, "r"] \arrow[d, "11i"'] \arrow[dr, bend left, "1"] \arrow[d, phantom, shift left=4ex, "1\rho\Nearrow"] & & \\
RRR \arrow[r, "1r"] \arrow[r, bend right=50, "1i^*1"'] \arrow[r, phantom, shift right=1ex, "1\kappa\Nearrow"] & RR \arrow[r, "r"'] & R
\end{tikzcd}
\overset{\text{(SKM4)}}{=}
\begin{tikzcd}[column sep=small]
 & R \arrow[d, "1i"] \arrow[ddr, bend left, "1"] \arrow[dr, phantom, "\rho\Nearrow"] & \\
RR \arrow[ur, "r"] \arrow[d, "11i"'] \arrow[r, phantom, "\cong"] & RR \arrow[dr, "r"] & \\
RRR \arrow[ur, "r1"] \arrow[dr, bend left, "1r"] \arrow[dr, bend right=50, "1i^*1"'] \arrow[r, phantom, "1r\alpha\Uparrow"] \arrow[dr, phantom, "1\kappa\Nearrow"] & & R \\
 & RR \arrow[ur, "r"'] &
\end{tikzcd}
$$

And axiom (OLA3) is precisely (SKM2'). ■

It is worth pointing out that in the presence of a right skew monoidale on an object R whose unit has a right adjoint $i \dashv i^*$ there are automatically three oplax actions defined on R.

- The regular oplax right action, see Remark 2.3.2.
- The oplax action induced by the adjunction $i \dashv i^*$.
- The "underlying oplax action" from the previous proposition.

Now we are ready to prove the main theorem of this section.

Theorem 3.3.9. *In a monoidal bicategory $\mathcal{M}$, monads of oplax actions are in bijection with right skew monoidales whose unit has a right adjoint.*

Proof.

In Proposition 3.3.5 we built a right skew monoidale out of a monad of oplax actions. Conversely, for a right skew monoidale $(R, i, m, \alpha, \lambda, \rho)$ whose unit has a right adjoint i^*, and where κ is the mate of λ under the adjunction; the following items (numbered as in the definition) constitute a monad of oplax actions.

(i). Take the underlying object R and the unit i with its right adjoint $(R, i \dashv i^*)$.

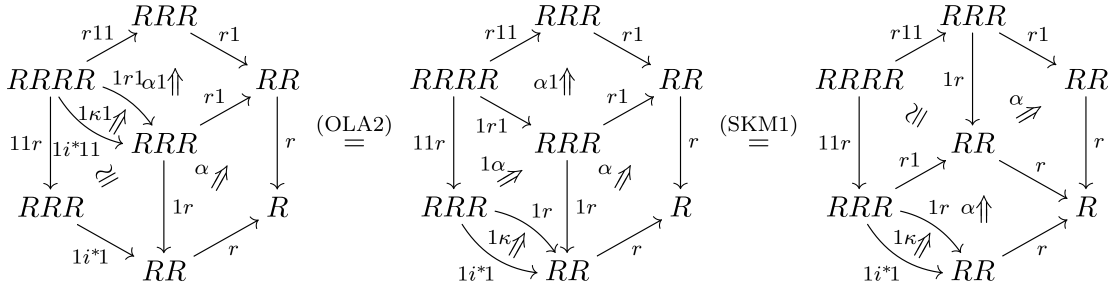
(ii). Take the "underlying oplax action" (r, r^0, r^2) of $(R, i, m, \alpha, \lambda, \rho)$ as constructed in Proposition 3.3.8. It is given by $r := m$, $r^0 := \rho$, and associator r^2 as shown.

$$
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "r^2\Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
RRR \arrow[r, "r1"] \arrow[d, bend right, "1i^*1"'] \arrow[d, bend left, "1r"] \arrow[d, phantom, "\overset{1\kappa}{\Rightarrow}"] \arrow[dr, phantom, "\alpha\Nearrow"] & RR \arrow[d, "r"] \\
RR \arrow[r, "r"'] & R
\end{tikzcd}
$$

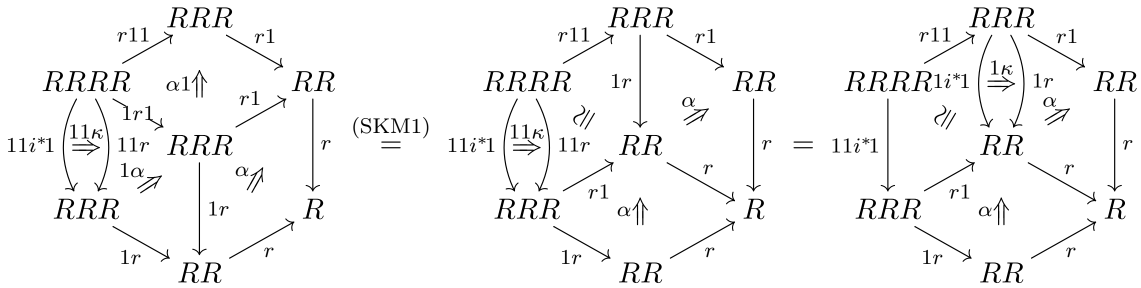
(iii). Take μ_2 to be the associator cell α and μ_0 the cell corresponding to the left unitor as in Lemma 3.3.7.

$$
\mu_2 := \alpha \qquad\qquad \mu_0 :=
\begin{tikzcd}
RRR \arrow[r, "i^*11"] \arrow[d, "1i^*1"'] \arrow[dr, phantom, "\cong"] & RR \arrow[d, bend right, "i^*1"'] \arrow[d, bend left, "r"] \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] \\
RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
$$

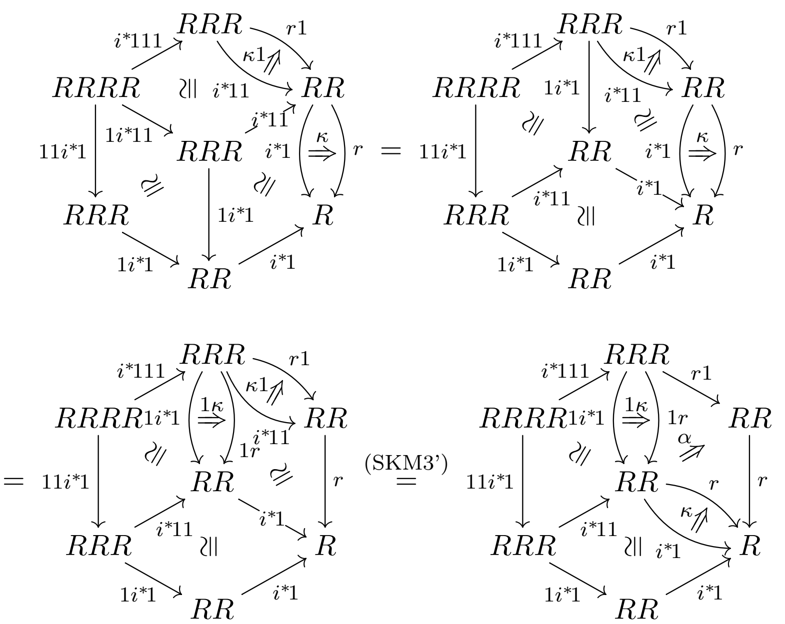
Now, one needs to prove that the cells μ_0 and μ_2 satisfy axioms (2SIM1), (2SIM2), and (2SIM3). The calculation below verifies axiom (2SIM1) for μ_2,



axiom (2SIM2) for μ_2 is a consequence of the following calculation,



and axiom (2SIM3) for μ_2 is literally axiom (SKM4) for the skew monoidale. Now, axiom (2SIM1) for μ_0 follows from the interchange law as we mentioned in Lemma 3.3.7 above, axiom (2SIM2) for μ_0 is verified as follows,



and axiom (2SIM3) for μ_0 holds true as one can see below.

$$
\begin{tikzcd}[column sep=small]
& R \arrow[ddd, bend left=60, "1"] & \\
RR \arrow[ur, "i^*1"] \arrow[d, "11i"'] \arrow[dr, "1i^*"] \arrow[ddr, bend left, "1"] & & \\
{} \arrow[r, phantom, "\cong"] & R \arrow[d, "1i"] \arrow[u, phantom, "1\varepsilon\Nearrow"] & \\
RRR \arrow[r, "1i^*1"'] & RR \arrow[r, "i^*1"'] & R
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& R \arrow[dd, "1i"'] \arrow[dr, "i^*"] \arrow[ddrr, bend left, "1"] & & \\
RR \arrow[ur, "i^*1"] \arrow[ddd, "11i"'] \arrow[r, phantom, "\cong"] & & R \arrow[dr, "i"] \arrow[ur, phantom, "\varepsilon\Nearrow"] \arrow[l, phantom, "\cong"] & \\
& RR \arrow[rr, "i^*1"] & & R \\
& {} \arrow[r, phantom, "\cong"] & & \\
RRR \arrow[uur, "i^*11"] \arrow[r, "1i^*1"'] & RR \arrow[uurr, "i^*1"'] & &
\end{tikzcd}
\overset{\text{(SKM5')}}{=}
\begin{tikzcd}[column sep=small]
& R \arrow[dd, "1i"] \arrow[ddr, bend left=60, "1"] & \\
RR \arrow[ur, "i^*1"] \arrow[ddd, "11i"'] \arrow[r, phantom, "\cong"] & & {} \arrow[dl, phantom, "\rho\Nearrow"] \\
& RR \arrow[r, bend left, "r"] \arrow[r, bend right, "i^*1"'] \arrow[r, phantom, "\kappa\Uparrow"] & R \\
& & \\
RRR \arrow[uur, "i^*11"] \arrow[r, "1i^*1"'] & RR \arrow[uur, "i^*1"'] &
\end{tikzcd}
$$

(iv). This item requires the existence of three 3-simplices, which amounts to verifying the three axioms (M1), (M2), and (M3) for the data defined previously. Axiom (M1) is nothing but the pentagon axiom (SKM1) for α, axiom (M2) happens to be the same as (OLA1) for r as verified in Proposition 3.3.8, and the following calculation proves axiom (M3).

$$
\begin{tikzcd}[column sep=small]
& RRR \arrow[dr, bend left, "r1"] \arrow[dr, bend right, "i^*11"'] \arrow[dr, phantom, "\kappa1\Uparrow"] & \\
RRRR \arrow[ur, "i^*111"] \arrow[dr, "1i^*11"] \arrow[dd, "11r"'] \arrow[r, phantom, "\cong"] & & RR \arrow[dd, "r"] \\
& RRR \arrow[ur, "i^*11"'] \arrow[dd, "1r"] \arrow[l, phantom, "\cong"] \arrow[r, phantom, "\cong"] & \\
RRR \arrow[dr, "1i^*1"'] & & R \\
& RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& RRR \arrow[dr, bend left, "r1"] \arrow[dr, bend right, "i^*11"'] \arrow[dr, phantom, "\kappa1\Uparrow"] \arrow[dd, "1r"'] & \\
RRRR \arrow[ur, "i^*111"] \arrow[dd, "11r"'] \arrow[r, phantom, "\cong"] & & RR \arrow[dd, "r"] \arrow[dl, phantom, "\cong"] \\
& RR \arrow[dr, "i^*1"] & \\
RRR \arrow[ur, "i^*11"] \arrow[dr, "1i^*1"'] & {} \arrow[phantom, "\cong"] & R \\
& RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
\overset{\text{(SKM3')}}{=}
\begin{tikzcd}[column sep=small]
& RRR \arrow[dr, "r1"] \arrow[dd, "1r"'] & \\
RRRR \arrow[ur, "i^*111"] \arrow[dd, "11r"'] \arrow[r, phantom, "\cong"] & & RR \arrow[dd, "r"] \arrow[dl, phantom, "\alpha\Nearrow"] \\
& RR \arrow[dr, bend left, "r"] \arrow[dr, bend right, "i^*1"'] \arrow[dr, phantom, "\kappa\Uparrow"] & \\
RRR \arrow[ur, "i^*11"] \arrow[dr, "1i^*1"'] & {} \arrow[phantom, "\cong"] & R \\
& RR \arrow[ur, "i^*1"'] &
\end{tikzcd}
$$

This completes the description of the monad of oplax actions induced by a right skew monoidale. The right skew monoidale induced by this monad of oplax actions as in Proposition 3.3.5 is the same as the original right skew monoidale; all the structure is literally the same, except maybe for the left unitor λ which by the bijection in Lemma 3.3.7 is verified to be the same as the induced one. Conversely, the monad of oplax actions built from a right skew monoidale (as done in this proof) which is induced by a monad of oplax actions using Proposition 3.3.5 is also the same monad as the original one: again, most of the structure is literally equal, except maybe for the associator of the 1-simplex oplax action r^2, which by Lemma 3.3.4 is verified to be the same as the induced one, and the cell μ_0 which by Proposition 3.3.5 is also verified to be the same as the induced one. ∎

3.4 Oplax Actions and Opmonoidal Arrows

In this section, we will address the point (2) that we made at the beginning of Subection 3.2.2, which states that under mild conditions on $\mathcal{M}$, a simplicial object of oplax actions is weakly equivalent in $[\Delta^{\mathrm{op}}, \mathsf{Cat}]$ to the lax-2-nerve of a bicategory of opmonoidal arrows. For a moment, allow us to be vague about the particular simplicial objects in question and look at Corollary 2.3.11; this result provides an equivalence of categories of the following form:

$$\mathsf{OpMon}(R^\circ R, S^\circ S) \simeq \mathsf{OplaxAct}(R; S).$$

Observe that on the left hand side we have a hom-category and on the right hand side we have a category of 1-simplices of oplax actions. Our goal is to extend this equivalence to a weak equivalence of simplicial objects in Cat in such a way that the weak equivalence at the level of 1-simplices is the equivalence of the aforementioned corollary. Thus, the rest of this section will consist of proving similar equivalences for the remaining categories of simplices. To achieve this, we impose the following conditions on $\mathcal{M}$:

(a) For every object R in $\mathcal{M}$ there is a chosen right bidual $R \dashv R^\circ$.

(b) For every object R in $\mathcal{M}$ there is a chosen adjunction $i \dashv i^*$ as shown.

$$\begin{tikzcd} R \arrow[d, bend left, "i^*"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i"] \end{tikzcd}$$

(c) For every object R the opposite of its chosen adjunction is opmonadic.

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}$$

(d) Tensoring with objects in $\mathcal{M}$ preserves opmonadicity.

(e) Composing with arrows in $\mathcal{M}$ preserves coequalisers of reflexive pairs in the hom categories of $\mathcal{M}$.

Using conditions (a) and (b) we can be explicit about which simplicial objects in Cat we are considering; it is not $\mathsf{OplaxAct}(\mathcal{M})$ nor the lax-2-nerve of $\mathsf{OpMon}(\mathcal{M})$, but a small alteration of each of them. This adjustment controls each of the collection of 0-simplices making them isomorphic to the set of objects of $\mathcal{M}$. Conditions (c)-(e) are a global version of the hypotheses of [LS12, Theorem 5.2] which are the same that those of Corollary 2.3.11, and they let us control the rest of the categories of simplices.

Condition (a) is the same as requiring that $\mathcal{M}$ is right autonomous, provided a suitable version of the axiom of choice holds, as it implies that taking chosen right biduals is a strong monoidal pseudofunctor which is a local equivalence, Remark 2.1.24.

$$\mathcal{M}^{\mathrm{rev\ op}} \xrightarrow{(\ )^\circ} \mathcal{M}$$

Using condition (a) consider the bicategory $\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M})$ of the enveloping monoidales $R^\circ R$ induced by the *chosen* bidualities, opmonoidal arrows between them, and opmonoidal cells between them; it is a full subbicategory of $\mathsf{OpMon}(\mathcal{M})$. We denote by $\mathsf{OpMon}^{\mathrm{e}}_n$ the category of n-simplices of the lax-2-nerve of $\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M})$. Furthermore, the collection of objects of $\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M})$ is isomorphic to that of $\mathcal{M}$, in other words $\mathsf{OpMon}^{\mathrm{e}}_0 \cong \mathsf{Ob}\,\mathcal{M}$.

Condition (b) allows us to consider the subsimplicial object $\mathsf{OplaxAct}^{\mathrm{e}}(\mathcal{M})$ of $\mathsf{OplaxAct}(\mathcal{M})$ whose simplices are all those simplices in $\mathsf{OplaxAct}(\mathcal{M})$ that have 0-faces $(R, i \dashv i^*)$, where $i \dashv i^*$ is the *chosen* adjunction for R. We denote by $\mathsf{OplaxAct}^{\mathrm{e}}_n$ the category of n-simplices of $\mathsf{OplaxAct}^{\mathrm{e}}(\mathcal{M})$. Now both the collection of 0-simplices of the lax-2-nerve of $\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M})$ and the collection of 0-simplices of $\mathsf{OplaxAct}^{\mathrm{e}}(\mathcal{M})$ are isomorphic to the collection of objects of $\mathcal{M}$.

$$\mathsf{OpMon}^{\mathrm{e}}_0 \cong \mathsf{Ob}\,\mathcal{M} \cong \mathsf{OplaxAct}^{\mathrm{e}}_0$$

Condition (c) might seem slightly artificial, but let us recall an example to see that it is quite reasonable. In Mod_k we may pick as our chosen adjunctions for k-algebras R the ones induced by the unit morphism $i : k \longrightarrow R$ which, as any other k-algebra morphism, defines an adjunction $i \dashv i^*$ in Mod_k. In this case, this adjunction and its opposite adjunction are monadic and opmonadic. In particular, opmonadicity of $i_\circ \dashv i^\circ$ states that right R°-modules may be viewed as arrows with target R° in Mod_k or as a k-algebra together with a left R°-action.

Conditions (d) and (e) ensure the compatibility between opmonadicity and the rest of the structure of $\mathcal{M}$, and are what we called opmonadic-friendly monoidal bicategory in Definition 2.2.1. Now, Corollary 2.3.11 implies that the categories of 1-simplices are equivalent, and furthermore, these equivalences commute with the face and degeneracy functors.

$$\begin{tikzcd}
\mathsf{OpMon}^{\mathrm{e}}_0 \arrow[r, "\cong"'] \arrow[d, "s_0" description] & \mathsf{OplaxAct}^{\mathrm{e}}_0 \arrow[d, "s_0" description] \\
\mathsf{OpMon}^{\mathrm{e}}_1 \arrow[r, "\simeq"] \arrow[u, shift left=3, "\partial_0"] \arrow[u, shift right=3, "\partial_1"'] & \mathsf{OplaxAct}^{\mathrm{e}}_1 \arrow[u, shift left=3, "\partial_0"] \arrow[u, shift right=3, "\partial_1"']
\end{tikzcd} \tag{3.4.1}$$

Remark 3.4.1. It is easy to see that square commutes strictly with respect to the face functors. However, it commutes only up to isomorphism with respect to the degeneracy functors. If we are to construct a genuine simplicial morphism, the square must commute strictly with the degeneracy functors too. For this to happen one must modify the definition of the degenerate 1-simplices of $\mathsf{OplaxAct}(\mathcal{M})$ in the following way: instead of having $s_0(R, i \dashv i^*) = i^*1 : RR \longrightarrow R$ one has to take the composite below.

$$s_0(R, i \dashv i^*) = RR \xrightarrow{1i_\circ 1} RR^\circ R \xrightarrow{1} RR^\circ R \xrightarrow{e1} R,$$

But this approach is inconvenient since it obscures our calculations. This behaviour continues to happen for the other dimensions and similar adjustments may be done for the higher degeneracy functors. Thus, we are going to show the existence of a pseudosimplicial morphism between the lax-2-nerve of $\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M})$ and $\mathsf{OplaxAct}^{\mathrm{e}}(\mathcal{M})$ which may be easily strictified by changing how the degeneracies of $\mathsf{OplaxAct}^{\mathrm{e}}(\mathcal{M})$ are defined.

To prove that the categories of n-simplices for $n \in \{2, 3\}$ of the lax-2-nerve of $\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M})$ are equivalent to those of $\mathsf{OplaxAct}^{\mathrm{e}}(\mathcal{M})$, no further assumptions are required on $\mathcal{M}$. We first simplify the task by showing that the categories of n-simplices *with fixed 0-faces* are equivalent. Then, these equivalences break down into two steps that are similar to those used for 1-simplices in Corollary 2.3.11; one using the transposition along bidualities, and another using the opmonadicity of certain adjunctions. In the case of 1-simplices, these two steps could be performed in any order, this fact manifests as the commutative square in Corollary 2.3.11. However, it seems that for n-simplices there is only one possible way to perform these steps, which is: starting from the opmonoidal simplices, transpose first and then use opmonadicity. We begin with the case of 2-simplices by introducing the categories that form part of this process.

Notation. Limited by the shortage of letters in the Latin and Greek alphabets, we introduce a naming convention for the opmonoidal arrows and oplax actions that are of the type that correspond to each other under Corollary 2.3.11, even when its hypotheses are not satisfied. We name them with the same letter and differentiate them with mathematical accents; for example,

$\bar{s} : R^\circ R \longrightarrow S^\circ S$	Opmonoidal arrow between enveloping monoidales.
$\widehat{s} : SR^\circ R \longrightarrow S$	Oplax Action with respect to an enveloping monoidale $R^\circ R$.
$s : SR \longrightarrow S$	Oplax action with respect to a skew monoidale R induced by an adjunction $i \dashv i^*$.

This will help the reader to figure out which things ought to be the same, as well as not to lose focus by going back to the definitions too often to figure out what is each thing with a new name.

Definition 3.4.2. For three bidualities $R \dashv R^\circ$, $S \dashv S^\circ$ and $T \dashv T^\circ$ the category denoted by $\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T)$ is the category of 2-simplices with fixed 0-faces $R^\circ R$, $S^\circ S$, and $T^\circ T$ of the lax-2-nerve of $\mathsf{OpMon}(\mathcal{M})$. An object in this category consists of three opmonoidal arrows $(\bar{s}, \bar{t}, \bar{v})$, and an opmonoidal cell $\bar{\alpha}$ with a triangular shape,

$$\begin{tikzcd}
R^\circ R \arrow[rr, bend left, "\bar{v}"] \arrow[dr, "\bar{s}"'] & \bar{\alpha}\Uparrow & T^\circ T \\
& S^\circ S \arrow[ur, "\bar{t}"'] &
\end{tikzcd}$$

while a morphism is a triple $(\bar{\sigma}, \bar{\tau}, \bar{\nu})$ of opmonoidal cells that satisfy the following equation.

$$\begin{tikzcd}
R^\circ R \arrow[rr, bend left, "\bar{v}'"] \arrow[dr, "\bar{s}'"] \arrow[dr, bend right, "\bar{s}"'] \arrow[dr, phantom, "\bar{\sigma}\Uparrow"] & \bar{\alpha}'\Uparrow & T^\circ T \\
& S^\circ S \arrow[ur, "\bar{t}'"] \arrow[ur, bend right, "\bar{t}"'] \arrow[ur, phantom, "\bar{\tau}\Uparrow"] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
R^\circ R \arrow[rr, bend left, "\bar{v}'"] \arrow[rr, "\bar{v}"] \arrow[rr, phantom, bend left=20, "\bar{\nu}\Uparrow"] \arrow[dr, bend right, "\bar{s}"'] & \bar{\alpha}\Uparrow & T^\circ T \\
& S^\circ S \arrow[ur, bend right, "\bar{t}"'] &
\end{tikzcd}
\tag{OM6}$$

Definition 3.4.3. For two enveloping monoidales $R^\circ R$ and $S^\circ S$ induced by bidualities $R \dashv R^\circ$ and $S \dashv S^\circ$, and an object T in $\mathcal{M}$, define a category $\mathcal{A}(R; S; T)$. An object consists of three oplax actions,

$$\widehat{s} : SR^\circ R \longrightarrow S \qquad \widehat{t} : TS^\circ S \longrightarrow T \qquad \widehat{v} : TR^\circ R \longrightarrow T$$

and a quadrangular cell $\widehat{\alpha}$,

$$
\begin{tikzcd}
TS^\circ SR^\circ R \arrow[r, "\widehat{t}11"] \arrow[d, "11\widehat{s}"'] \arrow[dr, phantom, "\widehat{\alpha}\Nearrow"] & TR^\circ R \arrow[d, "\widehat{v}"] \\
TS^\circ S \arrow[r, "\widehat{t}"'] & T
\end{tikzcd}
$$

satisfying three axioms,

$$
\begin{tikzcd}[column sep=small]
& TS^\circ SR^\circ R \arrow[dr, "\widehat{t}11"] & \\
TS^\circ SS^\circ SR^\circ R \arrow[ur, "\widehat{t}1111"] \arrow[dr, "11e111"'] \arrow[dd, "1111\widehat{s}"'] \arrow[r, phantom, "\widehat{t}^2 11\Uparrow"] & & TR^\circ R \arrow[dd, "\widehat{v}"] \\
& TS^\circ SR^\circ R \arrow[ur, "\widehat{t}11"'] \arrow[dd, "11\widehat{s}"] \arrow[dl, phantom, "\cong"] \arrow[r, phantom, "\widehat{\alpha}\Nearrow"] & \\
TS^\circ SS^\circ S \arrow[dr, "11e1"'] & & T \\
& TS^\circ S \arrow[ur, "\widehat{t}"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& TS^\circ SR^\circ R \arrow[dr, "\widehat{t}11"] \arrow[d, "11\widehat{s}"'] & \\
TS^\circ SS^\circ SR^\circ R \arrow[ur, "\widehat{t}1111"] \arrow[dd, "1111\widehat{s}"'] \arrow[r, phantom, "\cong"] & TS^\circ S \arrow[dr, "\widehat{t}"] \arrow[r, phantom, "\widehat{\alpha}\Nearrow"] & TR^\circ R \arrow[dd, "\widehat{v}"] \\
& & \\
TS^\circ SS^\circ S \arrow[uur, "\widehat{t}11"] \arrow[dr, "11e1"'] \arrow[r, phantom, "\widehat{t}^2\Uparrow"] & & T \\
& TS^\circ S \arrow[ur, "\widehat{t}"'] &
\end{tikzcd}
\tag{A1}
$$

$$
\begin{tikzcd}[column sep=small]
& TR^\circ RR^\circ R \arrow[dr, "\widehat{v}11"] & \\
TS^\circ SR^\circ RR^\circ R \arrow[ur, "\widehat{t}1111"] \arrow[dr, "11\widehat{s}11"'] \arrow[dd, "1111e1"'] \arrow[r, phantom, "\widehat{\alpha}11\Uparrow"] & & TR^\circ R \arrow[dd, "\widehat{v}"] \\
& TS^\circ SR^\circ R \arrow[ur, "\widehat{t}11"'] \arrow[dd, "11\widehat{s}"] \arrow[dl, phantom, "11\widehat{s}^2\Nearrow"] \arrow[r, phantom, "\widehat{\alpha}\Nearrow"] & \\
TS^\circ SR^\circ R \arrow[dr, "11\widehat{s}"'] & & T \\
& TS^\circ S \arrow[ur, "\widehat{t}"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& TR^\circ RR^\circ R \arrow[dr, "\widehat{v}11"] \arrow[dd, "11e1"'] & \\
TS^\circ SR^\circ RR^\circ R \arrow[ur, "\widehat{t}1111"] \arrow[dd, "1111e1"'] \arrow[r, phantom, "\cong"] & & TR^\circ R \arrow[dd, "\widehat{v}"] \arrow[dl, phantom, "\widehat{v}^2\Nearrow"] \\
& TR^\circ R \arrow[dr, "\widehat{v}"] & \\
TS^\circ SR^\circ R \arrow[ur, "\widehat{t}11"] \arrow[dr, "11\widehat{s}"'] \arrow[r, phantom, "\widehat{\alpha}\Uparrow"] & & T \\
& TS^\circ S \arrow[ur, "\widehat{t}"'] &
\end{tikzcd}
\tag{A2}
$$

$$
\begin{tikzcd}[column sep=small]
& T \arrow[dddr, bend left, "1"] & \\
TS^\circ S \arrow[ur, "\widehat{t}"] \arrow[dd, "111n"'] \arrow[dddr, bend left, "1"] & & \\
& & \\
TS^\circ SR^\circ R \arrow[dr, "11\widehat{s}"'] \arrow[ur, phantom, "11\widehat{t}^0\Nearrow"] & & T \\
& TS^\circ S \arrow[ur, "\widehat{t}"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& T \arrow[dddr, bend left, "1"] \arrow[dd, "1n"] & \\
TS^\circ S \arrow[ur, "\widehat{t}"] \arrow[dd, "111n"'] \arrow[r, phantom, "\cong"] & & \\
& TR^\circ R \arrow[dr, "\widehat{v}"] \arrow[ur, phantom, "\widehat{v}^0\Nearrow"] & \\
TS^\circ SR^\circ R \arrow[ur, "\widehat{t}11"] \arrow[dr, "11\widehat{s}"'] \arrow[r, phantom, "\widehat{\alpha}\Uparrow"] & & T \\
& TS^\circ S \arrow[ur, "\widehat{t}"'] &
\end{tikzcd}
\tag{A3}
$$

while the morphisms between them consist of three oplax actions cells

$$
\begin{tikzcd}
R^\circ R \arrow[r, bend left, "\widehat{s}'"] \arrow[r, bend right, "\widehat{s}"'] \arrow[r, phantom, "\widehat{\sigma}\Uparrow"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
S^\circ S \arrow[r, bend left, "\widehat{t}'"] \arrow[r, bend right, "\widehat{t}"'] \arrow[r, phantom, "\widehat{\tau}\Uparrow"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
R^\circ R \arrow[r, bend left, "\widehat{v}'"] \arrow[r, bend right, "\widehat{v}"'] \arrow[r, phantom, "\widehat{\nu}\Uparrow"] & T
\end{tikzcd}
$$

satisfying the following equation.

$$
\begin{tikzcd}
TS^\circ SR^\circ R \arrow[rr, bend left, "\widehat{t}'11"] \arrow[d, bend right, "11\widehat{s}"'] \arrow[d, bend left, "11\widehat{s}'"] \arrow[d, phantom, "\overset{11\widehat{\sigma}}{\Rightarrow}"] \arrow[drr, phantom, "\widehat{\alpha}'\nearrow"] & & TR^\circ R \arrow[d, bend left, "\widehat{v}'"] \\
TS^\circ S \arrow[rr, bend left, "\widehat{t}'"] \arrow[rr, bend right, "\widehat{t}"'] \arrow[rr, phantom, "\widehat{\tau}\Uparrow"] & & T
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
TS^\circ SR^\circ R \arrow[rr, bend left, "\widehat{t}'11"] \arrow[rr, bend right, "\widehat{t}11"'] \arrow[rr, phantom, "\widehat{\tau}11\Uparrow"] \arrow[d, bend right, "1\widehat{s}"'] \arrow[drr, phantom, "\widehat{\alpha}\nearrow"] & & TR^\circ R \arrow[d, bend right, "\widehat{v}"'] \arrow[d, bend left, "\widehat{v}'"] \arrow[d, phantom, "\overset{\widehat{\nu}}{\Rightarrow}"] \\
TS^\circ S \arrow[rr, bend right, "\widehat{t}"'] & & T
\end{tikzcd}
\tag{A4}
$$

Composition and identities are given as in the category

$$\mathsf{OplaxAct}(R^\circ R; S) \times \mathsf{OplaxAct}(S^\circ S; T) \times \mathsf{OplaxAct}(R^\circ R; T).$$

Remark 3.4.4. At this point it is worth mentioning that oplax actions with respect to enveloping monoidales $\hat{s} : SR^\circ R \longrightarrow S$ may be rewritten using the product

$$X \underset{R}{\circ} Y := XR^\circ Y$$

defined for a biduality $R \dashv R^\circ$ in $\mathcal{M}$. So, the data for an oplax action is now an arrow $\widehat{s} : S \underset{R}{\circ} R \longrightarrow S$ and cells as shown,

$$
\begin{tikzcd}
S \underset{R}{\circ} R \underset{R}{\circ} R \arrow[r, "\widehat{s} \underset{R}{\circ} 1"] \arrow[d, "1 \underset{R}{\circ} \ell"'] \arrow[dr, phantom, "\widehat{s}^2\nearrow"] & S \underset{R}{\circ} R \arrow[d, "\widehat{s}"] \\
S \underset{R}{\circ} R \arrow[r, "\widehat{s}"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
S \underset{R}{\circ} R \arrow[d, "\widehat{s}"'] \arrow[dr, phantom, "\Downarrow\widehat{s}^0"] & S \arrow[l, "\mathfrak{r}"'] \arrow[dl, "1"] \\
S &
\end{tikzcd}
$$

where $\ell = e1 : RR^\circ R \longrightarrow R$ and $\mathfrak{r} = 1n : S \longrightarrow SR^\circ R$. In the same way, we may also rewrite the objects $(\widehat{s}, \widehat{t}, \widehat{v}, \widehat{\alpha})$ of the categories $\mathcal{A}(R; S; T)$ defined above. Hence, the cell $\widehat{\alpha}$ becomes

$$
\begin{tikzcd}
T \underset{S}{\circ} S \underset{R}{\circ} R \arrow[r, "\widehat{t} \underset{R}{\circ} 1"] \arrow[d, "1 \underset{S}{\circ} \widehat{s}"'] \arrow[dr, phantom, "\widehat{\alpha}\nearrow"] & T \underset{R}{\circ} R \arrow[d, "\widehat{v}"] \\
T \underset{S}{\circ} S \arrow[r, "\widehat{t}"'] & T
\end{tikzcd}
$$

and the axioms change accordingly. For example, axiom (A1) becomes the following equation.

$$
\begin{tikzcd}[column sep=small]
& & T \underset{S}{\circ} S \underset{R}{\circ} R \arrow[drr, "\widehat{t} \underset{R}{\circ} 1"] \arrow[dd, phantom, "\widehat{t}^2 \underset{R}{\circ} 1 \Uparrow"] & & \\
T \underset{S}{\circ} S \underset{S}{\circ} S \underset{R}{\circ} R \arrow[urr, "\widehat{t} \underset{S}{\circ} 1 \underset{R}{\circ} 1"] \arrow[drr, "1 \underset{S}{\circ} \ell \underset{R}{\circ} 1"'] \arrow[dd, "1 \underset{S}{\circ} 1 \underset{S}{\circ} \widehat{s}"'] & & & & T \underset{R}{\circ} R \arrow[dd, "\widehat{v}"] \\
& & T \underset{S}{\circ} S \underset{R}{\circ} R \arrow[urr, "\widehat{t} \underset{R}{\circ} 1"'] \arrow[dd, "1 \underset{S}{\circ} \widehat{s}"] \arrow[dl, phantom, "\cong"] \arrow[dr, phantom, "\widehat{\alpha} \Nearrow"] & & \\
T \underset{S}{\circ} S \underset{S}{\circ} S \arrow[drr, "1 \underset{S}{\circ} \ell"'] & & & & T \\
& & T \underset{S}{\circ} S \arrow[urr, "\widehat{t}"'] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& & T \underset{S}{\circ} S \underset{R}{\circ} R \arrow[drr, "\widehat{t} \underset{R}{\circ} 1"] \arrow[dd, "1 \underset{S}{\circ} \widehat{s}"] & & \\
T \underset{S}{\circ} S \underset{S}{\circ} S \underset{R}{\circ} R \arrow[urr, "\widehat{t} \underset{S}{\circ} 1 \underset{R}{\circ} 1"] \arrow[dd, "1 \underset{S}{\circ} 1 \underset{S}{\circ} \widehat{s}"'] \arrow[r, phantom, "\cong"] & {} & & {} \arrow[r, phantom, "\widehat{\alpha} \Nearrow"] & T \underset{R}{\circ} R \arrow[dd, "\widehat{v}"] \\
& & T \underset{S}{\circ} S \arrow[drr, "\widehat{t}"] \arrow[dd, phantom, "\widehat{t}^2 \Uparrow"] & & \\
T \underset{S}{\circ} S \underset{S}{\circ} S \arrow[urr, "\widehat{t} \underset{S}{\circ} 1"] \arrow[drr, "1 \underset{S}{\circ} \ell"'] & & & & T \\
& & T \underset{S}{\circ} S \arrow[urr, "\widehat{t}"'] & &
\end{tikzcd}
$$

This approach is taken by Lack and Street in [LS12, Section 5] where they use the fact that the product $\underset{R}{\circ}$ turns the bicategory $\mathcal{M}$ into a *skew monoidal bicategory* which they call $\mathcal{M}_R$, where the skew unit is R. It is possible to define oplax actions in skew monoidal bicategories, and under this perspective, oplax actions with respect to an enveloping monoidale $R^\circ R$ in $\mathcal{M}$ are oplax actions with respect to the unit monoidale in $\mathcal{M}_R$. We refer the reader to Lack and Street's paper for more about skew monoidal bicategories. A full definition may be found in [Buc16, Definition 5.2].

We are now ready to address the equivalence between the categories of 2-simplices of opmonoidal arrows and oplax actions, and we begin with the transposition step mentioned earlier which in fact does not require any of the extra assumptions on the monoidal bicategory $\mathcal{M}$. This transposition step consists of an equivalence of categories, and the map that defines it is not trivial. The idea behind it comes from examining the equivalence in [LS12, Theorem 5.1] between opmonoidal monads on an enveloping monoidale $R^\circ R$ in $\mathcal{M}$ and right skew monoidales with a given skew unit on the object R in the skew monoidal bicategory $\mathcal{M}_R$. One has to look at how the multiplication of an opmonoidal monad is mapped to the associator of the corresponding right skew monoidale and generalise appropriately.

Theorem 3.4.5. *For every three bidualities $R \dashv R^\circ$, $S \dashv S^\circ$ and $T \dashv T^\circ$ in a monoidal bicategory $\mathcal{M}$ there is an equivalence of categories,*

$$\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T) \simeq \mathcal{A}(R; S; T)$$

given on objects by

$$
\begin{tikzcd}[column sep=small]
R^\circ R \arrow[rr, bend left, "\bar v"] \arrow[dr, "\bar s"'] & \arrow[d, phantom, "\bar\alpha\Uparrow"] & T^\circ T \\
& S^\circ S \arrow[ur, "\bar t"'] &
\end{tikzcd}
\overset{P}{\longmapsto} P(\bar\alpha) :=
\begin{tikzcd}
TS^\circ SR^\circ R \arrow[r, "1\bar t11"] \arrow[d, "111\bar s"] \arrow[rr, bend left, "\widehat t11"] \arrow[dd, bend right=60, "11\widehat s"'] \arrow[dr, phantom, "1\bar t\bar\alpha\Nearrow"] & TT^\circ TR^\circ R \arrow[r, "e111"] \arrow[d, "111\bar v"] \arrow[dr, phantom, "\cong"] & TR^\circ R \arrow[d, "1\bar v"] \arrow[dd, bend left=60, "\widehat v"] \\
TS^\circ SS^\circ S \arrow[r, "1\bar t\bar t"] \arrow[d, "11e1"] \arrow[dr, phantom, "\bar t^2\Nearrow"] & TT^\circ TT^\circ T \arrow[r, "e111"] \arrow[d, "11e1"] \arrow[dr, phantom, "\cong"] & TT^\circ T \arrow[d, "e1"] \\
TS^\circ S \arrow[r, "1\bar t"'] \arrow[rr, bend right, "\widehat t"'] & TT^\circ T \arrow[r, "e1"'] & T
\end{tikzcd}
$$

where $\widehat s$, $\widehat t$ and $\widehat v$ are obtained by transposition as in Theorem 2.3.7.

Proof.
By Theorem 2.3.7 one has the following equivalence of categories,

$$
\begin{aligned}
&\mathsf{OpMon}(R^\circ R, S^\circ S) \times \mathsf{OpMon}(S^\circ S, T^\circ T) \times \mathsf{OpMon}(R^\circ R, T^\circ T) \\
&\qquad \simeq \mathsf{OplaxAct}(R^\circ R; S) \times \mathsf{OplaxAct}(S^\circ S; T) \times \mathsf{OplaxAct}(R^\circ R; T) \qquad (3.4.2)
\end{aligned}
$$

which means that triples $(\bar s, \bar t, \bar v)$ of opmonoidal arrows as in the statement are already in equivalence with their transpose oplax actions $(\widehat s, \widehat t, \widehat v)$, so now one may focus on their fillings.

Hence, fix a triple $(\bar s, \bar t, \bar v)$ of opmonoidal arrows on the left hand side of (3.4.2), on the right hand side, fix the corresponding triple $(\widehat s, \widehat t, \widehat v)$ of oplax actions, and fix isomorphisms as shown below that witness the correspondence.

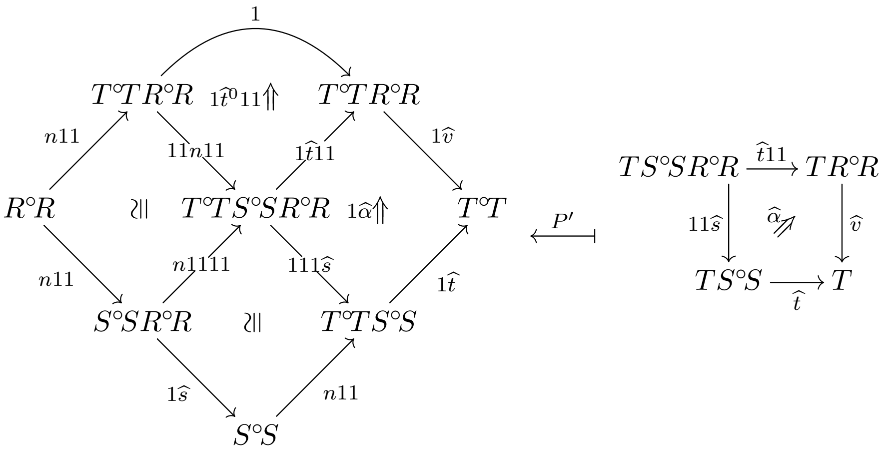
$$
\begin{tikzcd}[column sep=small]
R^\circ R \arrow[rr, "\bar s"] \arrow[dr, "n11"'] & \arrow[d, phantom, "\wr\|"] & S^\circ S \\
& S^\circ SR^\circ R \arrow[ur, "1\widehat s"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& SS^\circ S \arrow[dr, "e1"] \arrow[d, phantom, "\|\wr"] & \\
SR^\circ R \arrow[ur, "1\bar s"] \arrow[rr, "\widehat s"'] & & S
\end{tikzcd}
\qquad (3.4.3)
$$

Claim. *The assignation P defines a bijection between the set of opmonoidal cells $\bar\alpha$ and the set of cells $\widehat\alpha$ that satisfy* (A1)*,* (A2)*, and* (A3)*.*

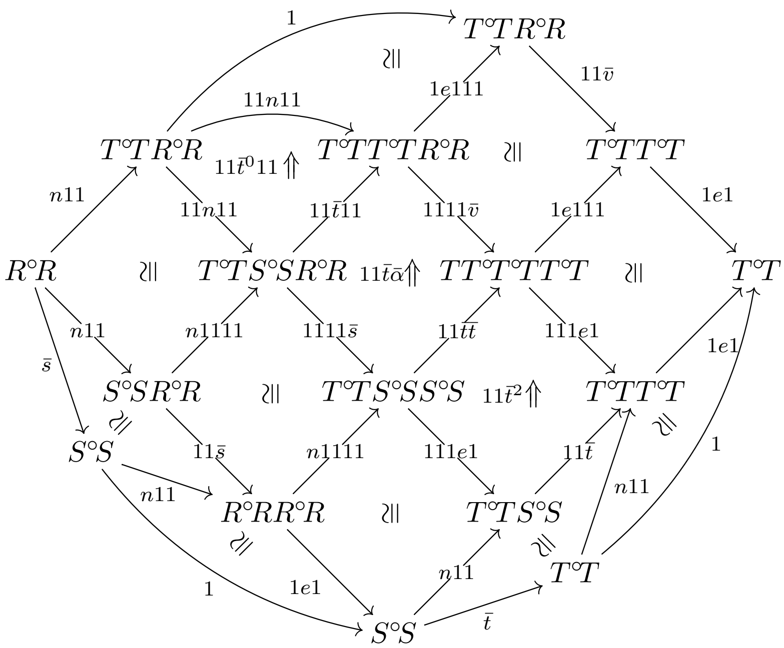
$$
\left\{
\begin{tikzcd}[column sep=small]
R^\circ R \arrow[rr, bend left, "\bar v"] \arrow[dr, "\bar s"'] & \arrow[d, phantom, "\bar\alpha\Uparrow"] & T^\circ T \\
& S^\circ S \arrow[ur, "\bar t"'] &
\end{tikzcd}
\;\middle|\; (\mathrm{OM4}), (\mathrm{OM5})
\right\}
\cong
\left\{
\begin{tikzcd}
TS^\circ SR^\circ R \arrow[r, "\widehat t11"] \arrow[d, "11\widehat s"'] \arrow[dr, phantom, "\widehat\alpha\Nearrow"] & TR^\circ R \arrow[d, "\widehat v"] \\
TS^\circ S \arrow[r, "\widehat t"'] & T
\end{tikzcd}
\;\middle|\; (\mathrm{A1}), (\mathrm{A2}), (\mathrm{A3})
\right\}
$$

Proof. [Sketch]

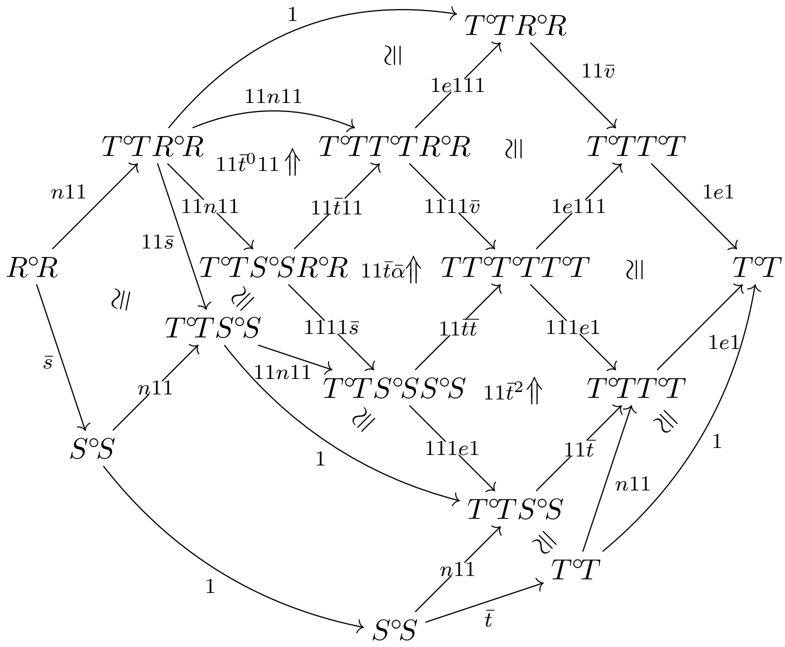
Let $\bar{\alpha}$ be a cell as in the statement of the claim, then axiom (OM4) for $\bar{\alpha}$ implies that $P(\bar{\alpha})$ satisfies axiom (A1), axiom (OM4) for $\bar{\alpha}$ implies axiom (A2) for $P(\bar{\alpha})$, and axiom (OM5) for $\bar{\alpha}$ implies axiom (A3) for $P(\bar{\alpha})$, thus P is well defined. Let P' be the function given as follows.



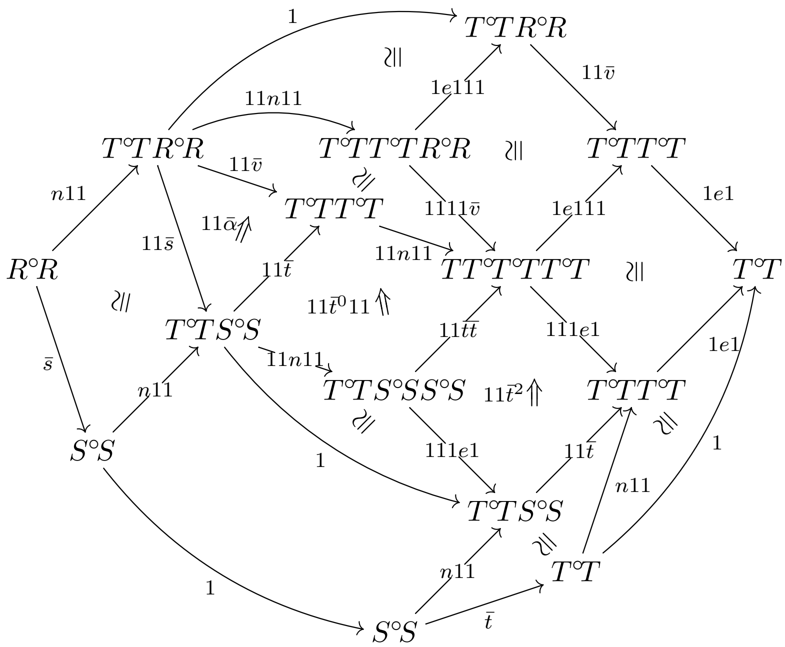
To show that P' is well defined let $\widehat{\alpha}$ be a cell in the domain of P', then $P'(\widehat{\alpha})$ satisfies axiom (OM4) as a consequence of (A2) for $\widehat{\alpha}$, and axiom (OM5) as a consequence of (A3) for $\widehat{\alpha}$. Finally, the assignations P and P' are inverse to each other; let $\bar{\alpha}$ in the domain of P and consider the calculation below.



$=$


$T^\circ TR^\circ R$
1
$11n11$
$1e111$
$11\bar{v}$
$T^\circ TR^\circ R$
$11\bar{t}^0 11 \Uparrow$
$T^\circ TT^\circ TR^\circ R$
$T^\circ TT^\circ T$
$n11$
$11n11$
$11\bar{t}11$
$111\bar{v}$
$1e111$
$1e1$
$11\bar{s}$
$R^\circ R$
$T^\circ TS^\circ SR^\circ R$
$11\bar{t}\bar{\alpha}\Uparrow$
$TT^\circ T^\circ TT^\circ T$
$T^\circ T$
$T^\circ TS^\circ S$
$111\bar{s}$
$11\bar{t}\bar{t}$
$11e1$
$1e1$
$\bar{s}$
$11n11$
$n11$
$T^\circ TS^\circ SS^\circ S$
$11\bar{t}^2\Uparrow$
$T^\circ TT^\circ T$
$S^\circ S$
1
$111e1$
$11\bar{t}$
1
$n11$
$T^\circ TS^\circ S$
$n11$
$T^\circ T$
1
$S^\circ S$
$\bar{t}$

$=$


$T^\circ TR^\circ R$
1
$11n11$
$1e111$
$11\bar{v}$
$T^\circ TR^\circ R$
$T^\circ TT^\circ TR^\circ R$
$T^\circ TT^\circ T$
$n11$
$11\bar{v}$
$T^\circ TT^\circ T$
$111\bar{v}$
$1e111$
$1e1$
$11\bar{\alpha}\Uparrow$
$11\bar{s}$
$11n11$
$R^\circ R$
$11\bar{t}$
$TT^\circ T^\circ TT^\circ T$
$T^\circ T$
$11\bar{t}^0 11 \Uparrow$
$T^\circ TS^\circ S$
$11\bar{t}\bar{t}$
$11e1$
$1e1$
$\bar{s}$
$11n11$
$n11$
$T^\circ TS^\circ SS^\circ S$
$11\bar{t}^2\Uparrow$
$T^\circ TT^\circ T$
$S^\circ S$
1
$111e1$
$11\bar{t}$
1
$n11$
$T^\circ TS^\circ S$
$n11$
$T^\circ T$
1
$S^\circ S$
$\bar{t}$

$$
= \quad
\begin{tikzcd}
 & & & T^\circ T R^\circ R \arrow[dr, "11\bar{v}"] & & \\
 & T^\circ T R^\circ R \arrow[urr, bend left, "1"] \arrow[dr, "11\bar{v}"] \arrow[ddd, "11\bar{s}"'] \arrow[dr, phantom, "11\bar{\alpha}\Uparrow", shift right=4] & & & T^\circ T T^\circ T \arrow[ddr, "1e1"] & \\
 & & T^\circ T T^\circ T \arrow[urr, bend left, "1"] \arrow[dr, "11n11"] \arrow[dddrr, bend right, "1"] \arrow[rr, phantom, "\cong"] & & & \\
R^\circ R \arrow[uur, "n11"] \arrow[ddd, "\bar{s}"'] \arrow[r, phantom, "\cong", shift right=6] & & & TT^\circ T^\circ TT^\circ T \arrow[uur, "1e111"] \arrow[ddr, "111e1"] \arrow[rr, phantom, "\cong"] & & T^\circ T \\
 & T^\circ T S^\circ S \arrow[uur, "11\bar{t}"] \arrow[dddrr, bend right, "1"] & & & & \\
 & & & & T^\circ T T^\circ T \arrow[uur, "1e1"] \arrow[r, phantom, "\cong", shift right=4] & \\
S^\circ S \arrow[uur, "n11"] \arrow[dddrr, bend right, "1"] & & & & & \\
 & & & T^\circ T S^\circ S \arrow[uur, "11\bar{t}"] \arrow[dr, phantom, "\cong"] & & \\
 & & & & T^\circ T \arrow[uuu, "n11"'] \arrow[uuuuur, bend right, "1"'] & \\
 & & S^\circ S \arrow[uur, "n11"] \arrow[uurr, "\bar{t}"'] & & &
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & & & & T^\circ T \\
 & T^\circ T R^\circ R \arrow[rr, "11\bar{v}"] \arrow[ddr, "11\bar{s}"'] \arrow[dr, phantom, "11\bar{\alpha}\Uparrow"] & & T^\circ T T^\circ T \arrow[ur, "1e1"] & \\
R^\circ R \arrow[ur, "n11"] \arrow[dddr, "\bar{s}"'] \arrow[r, phantom, "\cong", shift right=6] & & & & \\
 & & T^\circ T S^\circ S \arrow[ur, "11\bar{t}"] \arrow[d, phantom, "\cong"] & & \\
 & & & T^\circ T \arrow[uu, "n11"'] \arrow[uuur, bend right, "1"'] & \\
 & S^\circ S \arrow[uur, "n11"] \arrow[urr, "\bar{t}"'] & & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & T^\circ T R^\circ R \arrow[rr, bend left, "11\bar{v}"] \arrow[dr, phantom, "\cong"] & & T^\circ T T^\circ T \arrow[dr, "1e1"] \arrow[d, phantom, "\cong"] & \\
R^\circ R \arrow[ur, "n11"] \arrow[rr, bend left, "\bar{v}"] \arrow[dr, "\bar{s}"'] \arrow[rr, phantom, "\bar{\alpha}\Uparrow", shift right=3] & & T^\circ T \arrow[ur, "n11"] \arrow[rr, "1"'] & & T^\circ T \\
 & S^\circ S \arrow[ur, "\bar{t}"'] & & &
\end{tikzcd}
$$

This proves for $\bar{s}$, $\bar{t}$, and $\bar{v}$ that the triple of *fixed* isomorphisms of the following kind,

$$
\begin{tikzcd}
 & S^\circ S R^\circ R \arrow[rr, "11\bar{s}"] \arrow[dr, phantom, "\cong"] & & S^\circ S S^\circ S \arrow[dr, "1e1"] \arrow[d, phantom, "\cong"] & \\
R^\circ R \arrow[ur, "n11"] \arrow[rr, "\bar{s}"'] & & S^\circ S \arrow[ur, "n11"] \arrow[rr, "1"'] & & S^\circ S
\end{tikzcd}
$$

satisfies (OM6), creating a isomorphism of opmonoidal 2-simplices between $P'P\bar{\alpha}$ and $\bar{\alpha}$ which does not depend on the cell $\bar{\alpha}$ but only on its source and target. Now, let $\widehat{\alpha}$ in the domain of P', then $PP'(\widehat{\alpha}) = \widehat{\alpha}$ follows by axiom (A1). ∎

To continue with the proof of the theorem, fix two 2-simplices $(\bar{s}, \bar{t}, \bar{v}, \bar{\alpha})$ and $(\bar{s}', \bar{t}', \bar{v}', \bar{\alpha}')$ in $\mathsf{OpMon}(R, S, T)$, and let $(\widehat{s}, \widehat{t}, \widehat{v}, \widehat{\alpha})$ and $(\widehat{s}', \widehat{t}', \widehat{v}', \widehat{\alpha}')$ be their corresponding objects in $\mathcal{A}(R; S; T)$ under the equivalence (3.4.2) above and the isomorphism P of the claim.

Claim. *There is an isomorphism between the following hom sets.*

$$\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T)((\bar{s},\bar{t},\bar{v},\bar{\alpha}),(\bar{s}',\bar{t}',\bar{v}',\bar{\alpha}')) \cong \mathcal{A}(R;S;T)((\widehat{s},\widehat{t},\widehat{v},\widehat{\alpha}),(\widehat{s}',\widehat{t}',\widehat{v}',\widehat{\alpha}'))$$

Proof.
The only thing to verify is that the isomorphism on hom sets induced by (3.4.2) restricts to the isomorphism in this claim. Let $(\bar{\sigma},\bar{\tau},\bar{\nu}) : (\bar{s},\bar{t},\bar{v},\bar{\alpha}) \longrightarrow (\bar{s}',\bar{t}',\bar{v}',\bar{\alpha}')$ arrow of opmonoidal 2-simplices then its transpose $(P\bar{\sigma}, P\bar{\tau}, P\bar{\nu})$satisfies axiom (A4) as a consequence of axiom (OM4) for $\bar{\tau}$ and axiom (OM6) for $(\bar{\sigma},\bar{\tau},\bar{\nu})$. Now, let $(\widehat{\sigma},\widehat{\tau},\widehat{\nu}) : (\widehat{s},\widehat{t},\widehat{v},\widehat{\alpha}) \longrightarrow (\widehat{s}',\widehat{t}',\widehat{v}',\widehat{\alpha}')$ be an arrow in $\mathcal{A}(R;S;T)$ then its transpose satisfies axiom (OM6) as a consequence of axiom (OLA5) for $\widehat{\tau}$ axiom (A4) for $(\widehat{\sigma},\widehat{\tau},\widehat{\nu})$. ■

Thus with both claims we conclude the proof of the theorem. ■

Even though we call this the "transposition step", it is no longer mere transposition via the universal property of bidualities between the cells $\bar{\alpha}$ and cells $\widehat{\alpha}$ as in the case of 1-simplices. Perhaps one shall call this process "2-simplex transposition along $T \dashv T^\circ$" since this proof does not depend on the enveloping monoidal structure of $R^\circ R$ and $S^\circ S$. For a fully general version of Theorem 3.4.5, one might replace these enveloping monoidales for arbitrary right skew monoidales M and N.

Now, we perform the opmonadicity step.

Theorem 3.4.6. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every three bidualities $R \dashv R^\circ$, $S \dashv S^\circ$ and $T \dashv T^\circ$, and every two opmonadic adjunctions,*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd} \qquad \begin{tikzcd} S^\circ \arrow[d, bend left, "j^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "j_\circ"] \end{tikzcd}$$

there is an equivalence of categories,

$$\mathcal{A}(R;S;T) \simeq \mathsf{OplaxAct}(R;S;T)$$

given by precomposition with the opmonadic left adjoint $1j_\circ 1 i_\circ 1$.

$$\begin{tikzcd} & & & TSR \arrow[r, "1j_\circ 11"] \arrow[d, "11i_\circ 1"'] \arrow[dr, "1j_\circ 1 i_\circ 1"] \arrow[dr, phantom, shift left=3, "\cong"] & TS^\circ SR \arrow[r, "\widehat{t}1"] \arrow[d, "111i_\circ 1"] \arrow[dr, phantom, "\cong"] & TR \arrow[d, "1i_\circ 1"] \\ TS^\circ SR^\circ R \arrow[r, "\widehat{t}11"] \arrow[d, "11\widehat{s}"'] \arrow[dr, phantom, "\Nearrow\widehat{\alpha}"] & TR^\circ R \arrow[d, "\widehat{v}"] & \overset{Q}{\longmapsto} & TSR^\circ R \arrow[r, bend right, "1j_\circ 111"'] \arrow[d, "1\widehat{s}"'] \arrow[dr, phantom, "\cong"] & TS^\circ SR^\circ R \arrow[r, "\widehat{t}11"] \arrow[d, "11\widehat{s}"] \arrow[dr, phantom, "\Nearrow\widehat{\alpha}"] & TR^\circ R \arrow[d, "\widehat{v}"] \\ TS^\circ S \arrow[r, "\widehat{t}"'] & T & & TS \arrow[r, "1j_\circ 1"'] & TS^\circ S \arrow[r, "\widehat{t}"'] & T \end{tikzcd}$$

Proof.

The equivalence between the edges of the squares in the statement is determined by the equivalence below which obtained by three instances of the equivalence in Theorem 2.3.10.

$$\begin{aligned}&\mathsf{OplaxAct}(R^\circ R;S)\times\mathsf{OplaxAct}(S^\circ S;T)\times\mathsf{OplaxAct}(R^\circ R;T)\\ &\qquad\simeq \mathsf{OplaxAct}(R;S)\times\mathsf{OplaxAct}(S;T)\times\mathsf{OplaxAct}(R;T)\end{aligned}\tag{3.4.4}$$

Now, Q is well defined on objects because for a square $\widehat{\alpha}$, composing the appropriate left adjoint with each of the sides of axioms (A1), (A2), and (A3) turns them into the axioms (2SIM1), (2SIM2) and (2SIM3); for example, precomposing the arrow

$$1j_\circ 1j_\circ 1i_\circ 1 : TS^\circ SS^\circ SR^\circ R \longrightarrow TSSR$$

with both sides of axiom (A1) gives each of the sides of axiom (2SIM1). Likewise, Q is well defined on the arrows of $\mathcal{A}(R;S;T)$, because precomposing both sides of axiom (A4) with the arrow $1j_\circ 1i_\circ 1 : TS^\circ SR^\circ R \longrightarrow TSR$ gives each of the sides of axiom (2SIM4). Hence Q is a well defined functor, and because of the equivalence (3.4.4) above it is automatically faithful. To prove that Q is essentially surjective on objects and full, first remember the core technicalities of the equivalence (3.4.4) regarding opmonadicity. An oplax action in any of the categories on the right hand side of (3.4.4), comes equipped with the structure of a module for a monad on induced by an opmonadic adjunction Theorem 2.3.10. This module structure is expressed in terms of the oplax action constraints, for example, take an oplax right action $s : SR \longrightarrow S$ with respect to the skew monoidale induced by an adjunction $i \dashv i^*$, then the cell ψ below

$$\begin{tikzcd} SR \arrow[r, "1i_\circ 1"'] \arrow[rrr, bend left, "s", ""{name=A}] & SR^\circ R \arrow[r, "1i^\circ 1"'] \arrow[r, phantom, shift left=4ex, "\psi\Uparrow"] & SR \arrow[r, "s"'] & S \end{tikzcd} = \begin{tikzcd} & & SR \arrow[ddr, "s"] & \\ & SRR \arrow[ur, "s1"] \arrow[dr, "1i^*1"] \arrow[d, phantom, "\wr\|"] \arrow[rr, phantom, "s^2\Nearrow"] & & \\ SR \arrow[ur, "1i1"] \arrow[uurr, bend left, "1"] \arrow[r, "1i_\circ 1"'] \arrow[u, phantom, shift right=2ex, "s^0 1\Nwarrow"] & SR^\circ R \arrow[r, "1i^\circ 1"'] & SR \arrow[r, "s"'] & S \end{tikzcd}\tag{3.4.5}$$

is a module structure for s, with respect to the monad induced by the adjunction $1i_\circ 1 \dashv 1i^\circ 1$. And because this adjunction is opmonadic by hypothesis, there exists an oplax right action $\widehat{s} : SR^\circ R \longrightarrow S$ with respect to the enveloping monoidale $R^\circ R$, such that the precomposition with $1i_\circ 1 : SR \longrightarrow SR^\circ R$ is equal to s.

Now, to prove that Q is essentially surjective on objects let α be a 2-simplex in the category $\mathsf{OplaxAct}(R;S;T)$.

$$\begin{tikzcd} TSR \arrow[r, "t1"] \arrow[d, "1s"'] \arrow[dr, phantom, "\alpha\Nearrow"] & TR \arrow[d, "v"] \\ TS \arrow[r, "t"'] & T \end{tikzcd}$$

Then, because the arrows $1i1 : SR \longrightarrow SR^\circ R$, $1j1 : TS \longrightarrow TS^\circ S$, and $1i1 : TR \longrightarrow TR^\circ R$ are opmonadic, the module structures on s, t, and v given by the formula (3.4.5) (which by a slight abuse of notation will all be called ψ) induce three oplax actions $\widehat{s}$, $\widehat{t}$ and $\widehat{v}$ with isomorphisms as below,

$$
\begin{tikzcd}
SR \arrow[rr, "s"] \arrow[dr, "1i_\circ 1"'] \arrow[rr, phantom, shift right=3, "\cong"] & & S \\
& SR^\circ R \arrow[ur, "\widehat{s}"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
TS \arrow[rr, "t"] \arrow[dr, "1j_\circ 1"'] \arrow[rr, phantom, shift right=3, "\cong"] & & T \\
& TS^\circ S \arrow[ur, "\widehat{t}"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
TR \arrow[rr, "v"] \arrow[dr, "1i_\circ 1"'] \arrow[rr, phantom, shift right=3, "\cong"] & & T \\
& TR^\circ R \arrow[ur, "\widehat{v}"'] &
\end{tikzcd}
\tag{3.4.6}
$$

All this structure turns the cell α into a morphism of modules for the monad induced by the opmonadic adjunction below.

$$
\begin{tikzcd}
TS^\circ SR^\circ R \arrow[d, bend left, "1j^\circ 1i^\circ 1"] \arrow[d, phantom, "\dashv"] \\
TSR \arrow[u, bend left, "1j_\circ 1i_\circ 1"]
\end{tikzcd}
$$

but instead of proving that directly, it is simpler and equivalent to describe a module morphism structure for the monads induced by the two adjunctions below, which are also opmonadic by hypothesis.

$$
\begin{tikzcd}
TS^\circ SR \arrow[d, bend left, "1j^\circ 11"] \arrow[d, phantom, "\dashv"] \\
TSR \arrow[u, bend left, "1j_\circ 11"]
\end{tikzcd}
\qquad
\begin{tikzcd}
TSR^\circ R \arrow[d, bend left, "11i^\circ 1"] \arrow[d, phantom, "\dashv"] \\
TSR \arrow[u, bend left, "11i_\circ 1"]
\end{tikzcd}
$$

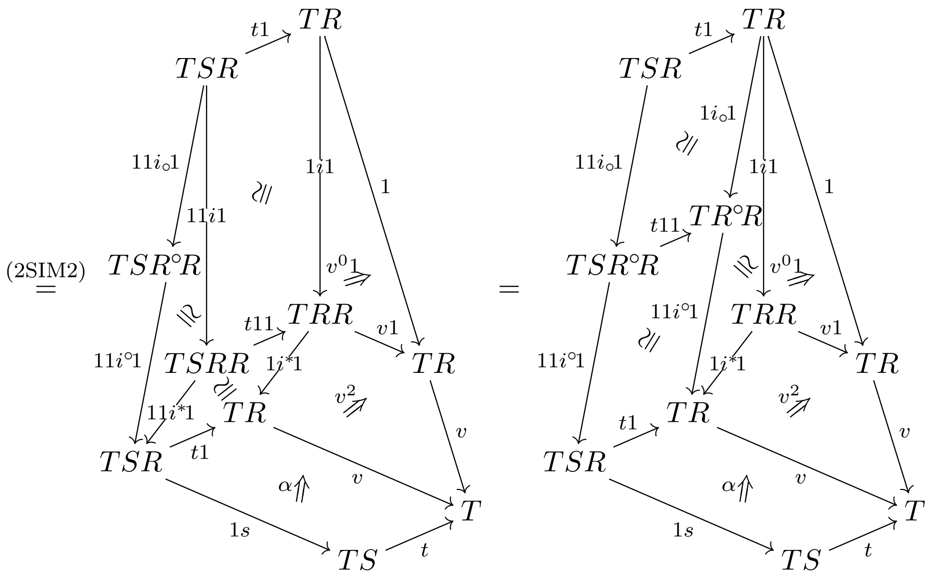
The structure of modules for the monads induced by these adjunctions on source and target of α is induced by the actions ψ on s, t, and v. And the fact that α is a module morphism means precisely that the following two equations are satisfied.

$$
\begin{tikzcd}
& TSR \arrow[r, "t1"] \arrow[dl, "11i_\circ 1"'] \arrow[ddr, "1s"] & TR \arrow[ddr, "v"] & \\
TSR^\circ R \arrow[d, "11i^\circ 1"'] \arrow[r, phantom, "1\psi\Uparrow"] & & \alpha\Uparrow & \\
TSR \arrow[rr, "1s"'] & & TS \arrow[r, "t"'] & T
\end{tikzcd}
=
\begin{tikzcd}
& TSR \arrow[r, "t1"] \arrow[dl, "11i_\circ 1"'] \arrow[d, phantom, "\cong"] & TR \arrow[dl, "1i_\circ 1"] \arrow[ddrr, "v"] & & \\
TSR^\circ R \arrow[r, "t11"] \arrow[d, "11i^\circ 1"'] \arrow[dr, phantom, "\cong"] & TR^\circ R \arrow[d, "1i^\circ 1"] & \psi\Uparrow & & \\
TSR \arrow[r, "t1"] \arrow[dr, "1s"'] & TR \arrow[rrr, "v"] \arrow[d, phantom, "\alpha\Uparrow"] & & & T \\
& TS \arrow[urrr, "t"'] & & &
\end{tikzcd}
\tag{3.4.7}
$$

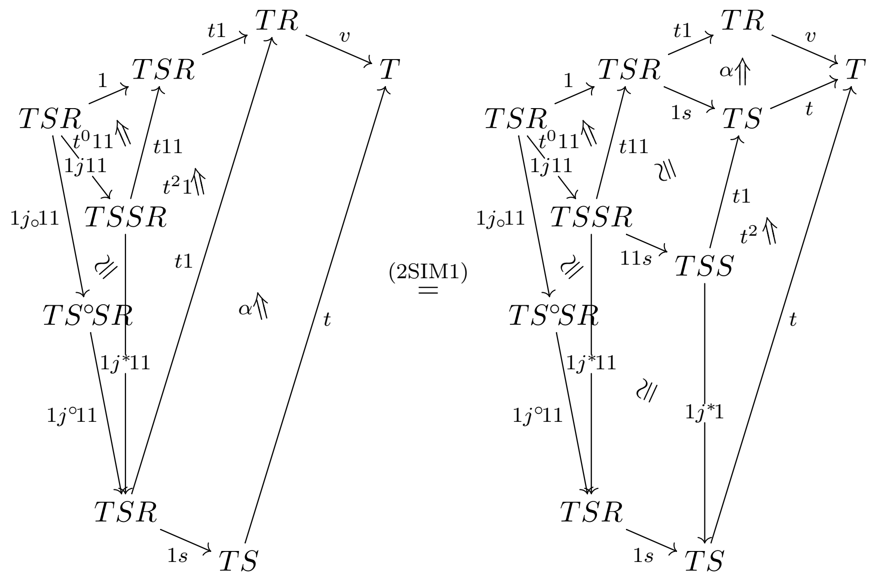
$$
\begin{tikzcd}
TSR \arrow[r, "t1"] \arrow[d, "1j_\circ 11"'] & TR \arrow[r, "v"] & T \\
TS^\circ SR \arrow[d, "1j^\circ 11"'] \arrow[ur, phantom, "\psi 1\Uparrow"] & & \\
TSR \arrow[uur, "t1"'] \arrow[r, "1s"'] & TS \arrow[uur, "t"'] \arrow[u, phantom, "\alpha\Uparrow"] &
\end{tikzcd}
=
\begin{tikzcd}
TSR \arrow[r, "t1"] \arrow[dr, "1s"'] \arrow[dd, "1j_\circ 11"'] & TR \arrow[r, "v"] \arrow[d, phantom, "\alpha\Uparrow"] & T \\
& TS \arrow[ur, "t"'] \arrow[d, "1j_\circ 1"] \arrow[dl, phantom, "\cong"] & \\
TS^\circ SR \arrow[r, "11s"'] \arrow[d, "1j^\circ 11"'] & TS^\circ S \arrow[d, "1j^\circ 1"] \arrow[dl, phantom, "\cong"] \arrow[ur, phantom, "\psi\Uparrow"] & \\
TSR \arrow[r, "1s"'] & TS \arrow[uuur, "t"'] &
\end{tikzcd}
\tag{3.4.8}
$$

Condition (3.4.7) is a consequence of the following calculation,

$$
\begin{tikzcd}
& TSR \arrow[rr, "t1"] \arrow[dl, "11i_\circ 1"'] \arrow[d, "11i1"] \arrow[dr, "1"] & & TR \arrow[d, "1"] \\
TSR^\circ R \arrow[r, phantom, "\cong"] \arrow[d, "11i^\circ 1"'] & TSRR \arrow[r, "1s1"] \arrow[dl, "11i^* 1"] \arrow[u, phantom, "1s^0 1\Rightarrow"] & TSR \arrow[r, "t1"] \arrow[d, "1s"] \arrow[dl, phantom, "1s^2\Rightarrow"] \arrow[dr, phantom, "\alpha\Rightarrow"] & TR \arrow[d, "v"] \\
TSR \arrow[rr, "1s"'] & & TS \arrow[r, "t"'] & T
\end{tikzcd}
\overset{(2\mathrm{SIM3})}{=}
\begin{tikzcd}
& TSR \arrow[r, "t1"] \arrow[ddl, "11i_\circ 1"'] \arrow[dd, "11i1"] & TR \arrow[d, "1i1"] \arrow[ddr, "1"] \arrow[dl, phantom, "\cong"] & \\
& & TRR \arrow[dr, "v1"] \arrow[d, phantom, "\alpha 1\Uparrow"] \arrow[r, phantom, "v^0 1\Rightarrow"] & \\
TSR^\circ R \arrow[r, phantom, "\cong"] \arrow[d, "11i^\circ 1"'] & TSRR \arrow[ur, "t11"] \arrow[r, "1s1"] \arrow[dl, "11i^* 1"] & TSR \arrow[r, "t1"] \arrow[d, "1s"] \arrow[dl, phantom, "1s^2\Rightarrow"] \arrow[dr, phantom, "\alpha\Rightarrow"] & TR \arrow[d, "v"] \\
TSR \arrow[rr, "1s"'] & & TS \arrow[r, "t"'] & T
\end{tikzcd}
$$



and equation (3.4.8) may be proved in the following way.



$$
= \quad
\begin{tikzcd}
 & & & & TR \arrow[dr, "v"] \arrow[d, phantom, "\alpha\Uparrow"] & \\
 & & TSR \arrow[urr, "t1"] \arrow[drr, "1s"] & & & T \\
TSR \arrow[urr, "1"] \arrow[dr, phantom, "\cong"] \arrow[drr, "1s"] \arrow[ddd, "1j_\circ 11"'] & & & & TS \arrow[ur, "t"'] & \\
 & & TS \arrow[urr, "1"] \arrow[dr, "1j1"] \arrow[ddd, "1j_\circ 1"] \arrow[ur, phantom, "t^0 1 \Uparrow"] & & & \\
 & & & TSS \arrow[uur, "t1"'] \arrow[dddd, "1j^* 1"] \arrow[ur, phantom, "t^2\Uparrow"] \arrow[dl, phantom, "\cong"] & & \\
 & TS^\circ SR \arrow[dr, "11s"] \arrow[dd, "1j^\circ 11"'] & & & & \\
 & & TS^\circ S \arrow[ddr, "1j^\circ 1"] \arrow[dl, phantom, "\cong"] & & & \\
 & TSR \arrow[dr, "1s"'] & & & & \\
 & & & TS \arrow[uuuuuuurr, "t"'] & &
\end{tikzcd}
$$

Therefore, by the opmonadicity of $1j_\circ 1i_\circ 1$, there exists a cell $\widehat{\alpha}$ such that the following equation is satisfied.

$$
\begin{tikzcd}
 & TSR \arrow[rr, "t1"] \arrow[dl, "11i_\circ 1"'] \arrow[dd, "1s"] & & TR \arrow[dd, "v"] \\
TSR^\circ R \arrow[r, phantom, "\cong"] \arrow[dr, "1\widehat{s}"'] & {} & \alpha\nearrow & \\
 & TS \arrow[rr, "t"] \arrow[dr, "1j_\circ 1"'] & \cong & T \\
 & & TS^\circ S \arrow[ur, "\widehat{t}"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
TSR \arrow[r, "1j_\circ 11"] \arrow[rr, bend left, "t1"] \arrow[d, "11i_\circ 1"'] \arrow[dr, "1j_\circ 1i_\circ 1"] & TS^\circ SR \arrow[r, "\widehat{t}1"] \arrow[d, "111i_\circ 1"] & TR \arrow[d, "1i_\circ 1"] \arrow[dd, bend left, "v"] \\
TSR^\circ R \arrow[r, bend right, "1j_\circ 111"'] \arrow[d, "1\widehat{s}"'] & TS^\circ SR^\circ R \arrow[r, "\widehat{t}11"] \arrow[d, "11\widehat{s}"] \arrow[dr, phantom, "\widehat{\alpha}\nearrow"] & TR^\circ R \arrow[d, "\widehat{v}"] \\
TS \arrow[r, "1j_\circ 1"'] & TS^\circ S \arrow[r, "\widehat{t}"'] & T
\end{tikzcd}
\tag{3.4.9}
$$

The cell $\widehat{\alpha}$ makes the quadruple $(\widehat{s}, \widehat{t}, \widehat{v}, \widehat{\alpha})$ into an object of $\mathcal{A}(R; S; T)$, because precomposing both sides of axioms (A1), (A2), and (A3) for $\widehat{\alpha}$ with the appropriate opmonadic left adjoint gives each of the sides of axioms (2SIM1), (2SIM2) and (2SIM3) for α which are pairwise equal. The three isomorphisms (3.4.6) become an morphism in $\mathcal{A}(R; S; T)$ as condition (3.4.9) is the appropriate instance of axiom (A4). Hence $Q(\widehat{s}, \widehat{t}, \widehat{v}, \widehat{\alpha}) \cong (s, t, v, \alpha)$ and so Q is essentially surjective on objects. To verify that Q is full let $(\sigma, \tau, \nu) : (s, t, v, \alpha) \longrightarrow (s', t', v', \alpha')$ be a morphism in $\mathsf{OplaxAct}(R; S; T)$ by the equivalence (3.4.4) we know that there exist a cell $(\widehat{\sigma}, \widehat{\tau}, \widehat{\nu})$ which is induced by opmonadicity, and precomposing both sides of axiom (A4) for $(\widehat{\sigma}, \widehat{\tau}, \widehat{\nu})$ with the opmonadic left adjoint $1j_\circ 1i_\circ 1$ gives each of the sides of axiom (2SIM4) for (σ, τ, ν) which are equal, therefore Q is full and so an equivalence. ■

Now we bring the equivalences in Theorems 3.4.5 and 3.4.6 together for any opmonadic-friendly monoidal bicategory, see Definition 2.2.1.

Corollary 3.4.7. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every three bidualities $R \dashv R^\circ$, $S \dashv S^\circ$, and $T \dashv T^\circ$, and opmonadic adjunctions*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd} \qquad \begin{tikzcd} S^\circ \arrow[d, bend left, "j^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "j_\circ"] \end{tikzcd}$$

there is an equivalence of categories as shown.

$$\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T) \simeq \mathsf{OplaxAct}(R; S; T)$$

Proof.

$$\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T) \simeq \mathcal{A}(R; S; T) \simeq \mathsf{OplaxAct}(R; S; T)$$

■

Corollary 3.4.8. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory such that every object R has a chosen right bidual, and a chosen adjunction $i \dashv i^*$ whose opposite adjunction is opmonadic. There is an equivalence of categories of 2-simplices,*

$$\mathsf{OpMon}^{\mathrm{e}}_2 \simeq \mathsf{OplaxAct}^{\mathrm{e}}_2$$

which commutes with faces and degeneracies.

$$\begin{tikzcd} \mathsf{OpMon}^{\mathrm{e}}_1 \arrow[rr, "\simeq"'] \arrow[d, "s_0"] \arrow[d, "s_1"] & & \mathsf{OplaxAct}^{\mathrm{e}}_1 \arrow[d, "s_0"] \arrow[d, "s_1"] \\ \mathsf{OpMon}^{\mathrm{e}}_2 \arrow[rr, "\simeq"] \arrow[u, "\partial_0"] \arrow[u, "\partial_1"] \arrow[u, "\partial_2"] & & \mathsf{OplaxAct}^{\mathrm{e}}_2 \arrow[u, "\partial_0"] \arrow[u, "\partial_1"] \arrow[u, "\partial_2"] \end{tikzcd}$$

Proof.

The face maps commute strictly with the equivalences, since the equivalence of 2-simplices was defined precisely as such. For the degeneracies, the square commutes up to isomorphism. These isomorphisms may be strictified in a similar way as is already discussed in Remark 3.4.1 by redefining the degeneracy functors. ■

We now continue with the case of 3-simplices by introducing the categories that will form part of this process.

Definition 3.4.9. Given four bidualities $R \dashv R^\circ$, $S \dashv S^\circ$, $T \dashv T^\circ$, and $U \dashv U^\circ$ denote by $\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T, U^\circ U)$ the category of 3-simplices of the lax-2-nerve of $\mathsf{OpMon}(\mathcal{M})$ with fixed 0-faces $R^\circ R$, $S^\circ S$, $T^\circ T$ and $U^\circ U$. It is a full subcategory of

$$\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T) \times \mathsf{OpMon}(R^\circ R, S^\circ S, U^\circ U) \\ \times \mathsf{OpMon}(R^\circ R, T^\circ T, U^\circ U) \times \mathsf{OpMon}(S^\circ S, T^\circ T, U^\circ U)$$

on the objects $(\bar\alpha, \bar\beta, \bar\gamma, \bar\zeta)$ satisfying the equation

$$
\begin{tikzcd}
R^\circ R \arrow[rr, bend left, "\bar w"] \arrow[dr, "\bar s"'] \arrow[drr, bend left=15, "\bar v"] \arrow[dr, phantom, shift left=3, "\bar\alpha\Uparrow"] & \arrow[d, phantom, "\bar\gamma\Uparrow"] & U^\circ U \\
& S^\circ S \arrow[r, "\bar t"'] & T^\circ T \arrow[u, "\bar u"']
\end{tikzcd}
=
\begin{tikzcd}
R^\circ R \arrow[rr, bend left, "\bar w"] \arrow[dr, "\bar s"'] \arrow[r, phantom, "\bar\beta\Uparrow"] & {} & U^\circ U \\
& S^\circ S \arrow[ur, bend left, "\bar x"] \arrow[r, "\bar t"'] \arrow[r, phantom, shift left=4, "\bar\zeta\Uparrow"] & T^\circ T \arrow[u, "\bar u"']
\end{tikzcd}
\tag{OM7}
$$

Remark 3.4.10. For the proofs below we conveniently rewrite condition (OM7), since it makes our pasting diagram calculations easier to follow.

$$
\begin{tikzcd}
& R^\circ R \arrow[dr, "1"] & \\
R^\circ R \arrow[ur, "1"] \arrow[dr, "1"] \arrow[dd, "\bar s"'] & & R^\circ R \arrow[dd, "\bar w"] \\
& R^\circ R \arrow[ur, "1"] \arrow[dd, "\bar v"] \arrow[dl, phantom, "\bar\alpha\Rightarrow"] \arrow[dr, phantom, "\bar\gamma\Uparrow"] & \\
S^\circ S \arrow[dr, "\bar t"'] & & U^\circ U \\
& T^\circ T \arrow[ur, "\bar u"'] &
\end{tikzcd}
=
\begin{tikzcd}
& R^\circ R \arrow[dr, "1"] \arrow[dd, "\bar s"] & \\
R^\circ R \arrow[ur, "1"] \arrow[dd, "\bar s"'] & & R^\circ R \arrow[dd, "\bar w"] \arrow[dl, phantom, "\bar\beta\Rightarrow"] \\
& S^\circ S \arrow[dr, "\bar x"] & \\
S^\circ S \arrow[ur, "1"] \arrow[dr, "\bar t"'] & \bar\zeta\Uparrow & U^\circ U \\
& T^\circ T \arrow[ur, "\bar u"'] &
\end{tikzcd}
\tag{OM7'}
$$

Definition 3.4.11. For three enveloping monoidales $R^\circ R$, $S^\circ S$, and $T^\circ T$ induced by bidualities $R \dashv R^\circ$, $S \dashv S^\circ$, and $T \dashv T^\circ$, and an object U in $\mathcal{M}$, define the category $\mathcal{A}(R;S;T;U)$ as the full subcategory of quadruples $(\widehat\alpha, \widehat\beta, \widehat\gamma, \widehat\zeta)$ in

$$\mathcal{A}(R;S;T) \times \mathcal{A}(R;S;U) \times \mathcal{A}(R;T;U) \times \mathcal{A}(S;T;U)$$

that satisfy the following condition,

$$
\begin{tikzcd}
& US^\circ SR^\circ R \arrow[dr, "\widehat x11"] & \\
UT^\circ TS^\circ SR^\circ R \arrow[ur, "\widehat u1111"] \arrow[dr, "11\widehat t11"] \arrow[dd, "1111\widehat s"'] \arrow[r, phantom, "\widehat\zeta11\Uparrow"] & {} & UR^\circ R \arrow[dd, "\widehat w"] \\
& UT^\circ TR^\circ R \arrow[ur, "\widehat u11"'] \arrow[dd, "11\widehat v"] \arrow[dl, phantom, "11\widehat\alpha\Rightarrow"] \arrow[dr, phantom, "\widehat\gamma\Uparrow"] & \\
UT^\circ TS^\circ S \arrow[dr, "11\widehat t"'] & & U \\
& UT^\circ T \arrow[ur, "\widehat u"'] &
\end{tikzcd}
=
\begin{tikzcd}
& US^\circ SR^\circ R \arrow[dr, "\widehat x11"] \arrow[dd, "11\widehat s"] & \\
UT^\circ TS^\circ SR^\circ R \arrow[ur, "\widehat u1111"] \arrow[dd, "1111\widehat s"'] \arrow[dr, phantom, "\cong"] & & UR^\circ R \arrow[dd, "\widehat w"] \arrow[dl, phantom, "\widehat\beta\Rightarrow"] \\
& US^\circ S \arrow[dr, "\widehat x"] & \\
UT^\circ TS^\circ S \arrow[ur, "\widehat u11"'] \arrow[dr, "11\widehat t"'] & \widehat\zeta\Uparrow & U \\
& UT^\circ T \arrow[ur, "\widehat u"'] &
\end{tikzcd}
\tag{A5}
$$

We now proceed with the transposition step.

Proposition 3.4.12. *For every four bidualities $R^\circ R$, $S^\circ S$, $T^\circ T$ and $U^\circ U$, there is an equivalence of categories*

$$\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T, U^\circ U) \simeq \mathcal{A}(R;S;T;U)$$

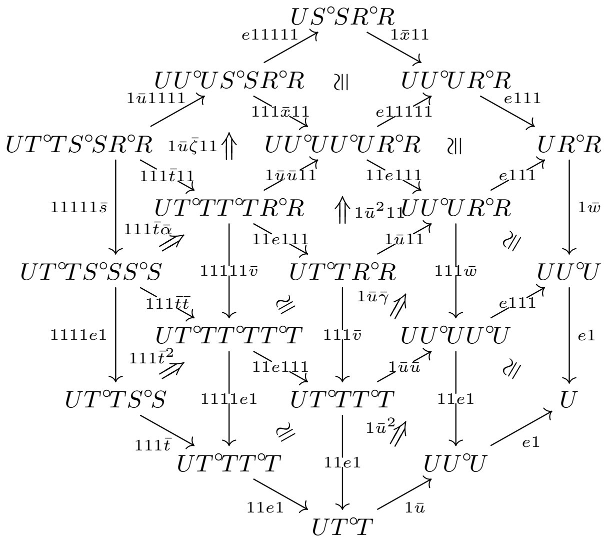
induced by the functor P defined in Theorem 3.4.5 and given on a 3-simplex $(\bar\alpha, \bar\beta, \bar\gamma, \bar\zeta)$ by $(P\bar\alpha, P\bar\beta, P\bar\gamma, P\bar\zeta)$

Proof.

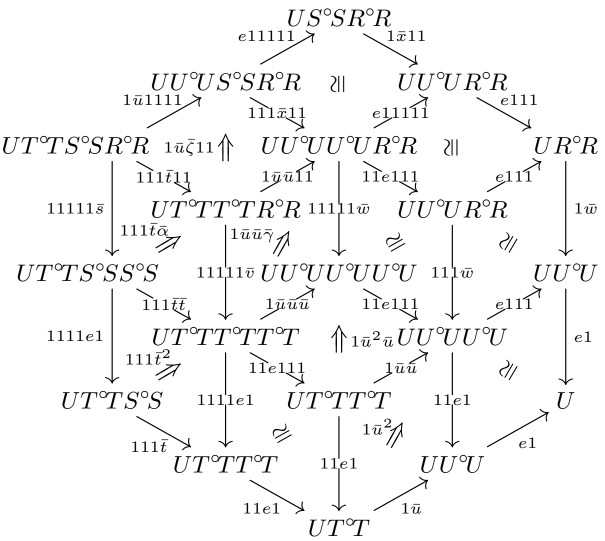
The categories in question are defined as full subcategories of 4-fold products of categories with the form $\mathsf{OpMon}(\_,\_,\_)$ and $\mathcal{A}(\_;\_;\_)$ respectively, and thus by Theorem 3.4.5 we have the equivalence of categories,

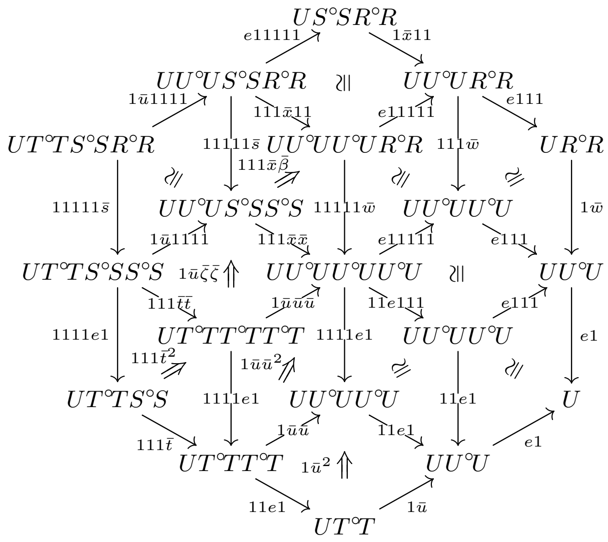
$$\begin{aligned}\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T) \times \mathsf{OpMon}(R^\circ R, S^\circ S, U^\circ U)&\\ \times \mathsf{OpMon}(R^\circ R, T^\circ T, U^\circ U) \times \mathsf{OpMon}(S^\circ S, T^\circ T, U^\circ U)&\\ \simeq \mathcal{A}(R;S;T) \times \mathcal{A}(R;S;U) \times \mathcal{A}(R;T;U) \times \mathcal{A}(S;T;U)&\end{aligned} \tag{3.4.10}$$

We verify that for a quadruple $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\zeta})$ in the category $\mathsf{OpMon}(R^\circ R, S^\circ S, T^\circ T, U^\circ U)$ the quadruple $(P\bar{\alpha}, P\bar{\beta}, P\bar{\gamma}, P\bar{\zeta})$ satisfies axiom (A5).

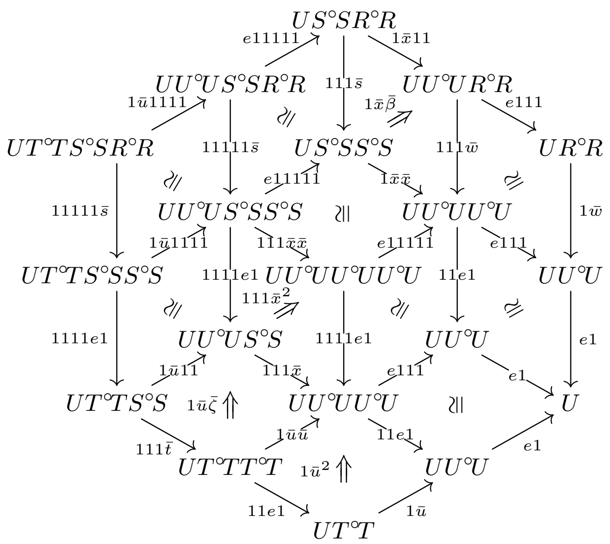


$=$

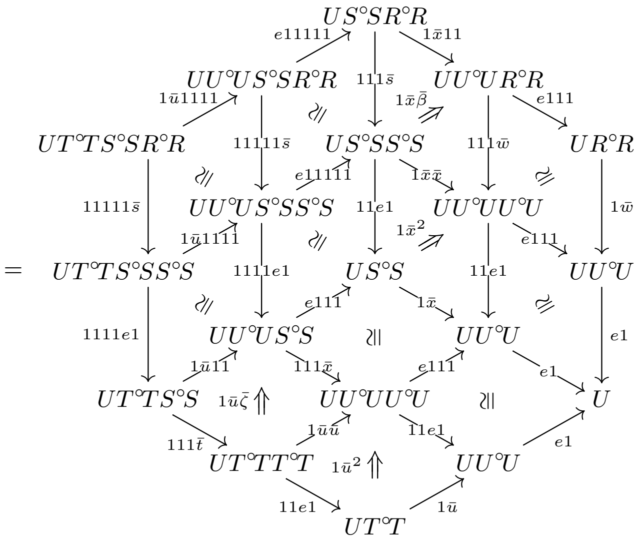




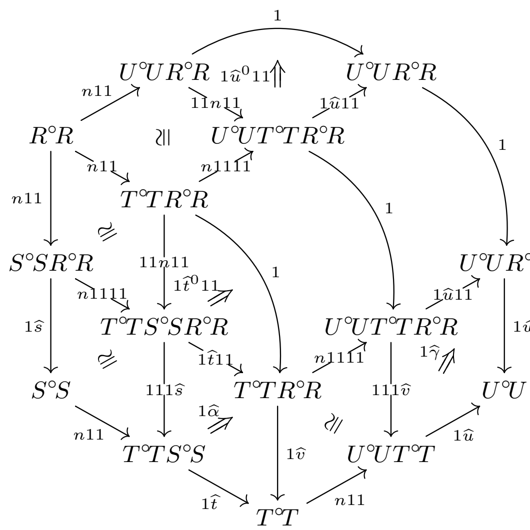
(OM1)
(OM7')
$=$



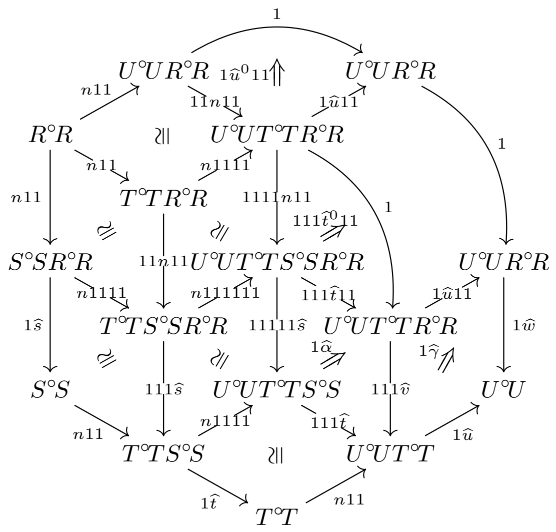
(OM4)
$=$



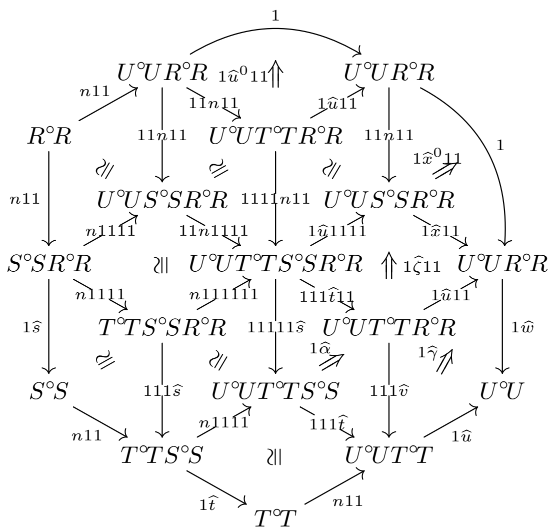
And for a quadruple $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\zeta})$ in $\mathcal{A}(R; S; T; U)$ the quadruple $(P^{-1}\widehat{\alpha}, P^{-1}\widehat{\beta}, P^{-1}\widehat{\gamma}, P^{-1}\widehat{\zeta})$ satisfies axiom (OM7).



$=$



$\overset{\text{(A3)}}{=}$



$$\overset{\text{(A5)}}{=}\quad\begin{tikzcd}[column sep=small]
 & U^\circ U R^\circ R \arrow[rr, bend left, "1"] \arrow[dd, "11n11"] & & U^\circ U R^\circ R \arrow[dd, "11n11"] \arrow[ddr, bend left, "1"] & \\
R^\circ R \arrow[ur, "n11"] \arrow[dd, "n11"'] \arrow[dr, phantom, "\cong"] & & & & \\
 & U^\circ U S^\circ S R^\circ R \arrow[rr, bend left, "1"] \arrow[dr, "11n1111"] \arrow[dd, "111\hat{s}"] \arrow[r, phantom, "1\hat{u}^0 1111\Uparrow"] & & U^\circ U S^\circ S R^\circ R \arrow[dd, "111\hat{s}"] \arrow[dr, "1\hat{x}11"] \arrow[r, phantom, "1\hat{x}^0 11 \Nearrow"] & \\
S^\circ S R^\circ R \arrow[ur, "n1111"] \arrow[dd, "1\hat{s}"'] \arrow[dr, phantom, "\cong"] & & U^\circ U T^\circ T S^\circ S R^\circ R \arrow[ur, "1\hat{u}1111"] \arrow[dd, "11111\hat{s}"] & & U^\circ U R^\circ R \arrow[dd, "1\hat{w}"] \\
 & U^\circ U S^\circ S \arrow[dr, "11n11"] & & U^\circ U S^\circ S \arrow[dr, "1\hat{x}"] \arrow[ur, phantom, "1\hat{\beta}\Nearrow"] & \\
S^\circ S \arrow[ur, "n11"] \arrow[dr, "n11"'] \arrow[r, phantom, "\cong"] & {} & U^\circ U T^\circ T S^\circ S \arrow[ur, "1\hat{u}11"] \arrow[dr, "111\hat{t}"] \arrow[r, phantom, "1\hat{\zeta}\Uparrow"] & {} & U^\circ U \\
 & T^\circ T S^\circ S \arrow[ur, "n1111"] \arrow[dr, "1\hat{t}"'] \arrow[r, phantom, "\cong"] & {} & U^\circ U T^\circ T \arrow[ur, "1\hat{u}"'] & \\
 & & T^\circ T \arrow[ur, "n11"'] & &
\end{tikzcd}$$

$$=\quad\begin{tikzcd}[column sep=small]
 & U^\circ U R^\circ R \arrow[rr, bend left, "1"] \arrow[dd, "11n11"] & & U^\circ U R^\circ R \arrow[dd, "11n11"] \arrow[ddr, bend left, "1"] & \\
R^\circ R \arrow[ur, "n11"] \arrow[dd, "n11"'] \arrow[dr, phantom, "\cong"] & & & & \\
 & U^\circ U S^\circ S R^\circ R \arrow[rr, bend left, "1"] \arrow[dd, "111\hat{s}"] & & U^\circ U S^\circ S R^\circ R \arrow[dd, "111\hat{s}"] \arrow[dr, "1\hat{x}11"] \arrow[r, phantom, "1\hat{x}^0 11 \Nearrow"] & \\
S^\circ S R^\circ R \arrow[ur, "n1111"] \arrow[dd, "1\hat{s}"'] \arrow[dr, phantom, "\cong"] & & & & U^\circ U R^\circ R \arrow[dd, "1\hat{w}"] \\
 & U^\circ U S^\circ S \arrow[rr, bend left, "1"] \arrow[dr, "11n11"] \arrow[r, phantom, "1\hat{u}^0 11\Uparrow"] & & U^\circ U S^\circ S \arrow[dr, "1\hat{x}"] \arrow[ur, phantom, "1\hat{\beta}\Nearrow"] & \\
S^\circ S \arrow[ur, "n11"] \arrow[dr, "n11"'] \arrow[r, phantom, "\cong"] & {} & U^\circ U T^\circ T S^\circ S \arrow[ur, "1\hat{u}11"] \arrow[dr, "111\hat{t}"] \arrow[r, phantom, "1\hat{\zeta}\Uparrow"] & {} & U^\circ U \\
 & T^\circ T S^\circ S \arrow[ur, "n1111"] \arrow[dr, "1\hat{t}"'] \arrow[r, phantom, "\cong"] & {} & U^\circ U T^\circ T \arrow[ur, "1\hat{u}"'] & \\
 & & T^\circ T \arrow[ur, "n11"'] & &
\end{tikzcd}$$

Therefore the equivalence 3.4.10 restricts its domain and codomain to the desired equivalence of categories. ■

Now we continue with the opmonadicity step.

Theorem 3.4.13. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every four bidualities $R \dashv R^\circ$, $S \dashv S^\circ$, $T \dashv T^\circ$ and $U \dashv U^\circ$, and every three opmonadic adjunctions*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd} \qquad \begin{tikzcd} S^\circ \arrow[d, bend left, "j^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "j_\circ"] \end{tikzcd} \qquad \begin{tikzcd} T^\circ \arrow[d, bend left, "k^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "k_\circ"] \end{tikzcd}$$

there is an equivalence of categories as shown.

$$\mathcal{A}(R; S; T; U) \simeq \mathsf{OplaxAct}(R; S; T; U)$$

Proof.

By Theorem 3.4.6 there is an equivalence of categories as follows.

$$\begin{aligned}\mathcal{A}(R;S;T)\times\mathcal{A}(R;S;U)\times\mathcal{A}(R;T;U)\times\mathcal{A}(S;T;U)&\\ \simeq \mathsf{OplaxAct}(R;S;T)\times\mathsf{OplaxAct}(R;S;U)&\\ \times\mathsf{OplaxAct}(R;T;U)\times\mathsf{OplaxAct}(S;T;U)&\end{aligned}\tag{3.4.11}$$

This equivalence restricts to $\mathcal{A}(R;S;T;U)$ because precomposition with the arrow

$$1k_\circ 1j_\circ 1i_\circ 1 : UT^\circ TS^\circ SR^\circ R \longrightarrow UTSR$$

takes axiom (A5) to axiom (3SIM). Furthermore, for any quadruple $(\alpha,\beta,\gamma,\zeta)$ in the category $\mathsf{OplaxAct}(R;S;T;U)$, the quadruple induced by opmonadicity $(\widehat{\alpha},\widehat{\beta},\widehat{\gamma},\widehat{\zeta})$ in the category $\mathcal{A}(R;S;T)\times\mathcal{A}(R;S;U)\times\mathcal{A}(R;T;U)\times\mathcal{A}(S;T;U)$ satisfies axiom (A5). Indeed, if both sides of axiom (A5) for $(\widehat{\alpha},\widehat{\beta},\widehat{\gamma},\widehat{\zeta})$ are precomposed with the opmonadic arrow $1k_\circ 1j_\circ 1i_\circ 1$, one ends up with each of the sides of axiom (3SIM) for $(\alpha,\beta,\gamma,\zeta)$, which are equal. ∎

With Proposition 3.4.12 and Theorem 3.4.13 we deduce the following result.

Corollary 3.4.14. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory. For every four bidualities $R\dashv R^\circ$, $S\dashv S^\circ$, $T\dashv T^\circ$ and $U\dashv U^\circ$, and every three opmonadic adjunctions*

$$\begin{tikzcd} R^\circ \arrow[d, bend left, "i^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "i_\circ"] \end{tikzcd}\qquad \begin{tikzcd} S^\circ \arrow[d, bend left, "j^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "j_\circ"] \end{tikzcd}\qquad \begin{tikzcd} T^\circ \arrow[d, bend left, "k^\circ"] \arrow[d, phantom, "\dashv"] \\ I \arrow[u, bend left, "k_\circ"] \end{tikzcd}$$

there is an equivalence of categories as shown.

$$\mathsf{OplaxAct}(R;S;T;U)\simeq\mathsf{OpMon}(R^\circ R,S^\circ S,T^\circ T,U^\circ U)$$

Proof.

$$\mathsf{OplaxAct}(R;S;T;U)\simeq\mathcal{A}(R;S;T;U)\simeq\mathsf{OpMon}(R^\circ R,S^\circ S,T^\circ T,U^\circ U)$$

∎

Corollary 3.4.15. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory such that every object R has a chosen right bidual, and a chosen adjunction $i\dashv i^*$ whose opposite adjunction is opmonadic. There is an equivalence of categories of 3-simplices,*

$$\mathsf{OplaxAct}^{\mathrm{e}}_3\simeq\mathsf{OpMon}^{\mathrm{e}}_3$$

which commutes with faces and degeneracies.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{OpMon}^{\mathrm{e}}_2 \arrow[r, "\simeq"'] \arrow[d, shift left=2, "s_0"] \arrow[d, shift left=6, "s_1"] \arrow[d, shift left=10, "s_2"] & \mathsf{OplaxAct}^{\mathrm{e}}_2 \arrow[d, shift left=2, "s_0"] \arrow[d, shift left=6, "s_1"] \arrow[d, shift left=10, "s_2"] \\
\mathsf{OpMon}^{\mathrm{e}}_3 \arrow[r, "\simeq"] \arrow[u, shift left=0, "\partial_0"] \arrow[u, shift right=4, "\partial_1"] \arrow[u, shift right=8, "\partial_2"] \arrow[u, shift right=12, "\partial_3"] & \mathsf{OplaxAct}^{\mathrm{e}}_3 \arrow[u, shift left=0, "\partial_0"] \arrow[u, shift right=4, "\partial_1"] \arrow[u, shift right=8, "d_2"] \arrow[u, shift right=12, "d_3"]
\end{tikzcd}$$

∎

And finally we put together (3.4.1), Corollary 3.4.8, and Corollary 3.4.15 which form a pseudo-simplicial morphism of simplicial objects in Cat, which as we mentioned in Remark 3.4.1 may be strictified by making the right choices in the definition of the degeneracy functors of $\mathsf{OplaxAct}(\mathcal{M})$ to get an actual simplicial map.

Theorem 3.4.16. *Let $\mathcal{M}$ be an opmonadic-friendly monoidal bicategory such that every object R has a chosen right bidual, and a chosen adjunction $i \dashv i^*$ whose opposite adjunction is opmonadic. There is a weak equivalence of simplicial objects in Cat as shown.*

$$\mathsf{OplaxAct}^{\mathrm{e}}(\mathcal{M}) \simeq N(\mathsf{OpMon}^{\mathrm{e}}(\mathcal{M}))$$

■

4
Enriched Icons

In [Moe02] Moerdijk proves that a monad on a monoidal category is opmonoidal if and only if the category of Eilenberg-Moore algebras for the monad has a monoidal structure for which the forgetful functor is strong monoidal. In a sister paper [McC02], McCrudden presents a 2-categorical analysis of Moerdijk's result. It is stated in terms of the 2-category $\mathsf{OpMon}(\mathsf{Cat})$ of monoidal categories, opmonoidal functors, and opmonoidal natural transformations.

Theorem (Moerdijk-McCrudden)**.** *The 2-category* $\mathsf{OpMon}(\mathsf{Cat})$ *has Eilenberg-Moore objects for monads, and the forgetful functor to* Cat *preserves them.*

In this chapter we generalise this theorem in two different directions. On the one hand, Theorem 4.1.7 is a multiobject version of the Moerdijk-McCrudden Theorem, obtained by replacing monoidal categories with bicategories, opmonoidal functors with oplax functors, and opmonoidal natural transformations with icons. And on the other hand, Theorem 4.2.10 is a version of the Moerdijk-McCrudden Theorem internal to a monoidal bicategory $\mathcal{M}$ with Eilenberg-Moore objects. This means that $\mathsf{OpMon}(\mathsf{Cat})$ is replaced by $\mathsf{OpMon}(\mathcal{M})$; so replacing monoidal categories with monoidales in $\mathcal{M}$, opmonoidal functors with opmonoidal arrows in $\mathcal{M}$ and opmonoidal natural transformations with opmonoidal cells in $\mathcal{M}$. We then combine these two generalisations in Theorem 4.3.27 into a version for bicategories enriched in a monoidal bicategory $\mathcal{M}$ with Eilenberg-Moore objects. It is easy to recover Theorems 4.2.10 and 4.1.7 from this perspective since $\mathcal{M}$-enriched bicategories with one object are monoidales in $\mathcal{M}$ and Cat-enriched bicategories are ordinary bicategories.

The methods used here are tricategorical but direct proofs, similar to those of Moerdijk and McCrudden, are also possible. We give a description of the constructions obtained this way at the end of the chapter.

4.1 Icons and Eilenberg Moore Objects

We begin our discussion with an analysis of the multiobject analogue of the 2-category $\mathsf{OpMon(Cat)}$: this is a 2-category whose objects are bicategories, arrows are oplax functors, and cells are icons.

Icons were introduced by Lack in [Lac10b] to show the existence of a 2-category whose objects are bicategories. This happens to be not as straightforward as discarding the last dimension in the tricategory Bicat of bicategories, pseudofunctors, pseudonatural transformations, and modifications. The reason is because the middle four interchange law in terms of whiskerings, used to define horizontal composition, holds up to an invertible modification which is no longer there when we disregard modifications. Hence, horizontal composition would not exist. Other problems are present if one prefers to relax pseudofunctors or pseudonatural transformations [Lac10b, Section 3]. So, instead of arbitrary oplax natural transformations between a pair of lax functors, what Lack considers are those oplax natural transformations with identity components for objects; he names them *icons* as an acronym for *i*dentity *c*omponent *o*plax *n*atural transformations. In his paper, Lack sets icons as the cells of a 2-category $\mathsf{Bicat}_\mathbf{2}$ of bicategories, lax functors, and icons, which has the property that the full sub-2-category of one object bicategories is the 2-category $\mathsf{Mon(Cat)}$ of monoidal categories, monoidal functors, and monoidal natural transformations. Thus, we can think of $\mathsf{Bicat}_\mathbf{2}$ as a multiobject version of $\mathsf{Mon(Cat)}$. But since we are more interested in $\mathsf{OpMon(Cat)}$, we adjust Lack's approach by taking oplax functors instead of lax ones and the appropriate notion of icons between them.

Definition 4.1.1. For two bicategories $\mathcal{B}$ and $\mathcal{C}$ an *oplax functor* $F : \mathcal{B} \longrightarrow \mathcal{C}$ consists of a function $F : \mathsf{Ob}\,\mathcal{B} \longrightarrow \mathsf{Ob}\,\mathcal{C}$, for each pair of objects X and Y in $\mathcal{B}$ a functor between the hom categories $F : \mathcal{B}(X,Y) \longrightarrow \mathcal{C}(FX,FY)$, for each object X in $\mathcal{B}$ a natural transformation F^0 called the *oplax identities constraint* pictured below, and for each triple of objects X, Y and Z a natural transformation F^2 called the *oplax composition constraint* as follows.

$$
\begin{tikzcd}
\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \arrow[r, "FF"] \arrow[d, "m"'] \arrow[dr, phantom, "F^2"] & \mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY} \arrow[d, "m"] \\
\mathcal{B}_{X,Z} \arrow[r, "F"'] & \mathcal{C}_{FX,FZ}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "F^0"] & \\
\mathcal{B}_{X,X} \arrow[rr, "F"'] & & \mathcal{C}_{FX,FX}
\end{tikzcd}
$$

These are subject to the following three axioms.

$$
\begin{tikzcd}[column sep=small]
& \mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY}\mathcal{C}_{FW,FX} \arrow[dr, "m1"] \arrow[dd, phantom, "F^2F\Uparrow"] & \\
\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}\mathcal{B}_{W,X} \arrow[ur, "FFF"] \arrow[dr, "m1"'] \arrow[dd, "1m"'] \arrow[ddr, phantom, "\alpha"] & & \mathcal{C}_{FX,FZ}\mathcal{C}_{FW,FX} \arrow[dd, "m"] \arrow[ddl, phantom, "F^2"] \\
& \mathcal{B}_{X,Z}\mathcal{B}_{W,X} \arrow[ur, "FF"'] \arrow[dd, "m"] & \\
\mathcal{B}_{Y,Z}\mathcal{B}_{W,Y} \arrow[dr, "m"'] & & \mathcal{C}_{FW,FZ} \\
& \mathcal{B}_{W,Z} \arrow[ur, "F"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& \mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY}\mathcal{C}_{FW,FX} \arrow[dr, "m1"] \arrow[dd, "1m"'] & \\
\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}\mathcal{B}_{W,X} \arrow[ur, "FFF"] \arrow[dd, "1m"'] \arrow[r, phantom, "FF^2"] & & \mathcal{C}_{FX,FZ}\mathcal{C}_{FW,FX} \arrow[dd, "m"] \arrow[l, phantom, "\alpha"] \\
& \mathcal{C}_{FY,FZ}\mathcal{C}_{FW,FY} \arrow[dr, "m"] \arrow[dd, phantom, "F^2\Uparrow"] & \\
\mathcal{B}_{Y,Z}\mathcal{B}_{W,Y} \arrow[ur, "FF"'] \arrow[dr, "m"'] & & \mathcal{C}_{FW,FZ} \\
& \mathcal{B}_{W,Z} \arrow[ur, "F"'] &
\end{tikzcd}
\tag{OLF1}
$$

$$
\begin{tikzcd}
& \mathcal{C}_{FX,FY} \arrow[rdd, bend left, "1"] & \\
\mathcal{B}_{X,Y} \arrow[ru, "F"] \arrow[d, "1u"'] \arrow[rdd, bend left, "1"] & & \\
\mathcal{B}_{X,Y}\mathcal{B}_{X,X} \arrow[rd, "m"'] \arrow[r, phantom, "\rho\Rightarrow"] & {} & \mathcal{C}_{FX,FY} \\
& \mathcal{B}_{X,Y} \arrow[ru, "F"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{C}_{FX,FY} \arrow[dd, "1u"'] \arrow[rddd, bend left, "1"] \arrow[rd, phantom, "\rho\Rightarrow"] & \\
\mathcal{B}_{X,Y} \arrow[ru, "F"] \arrow[dd, "1u"'] \arrow[rd, phantom, "FF^0\Rightarrow"] & {} & {} \\
& \mathcal{C}_{FX,FY}\mathcal{C}_{FX,FX} \arrow[rd, "m"] \arrow[dd, phantom, "F^2\Uparrow"] & \\
\mathcal{B}_{X,Y}\mathcal{B}_{X,X} \arrow[ru, "FF"] \arrow[rd, "m"'] & & \mathcal{C}_{FX,FY} \\
& \mathcal{B}_{X,Y} \arrow[ru, "F"'] &
\end{tikzcd}
\tag{OLF2}
$$

$$
\begin{tikzcd}
& \mathcal{C}_{FX,FY} \arrow[rd, "u1"] \arrow[dd, phantom, "F^0F\Uparrow"] & \\
\mathcal{B}_{X,Y} \arrow[ru, "F"] \arrow[rd, "u1"'] \arrow[rddd, bend right, "1"'] & {} & \mathcal{C}_{FY,FY}\mathcal{C}_{FX,FY} \arrow[dd, "m"] \\
{} & \mathcal{B}_{Y,Y}\mathcal{B}_{X,Y} \arrow[ru, "FF"] \arrow[dd, "m"] \arrow[ld, phantom, "\lambda\Rightarrow"] \arrow[rd, phantom, "F^2\Rightarrow"] & \\
{} & & \mathcal{C}_{FX,FY} \\
& \mathcal{B}_{X,Y} \arrow[ru, "F"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& \mathcal{C}_{FX,FY} \arrow[rd, "u1"] \arrow[rdd, bend right, "1"'] & \\
\mathcal{B}_{X,Y} \arrow[ru, "F"] \arrow[rddd, bend right, "1"'] & & \mathcal{C}_{FY,FY}\mathcal{C}_{FX,FY} \arrow[dd, "m"] \arrow[ld, phantom, "\lambda\Rightarrow"] \\
& {} & \\
& & \mathcal{C}_{FX,FY} \\
& \mathcal{B}_{X,Y} \arrow[ru, "F"'] &
\end{tikzcd}
\tag{OLF3}
$$

Definition 4.1.2 (Lack)**.** Let $\mathcal{B}$ and $\mathcal{C}$ be two bicategories, and F and $G : \mathcal{B} \longrightarrow \mathcal{C}$ a parallel pair of oplax functors such that $FX = GX$ for every object X of $\mathcal{B}$. An *icon* α with source and target as shown,

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha\Uparrow"] & \mathcal{C}
\end{tikzcd}
$$

consists of a natural transformation α_{XY} for every pair of objects X and Y of $\mathcal{B}$,

$$
\begin{tikzcd}
\mathcal{B}_{X,Y} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha_{XY}\Uparrow"] & \mathcal{C}_{FX,FY}
\end{tikzcd}
$$

which satisfies the following two compatibility conditions with respect to composition and identities.

$$
\begin{tikzcd}
\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \arrow[r, bend left, "GG"] \arrow[d, "m"'] \arrow[rd, phantom, "G^2\Rightarrow"] & \mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY} \arrow[d, "m"] \\
\mathcal{B}_{X,Z} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha_{XY}\Uparrow"] & \mathcal{C}_{FX,FZ}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \arrow[r, bend left, "GG"] \arrow[r, bend right, "FF"'] \arrow[r, phantom, "\alpha_{YZ}\alpha_{XY}\Uparrow"] \arrow[d, "m"'] \arrow[rd, phantom, "F^2\Rightarrow"] & \mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY} \arrow[d, "m"] \\
\mathcal{B}_{X,Z} \arrow[r, bend right, "F"'] & \mathcal{C}_{FX,FZ}
\end{tikzcd}
\tag{ICON1}
$$

$$\begin{tikzcd} & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "G^0 \Nearrow"] & \\ \mathcal{B}_{X,X} \arrow[rr, bend left, "G"] \arrow[rr, bend right, "F"'] \arrow[rr, phantom, "\alpha_{XY} \Uparrow"] & & \mathcal{C}_{FX,FX} \end{tikzcd} = \begin{tikzcd} & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "F^0 \Nearrow"] & \\ \mathcal{B}_{X,X} \arrow[rr, bend right, "F"'] & & \mathcal{C}_{FX,FX} \end{tikzcd} \tag{ICON2}$$

Remark 4.1.3. For a pair of fixed bicategories $\mathcal{B}$ and $\mathcal{C}$ there is a bijection between icons as defined above and oplax natural transformations whose arrow components are identities. This bijection is obtained by composing and precomposing the cell components α_{XY} with the appropriate coherence isomorphisms. In the case that $\mathcal{C}$ is a 2-category, the bijection is an equality. If one chooses to work with oplax natural transformations whose arrow components are identities, the usual composition between oplax transformations does not make them the cells of a 2-category of bicategories. This is because the bicategory $\mathcal{C}$ may not have strict identity laws. But if one works with icons instead, horizontal composition may be defined by horizontally composing their components α_{XY}. And this horizontal composition makes icons the cells of a 2-category of bicategories in the same way that in [Lac10b] bicategories, lax functors, and icons between them constitute a 2-category $\mathsf{Bicat}_\mathbf{2}$.

Theorem 4.1.4. *There is a 2-category* Icon *that has bicategories as objects, oplax functors as arrows, and icons as cells. Compositions and identities are calculated in the usual way.*

The full sub-2-category of Icon consisting of the one object bicategories is isomorphic to $\mathsf{OpMon}(\mathsf{Cat})$; this is witnessed by the fully faithful *suspension* 2-functor.

$$\Sigma : \mathsf{OpMon}(\mathsf{Cat}) \longrightarrow \mathsf{Icon}$$

There is a 2-functor $U : \mathsf{OpMon}(\mathsf{Cat}) \longrightarrow \mathsf{Cat}$ which forgets about the monoidal structure. Its multiobject analogue is a 2-functor from Icon that takes a bicategory $\mathcal{B}$ and forgets about the horizontal composition. This process leaves us with another structure called *graph enriched in* Cat [Wol74].

Definition 4.1.5. A *graph enriched in* Cat or Cat-*graph* $\mathcal{B}$ consists of a collection of vertices $\mathsf{Ob}\,\mathcal{B}$ and, for each pair of vertices X and Y, a category $\mathcal{B}(X,Y)$ of $\mathcal{B}$. A *morphism of* Cat-*graphs* $F : \mathcal{B} \longrightarrow \mathcal{C}$ consists of a function $F : \mathsf{Ob}\,\mathcal{B} \longrightarrow \mathsf{Ob}\,\mathcal{C}$ and, for each pair of vertices in $\mathcal{B}$, a functor $F : \mathcal{B}(X,Y) \longrightarrow \mathcal{C}(FX,FY)$.

In [LP08, Section 4] the authors define a 2-category of Cat-graphs, we call it $\mathsf{Grph}(\mathsf{Cat})$. A cell $\alpha : F \longrightarrow G$ in $\mathsf{Grph}(\mathsf{Cat})$ may exist only if F and G coincide on vertices, and it is composed of natural transformations

$$\begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha_{XY} \Uparrow"] & \mathcal{C}_{FX,FY} \end{tikzcd}$$

for each pair of vertices X and Y of $\mathcal{B}$. Hence, there is a strict functor that forgets about the horizontal composition.

$$U : \mathsf{Icon} \longrightarrow \mathsf{Grph}(\mathsf{Cat})$$

Every category may be thought as the hom category of a one vertex Cat-graph, functors as morphisms between one vertex Cat-graphs, and natural transformations as transformations of Cat-graphs between these. This is witnessed by the fully faithful *suspension* 2-functor $\Sigma : \mathsf{Cat} \longrightarrow \mathsf{Grph}(\mathsf{Cat})$.

The definition of the elements of $\mathsf{OpMon}(\mathsf{Cat})$ are given by 2-dimensional operations and relations, so there is a presentation for a finitary 2-monad T_1 in the sense of [KP93]. This 2-monad is such that the 2-category $\mathsf{OpMon}(\mathsf{Cat})$ is its 2-category of strict algebras, oplax algebra morphisms, and algebra transformations.

$$T_1\text{-}\mathsf{Alg}_{o\ell} = \mathsf{OpMon}(\mathsf{Cat})$$

Now, [Lac05, Proposition 4.11] guarantees, for every 2-monad T on a 2-category $\mathcal{B}$ with Eilenberg-Moore objects for monads, the existence of Eilenberg-Moore objects for monads in the 2-category $T\text{-}\mathsf{Alg}_{o\ell}$ and the fact that these are preserved by the forgetful functor. Since Cat has Eilenberg-Moore objects for monads, the Moerdijk-McCrudden Theorem follows. A similar proof shows that Icon has Eilenberg-Moore objects for monads, but first we need to prove that $\mathsf{Grph}(\mathsf{Cat})$ does.

Theorem 4.1.6. $\mathsf{Grph}(\mathsf{Cat})$ *has Eilenberg-Moore objects for monads and* $\Sigma : \mathsf{Cat} \longrightarrow \mathsf{Grph}(\mathsf{Cat})$ *creates them.*

Proof.

Eilenberg-Moore objects for monads in $\mathsf{Grph}(\mathsf{Cat})$ are calculated locally in the following sense. Let $(\mathcal{B}, T)$ be monad in $\mathsf{Grph}(\mathsf{Cat})$. Since the source and target of the unit of the monad must coincide on vertices, the morphism of Cat-graphs T is the identity on vertices. Ergo, a monad $(\mathcal{B}, T)$ in $\mathsf{Grph}(\mathsf{Cat})$ consists of a collection of monads on each hom category $T : \mathcal{B}(X, Y) \longrightarrow \mathcal{B}(X, Y)$. If for each of these monads we denote their respective categories of Eilenberg-Moore algebras by $\mathcal{B}^T_{X,Y}$, then the Cat-graph with the same objects as $\mathcal{B}$ and hom categories $\mathcal{B}^T_{X,Y}$ is the Eilenberg-Moore object of $(\mathcal{B}, T)$.

$$\begin{aligned}
\mathsf{Grph}(\mathsf{Cat})(\mathcal{X}, \mathcal{B}^T) &= \coprod_{F\,:\,\mathsf{Ob}\,\mathcal{X} \longrightarrow \mathsf{Ob}\,\mathcal{B}^T} \left(\prod_{A,B \in \mathsf{Ob}\,\mathcal{X}} \mathsf{Cat}(\mathcal{X}_{A,B}, \mathcal{B}^T_{FA,FB}) \right) \\
&\simeq \coprod_{F\,:\,\mathsf{Ob}\,\mathcal{X} \longrightarrow \mathsf{Ob}\,\mathcal{B}} \left(\prod_{A,B \in \mathsf{Ob}\,\mathcal{X}} \mathsf{Cat}(\mathcal{X}_{A,B}, \mathcal{B}_{FA,FB})^{\mathsf{Cat}(\mathcal{X}_{A,B}, T)} \right) \\
&= \mathsf{Grph}(\mathsf{Cat})(\mathcal{X}, \mathcal{B})^{\mathsf{Grph}(\mathsf{Cat})(\mathcal{X}, T)}
\end{aligned}$$

■

Theorem 4.1.7. *The 2-category* Icon *has Eilenberg-Moore objects for monads. These are preserved by the forgetful 2-functor* $U : \mathsf{Icon} \longrightarrow \mathsf{Grph}(\mathsf{Cat})$*, and created by the suspension 2-functor* $\Sigma : \mathsf{OpMon}(\mathsf{Cat}) \longrightarrow \mathsf{Icon}$.

Proof.

Since the elements of Icon are given by 2-dimensional operations and relations, there exists a presentation in the sense of [KP93] for a finitary 2-monad T_2 on $\mathsf{Grph}(\mathsf{Cat})$ such that $T_2\text{-}\mathsf{Alg}_{o\ell} = \mathsf{Icon}$. And since $\mathsf{Grph}(\mathsf{Cat})$ has Eilenberg-Moore objects for monads, the theorem follows by [Lac05, Proposition 4.11]. ∎

Hence, we can give a description of how to calculate Eilenberg-Moore objects in Icon. For a monad $(\mathcal{B}, T)$ in Icon the oplax functor T is forced to be the identity on objects because of the existence of the unit icon η. The Eilenberg-Moore object $\mathcal{B}^T$ is a bicategory that has the same objects as $\mathcal{B}$, its hom categories $\mathcal{B}^T(X, Y)$ are the categories of algebras for the monads induced by T on the hom categories $T : \mathcal{B}(X, Y) \longrightarrow \mathcal{B}(X, Y)$. Horizontal composition and identities are induced by the oplax constraints of T in the same way that a monad morphism induces a functor between the categories of algebras. Theorem 4.1.7 above may also be deduced as a special case of Theorem 4.3.27.

4.2 Internal Moerdijk-McCrudden Theorem

In this section we present an alternative proof of the Moerdijk-McCrudden Theorem. This proof has two purposes: first, it takes no effort to generalise from the language of monoidal categories and opmonoidal functors to the internal language of a monoidal bicategory $\mathcal{M}$. Second, and perhaps more importantly, it illustrates the general process that takes part during the next section without too much technicalities. Thus, it provides a motivation for all the tricategorical technology and the even more general version of Moerdijk-McCrudden Theorem that takes place in the next section. As a consequence, we intentionally leave this proof with some of its key points unproven since they easily follow from their more general counterparts. We begin by recalling some of the theory of monads for bicategories but with a tricategorical touch.

Definition 4.2.1. For a bicategory $\mathcal{B}$ there is a bicategory $\mathsf{Mnd}\,\mathcal{B}$ whose objects are monads (X, t) in $\mathcal{B}$; that is, an object X in $\mathcal{B}$, an arrow $t : X \longrightarrow X$, a multiplication cell μ, and a unit cell η as below, that satisfy the associative and unit laws.

$$\begin{tikzcd} X \arrow[rr, bend left, "t"] \arrow[dr, "t"'] & \Uparrow\mu & X \\ & X \arrow[ur, "t"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, bend left, "t"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\eta\Uparrow"] & X \end{tikzcd}$$

Arrows $(f, \varphi) : (X, t) \longrightarrow (X', t')$ in $\mathsf{Mnd}\,\mathcal{B}$ are *monad morphisms* between monads; that is, an arrow $f : X \longrightarrow X'$ and a cell φ in $\mathcal{B}$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow\varphi"] & X' \arrow[d, "t'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

satisfying the two axioms below.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dd, bend right=60, "t"'] \arrow[dr, phantom, "\swarrow\varphi"] & X' \arrow[d, "t'"] \\
X \arrow[r, "f"] \arrow[d, "t"'] \arrow[l, phantom, "\Leftarrow\mu"] \arrow[dr, phantom, "\swarrow\varphi"] & X' \arrow[d, "t'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dd, bend right=40, "t"'] \arrow[ddr, phantom, "\swarrow\varphi"] & X' \arrow[d, "t'"] \arrow[dd, bend right=40, "t'"'] \\
 & X' \arrow[d, "t'"] \arrow[l, phantom, "\Leftarrow\mu'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, bend right=50, "t"'] \arrow[d, bend left=50, "1"] \arrow[d, phantom, "\Leftarrow\eta"] & X' \arrow[d, bend left=50, "1"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, bend right=50, "t"'] \arrow[dr, phantom, "\swarrow\varphi"] & X' \arrow[d, bend right=50, "t'"'] \arrow[d, bend left=50, "1"] \arrow[d, phantom, "\Leftarrow\eta'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}
$$

Cells $\alpha : (f, \varphi) \longrightarrow (f', \varphi')$ in $\mathsf{Mnd}\,\mathcal{B}$ are *monad transformations* between monad morphisms in $\mathcal{B}$; that is, a cell α in $\mathcal{B}$,

$$
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & X'
\end{tikzcd}
$$

satisfying the axiom below.

$$
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow\varphi"] & X' \arrow[d, "t'"] \\
X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & X'
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow\varphi'"] & X' \arrow[d, "t'"] \\
X \arrow[r, bend right, "f'"'] & X'
\end{tikzcd}
$$

Vertical composition and identities are calculated as the ones in $\mathcal{B}(X, Y)$ and horizontal composition and identities are computed in the following way.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "\swarrow\varphi"] & X' \arrow[r, "g"] \arrow[d, "t'"'] \arrow[dr, phantom, "\swarrow\psi"] & X'' \arrow[d, "t''"] \\
X \arrow[r, "f"'] & X' \arrow[r, "g"'] & X''
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "t"'] & X \arrow[d, "t"] \\
X \arrow[r, "1"'] & X
\end{tikzcd}
$$

Remark 4.2.2. For a monoidal bicategory $\mathcal{M}$ the bicategory $\mathsf{Mnd}\,\mathcal{M}$ is also monoidal, with tensor product of monads, their morphisms, and transformations taken pointwise on the data and structure. For example, the tensor product of two monads (X, t) and (X', t') is given by,

$(X,t)(X',t') = (XX',tt')$ with unit $\eta\eta'$ and multiplication as shown.

$$\begin{tikzcd}
XX' \arrow[rr, bend left, "1t'"] \arrow[dr, "1t'"'] & \arrow[d, phantom, "1\mu'\Uparrow"] & XX' \arrow[rr, bend left, "t1"] \arrow[dr, "t1"'] & \arrow[d, phantom, "\mu 1\Uparrow"] & XX' \\
& XX' \arrow[ur, "1t'"'] \arrow[dr, "t1"'] & \cong & XX' \arrow[ur, "t1"'] & \\
& & XX' \arrow[ur, "1t'"'] & &
\end{tikzcd}$$

For the notation $\mathcal{F}^{\square}\varphi$ used in the following lemma, we refer the reader to the beginning of the present thesis in the background, notations, and conventions section.

Lemma 4.2.3. *For every pseudofunctor $\mathcal{F} : \mathcal{B} \longrightarrow \mathcal{C}$ there is a pseudofunctor $\mathsf{Mnd}\,\mathcal{F}$ between the bicategories of monads,*

$$\mathsf{Mnd}\,\mathcal{F} : \mathsf{Mnd}\,\mathcal{B} \longrightarrow \mathsf{Mnd}\,\mathcal{C}$$

defined as $\mathsf{Mnd}\,\mathcal{F}(X,t) = (\mathcal{F}X, \mathcal{F}t)$ for monads (X,t) in $\mathcal{B}$, as $(\mathcal{F}f, \mathcal{F}^{\square}\varphi)$ for monad morphisms $(f,\varphi) : (X,t) \longrightarrow (Y,u)$,

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\mathcal{F}f"] \arrow[d, "\mathcal{F}t"'] \arrow[dr, phantom, "\swarrow \mathcal{F}^{\square}\varphi"] & \mathcal{F}X' \arrow[d, "\mathcal{F}t'"] \\
\mathcal{F}X \arrow[r, "\mathcal{F}f"'] & \mathcal{F}X'
\end{tikzcd}$$

and for monad transformations α by $\mathsf{Mnd}\,\mathcal{F}(\alpha) = \mathcal{F}\alpha$. The pseudofunctoriality isomorphisms for $\mathsf{Mnd}\,\mathcal{F}$ are those of $\mathcal{F}$ on the underlying arrows of the monad morphisms. ■

Lemma 4.2.4. *For every pseudonatural transformation a,*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{C}
\end{tikzcd}$$

there is a pseudonatural transformation $\mathsf{Mnd}\,a$ as shown,

$$\begin{tikzcd}
\mathsf{Mnd}\,\mathcal{B} \arrow[r, bend left, "\mathsf{Mnd}\,\mathcal{F}"] \arrow[r, bend right, "\mathsf{Mnd}\,\mathcal{G}"'] \arrow[r, phantom, "\Downarrow \mathsf{Mnd}\,a"] & \mathsf{Mnd}\,\mathcal{C}
\end{tikzcd}$$

with components $(a_X, a_t) : (\mathcal{F}X, \mathcal{F}t) \longrightarrow (\mathcal{G}X, \mathcal{G}t)$ for monads (X,t) in $\mathsf{Mnd}\,\mathcal{B}$, and pseudonaturality components a_f for monad morphisms $(f,\varphi) : (X,t) \longrightarrow (X',t')$,

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "a_X"] \arrow[d, "\mathcal{F}f"'] \arrow[dr, phantom, "\swarrow\!\!\wr\, a_f"] & \mathcal{G}X \arrow[d, "\mathcal{G}f"] \\
\mathcal{F}X' \arrow[r, "a_{X'}"'] & \mathcal{G}X'
\end{tikzcd}$$

which are monad transformations by the pseudonaturality coherence for a. ■

Lemma 4.2.5. *For a modification*

$$\begin{tikzcd} \mathcal{B} \arrow[rr, bend left, "\mathcal{F}"] \arrow[rr, bend right, "\mathcal{G}"'] \arrow[rr, phantom, "a \Downarrow \overset{\Xi}{\Rrightarrow} \Downarrow b"] & & \mathcal{C} \end{tikzcd}$$

there is a modification $\mathsf{Mnd}\,\Xi$ *with components* $\Xi_X : (a_X, a_t) \longrightarrow (b_X, b_t)$. ■

Remark 4.2.6. What we have described above is a trifunctor.

$$\mathsf{Mnd} : \mathsf{Bicat} \longrightarrow \mathsf{Bicat}$$

In fact this functor lifts to the tricategory $\mathsf{MonBicat}$ of monoidal bicategories, monoidal pseudofunctors, monoidal pseudonatural transformations, and monoidal modifications.

$$\mathsf{Mnd} : \mathsf{MonBicat} \longrightarrow \mathsf{MonBicat}$$

Since we intend to generalise the Moerdijk-McCrudden Theorem and it involves Eilenberg-Moore objects for monads in Cat, we recall from [Str72] how to globally define Eilenberg-Moore objects in a bicategory $\mathcal{B}$.

Notation. For a bicategory $\mathcal{B}$ with Eilenberg-Moore objects, denote by X^t the Eilenberg-Moore object for a monad t on an object X, $f^{\varphi} : X^t \longrightarrow Y^s$ the arrow induced by a monad morphism $(f, \varphi) : (X, t) \longrightarrow (Y, s)$ between the Eilenberg-Moore objects, and $\widehat{\alpha} : f^{\varphi} \longrightarrow g^{\psi}$ for the cell induced by a monad transformation $\alpha : (f, \varphi) \longrightarrow (g, \psi)$.

If a bicategory $\mathcal{B}$ has Eilenberg-Moore objects for monads the axiom of choice allows us to form a pseudofunctor EM as below.

$$\mathsf{Mnd}\,\mathcal{B} \xrightarrow{\quad \mathsf{EM} \quad} \mathcal{B}$$

$$\begin{tikzcd} (X,t) \arrow[d, bend right, "(f,\varphi)"'] \arrow[d, bend left, "(g,\psi)"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ (Y,s) \end{tikzcd} \longmapsto \begin{tikzcd} X^t \arrow[d, bend right, "f^{\varphi}"'] \arrow[d, bend left, "g^{\psi}"] \arrow[d, phantom, "\overset{\widehat{\alpha}}{\Rightarrow}"] \\ Y^s \end{tikzcd}$$

Now, for a global definition in a bicategory $\mathcal{B}$ let $\mathsf{Inc} : \mathcal{B} \longrightarrow \mathsf{Mnd}\,\mathcal{B}$ denote the inclusion strict functor that takes an object X to the identity monad (X, id_X).

Proposition 4.2.7. *A bicategory $\mathcal{B}$ has Eilenberg-Moore objects for monads if and only if the pseudofunctor* Inc *has a right biadjoint in the tricategory* Bicat.

$$\begin{tikzcd} \mathsf{Mnd}\,\mathcal{B} \arrow[d, bend left, "\mathsf{EM}"] \arrow[d, phantom, "\dashv"] \\ \mathcal{B} \arrow[u, bend left, "\mathsf{Inc}"] \end{tikzcd}$$

Proof.
The result follows from the equivalence $\mathsf{Mnd}\,\mathcal{B}(\mathsf{Inc}(X),(Y,s)) \simeq \mathcal{B}(X,Y)^{\mathcal{B}(X,s)}$ [Str72, Theorem 8]. ∎

Remark 4.2.8. The inclusion pseudofunctor always has a left pseudoadjoint Und which takes the underlying object X of a monad (X,t).

$$\mathcal{B}(\mathsf{Und}(X,t),Y) = \mathcal{B}(X,Y) \cong \mathsf{Mnd}\,\mathcal{B}((X,t),\mathsf{Inc}(Y))$$

$$\begin{tikzcd}
\mathsf{Mnd}\,\mathcal{B} \arrow[d, bend right=60, "\mathsf{Und}"', ""{name=U}] \arrow[d, bend left=60, "\mathsf{EM}", ""{name=E}] \\
\mathcal{B} \arrow[u, "\mathsf{Inc}"{description}, ""{name=I}]
\arrow[from=U, to=I, phantom, "\dashv"] \arrow[from=I, to=E, phantom, "\dashv"]
\end{tikzcd}$$

We now provide the layout of an alternative proof of the Moerdijk-McCrudden Theorem which is then easy to generalise to the multiobject case.

Theorem (Moerdijk-McCrudden)**.** $\mathsf{OpMon(Cat)}$ *has Eilenberg-Moore objects for monads, and the forgetful functor* $U : \mathsf{OpMon(Cat)} \longrightarrow \mathsf{Cat}$ *preserves them.*

Proof. [Idea]
The 2-category Cat has Eilenberg-Moore objects for monads. Now, if $\mathsf{OpMon}(\mathcal{M})$ is trifunctorial in $\mathcal{M}$, we may apply it to the 2-adjunction $\mathsf{Inc} \dashv \mathsf{EM}$.

$$\begin{tikzcd}
\mathsf{Mnd(Cat)} \arrow[d, bend left, "\mathsf{EM}"] \arrow[d, phantom, "\dashv"] & & \mathsf{OpMon(Mnd(Cat))} \arrow[d, bend left, "\mathsf{OpMon(EM)}"] \arrow[d, phantom, "\dashv"] \\
\mathsf{Cat} \arrow[u, bend left, "\mathsf{Inc}"] & \arrow[r, mapsto, squiggly] & \mathsf{OpMon(Cat)} \arrow[u, bend left, "\mathsf{OpMon(Inc)}"]
\end{tikzcd}$$

Moreover, there is an isomorphism of 2-categories $\mathsf{OpMon\,Mnd(Cat)} \cong \mathsf{Mnd(OpMon\,Cat)}$, and it makes the triangle below commute.

$$\begin{tikzcd}
\mathsf{OpMon(Cat)} \arrow[r, "\mathsf{OpMon(Inc)}"] \arrow[rr, bend right=15, "\mathsf{Inc}"'] & \mathsf{OpMon(Mnd(Cat))} \cong & \mathsf{Mnd(OpMon(Cat))}
\end{tikzcd}$$

And so one obtain a new 2-adjunction as shown below which witnesses the existence of Eilenberg-Moore objects in $\mathsf{OpMon(Cat)}$, where the dotted arrow is the composite of the pictured isomorphism followed by $\mathsf{OpMon(EM)}$.

$$\begin{tikzcd}
\mathsf{Mnd(OpMon(Cat))} \arrow[dd, bend left=70, dotted, "\mathsf{EM}"] \\
\mathsf{OpMon(Mnd(Cat))} \arrow[u, phantom, "\cong" sloped] \arrow[d, bend left, "\mathsf{OpMon(EM)}"] \arrow[d, phantom, "\dashv"] \\
\mathsf{OpMon(Cat)} \arrow[u, bend left, "\mathsf{OpMon(Inc)}"] \arrow[uu, bend left=70, "\mathsf{Inc}"]
\end{tikzcd}$$

Finally, if $U : \mathsf{OpMon}(\mathcal{M}) \longrightarrow \mathcal{M}$ is trinatural in $\mathcal{M}$, then by naturality one may obtain the following commutative diagram.

$$\begin{tikzcd}
\mathsf{Mnd}(\mathsf{OpMon}(\mathsf{Cat})) \arrow[r, "\mathsf{Mnd}(U)"] \arrow[d, phantom, "\cong"] \arrow[dd, bend right=60, "\mathsf{EM}"'] & \mathsf{Mnd}(\mathsf{Cat}) \arrow[dd, "\mathsf{EM}"] \\
\mathsf{OpMon}(\mathsf{Mnd}(\mathsf{Cat})) \arrow[ur, "U"'] \arrow[d, "\mathsf{OpMon}(\mathsf{EM})"'] & \\
\mathsf{OpMon}(\mathsf{Cat}) \arrow[r, "U"'] & \mathsf{Cat}
\end{tikzcd}$$

The outer square of the diagram witnesses that the forgetful functor $U : \mathsf{OpMon}(\mathsf{Cat}) \longrightarrow \mathsf{Cat}$ preserves Eilenberg-Moore objects for monads. ■

Remark 4.2.9. For the proof above to be complete one has to verify that the following three key points hold true.

- There is a trifunctor $\mathsf{OpMon}(\_) : \mathsf{MonBicat} \longrightarrow \mathsf{Bicat}$.
- There is a trinatural transformation

$$\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[r, bend right, "(_)"'] \arrow[r, phantom, "\Downarrow U"] & \mathsf{Bicat}
\end{tikzcd}$$

  where $(\_) : \mathsf{MonBicat} \longrightarrow \mathsf{Bicat}$ denotes the trifunctor that forgets the monoidal structure.

- There is an isomorphism $\mathsf{OpMon}(\mathsf{Mnd}(\mathsf{Cat})) \cong \mathsf{Mnd}(\mathsf{OpMon}(\mathsf{Cat}))$.

Since our intention is to generalise this theorem, we are not going to prove these points until the next section in Theorems 4.3.19, 4.3.21, and 4.3.22 where they easily follow from their more general variants.

One may replace every appearance of Cat in the proof above by a monoidal bicategory $\mathcal{M}$ with Eilenberg-Moore objects for monads, and the proof follows in the same way but this time using the following isomorphism instead $\mathsf{OpMon}(\mathsf{Mnd}(\mathcal{M})) \cong \mathsf{Mnd}(\mathsf{OpMon}(\mathcal{M}))$.

Theorem 4.2.10. *For a monoidal bicategory $\mathcal{M}$, the bicategory $\mathsf{OpMon}(\mathcal{M})$ has Eilenberg-Moore objects for monads if $\mathcal{M}$ does so, and the forgetful functor to $U : \mathsf{OpMon}(\mathcal{M}) \longrightarrow \mathcal{M}$ preserves them.* ■

Remark 4.2.11. The theorem above appears in [DS04, Lemma 3.2]. A version of it where the domain of $\mathsf{OpMon}(\_)$ is a 3-category of cartesian monoidal 2-categories may be found in [Zaw12, Theorem 5.1].

4.3 Enriched Icons

In this section we combine the two generalisations of the Moerdijk-McCrudden Theorem in Theorems 4.1.7 and 4.2.10 together into a common framework in Theorem 4.3.27 using bicategories enriched in monoidal bicategories. One recovers Theorem 4.1.7 by enriching in the cartesian monoidal 2-category Cat, and Theorem 4.2.10 by considering one object enriched bicategories. We prove the analogues of each key point in Remark 4.2.9 in this context. First, we show the existence of a trifunctor in Theorem 4.3.16 that one may think as an "enrichment trifunctor": it takes a monoidal bicategory $\mathcal{M}$ to the bicategory $\mathsf{Icon}\,\mathcal{M}$ of $\mathcal{M}$-enriched bicategories, $\mathcal{M}$-enriched oplax functors, and $\mathcal{M}$-enriched icons. And second, we prove in Theorem 4.3.22 that this "enrichment trifunctor" commutes with the monad trifunctor described in Remark 4.2.6.

Enriched bicategories date back to [Car95], enriched pseudofunctors, enriched transformations, and enriched modifications are considered in [Lac95] together with change of base along monoidal pseudofunctors. In [CG14], under the name of weakly enriched bicategories, the authors consider an enriched version of Lack's 2-category of bicategories $\mathsf{Bicat}_{\mathbf{2}}$. For that, the authors define enriched icons which play a fundamental role in this chapter. They iterate the enrichment process in the case that the enriching monoidal bicategory is symmetric, and with this iteration they study the structure of k-monoidal n-categories. In [GS16] the authors give a comprehensive account to the theory of enriched bicategories to date and develop the theory to the extent of free cocompletions under a certain kind of weighted bicolimits. In [GS16] and [CG14], as in [Lac10b] enriched icons are between lax enriched functors whereas our enriched icons are between oplax enriched functors.

Definition 4.3.1. Let $\mathcal{M}$ be a monoidal bicategory. An *$\mathcal{M}$-bicategory* (or *$\mathcal{M}$-enriched bicategory*) $\mathcal{B}$ consists of the following data.

- A collection $\mathsf{Ob}\,\mathcal{B}$ of objects.
- For each pair of objects X and Y of $\mathcal{B}$, an object $\mathcal{B}_{X,Y}$ in $\mathcal{M}$ called the $\mathcal{M}$-hom object.
- For each triple of objects X, Y, and Z of $\mathcal{B}$, an arrow m in $\mathcal{M}$ called the composition arrow.
$$\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \xrightarrow{m} \mathcal{B}_{X,Z}$$
- For each object X of $\mathcal{B}$, an arrow u in $\mathcal{M}$ called the identities arrow.
$$I \xrightarrow{u} \mathcal{B}_{X,X}$$
- For each quadruple objects W, X, Y and Z in $\mathcal{B}$ an invertible cell α in $\mathcal{M}$ called the associator cell.
$$\begin{tikzcd} \mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}\mathcal{B}_{W,X} \arrow[r, "1m"] \arrow[d, "m1"'] \arrow[dr, phantom, "\alpha"] & \mathcal{B}_{X,Z}\mathcal{B}_{W,X} \arrow[d, "m"] \\ \mathcal{B}_{X,Z}\mathcal{B}_{W,X} \arrow[r, "m"'] & \mathcal{B}_{W,Z} \end{tikzcd}$$

- For each pair of objects X and Y of $\mathcal{B}$, an invertible cell λ in $\mathcal{M}$ called the left unitor cell.

$$\begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, "u1"] \arrow[dr, "1"'] & \mathcal{B}_{Y,Y}\mathcal{B}_{X,Y} \arrow[d, "m"] \arrow[dl, phantom, "\lambda\cong"] \\ & \mathcal{B}_{X,Y} \end{tikzcd}$$

- For each pair of objects X and Y of $\mathcal{B}$, an invertible cell ρ in $\mathcal{M}$ called the right unitor cell.

$$\begin{tikzcd} \mathcal{B}_{X,Y}\mathcal{B}_{X,X} \arrow[d, "m"'] & \mathcal{B}_{X,Y} \arrow[l, "1u"'] \arrow[dl, "1"] \arrow[dl, phantom, "\cong\rho"] \\ \mathcal{B}_{X,Y} \end{tikzcd}$$

These are subject to the pentagon and triangle axioms for α, λ and ρ.

Definition 4.3.2. Let $\mathcal{M}$ be a monoidal bicategory, and let $\mathcal{B}$ and $\mathcal{C}$ be a pair $\mathcal{M}$-bicategories. An *oplax $\mathcal{M}$-functor* (or $\mathcal{M}$-enriched oplax functor) $F : \mathcal{B} \longrightarrow \mathcal{C}$ consists of the following data.

- A function $F : \mathsf{Ob}\,\mathcal{B} \longrightarrow \mathsf{Ob}\,\mathcal{C}$.

- For each pair of objects X and Y in $\mathcal{B}$, an arrow $\mathcal{B}_{X,Y} \xrightarrow{F} \mathcal{C}_{FX,FY}$ in $\mathcal{M}$ referred to as the action of F on $\mathcal{M}$-hom objects.

- For each triple of objects X, Y and Z in $\mathcal{B}$, a cell F^2 in $\mathcal{M}$ referred to as the oplax composition constraint.

$$\begin{tikzcd} \mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \arrow[r, "FF"] \arrow[d, "m"'] \arrow[dr, phantom, "F^2\Nearrow"] & \mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY} \arrow[d, "m"] \\ \mathcal{B}_{X,Z} \arrow[r, "F"'] & \mathcal{C}_{FX,FZ} \end{tikzcd}$$

- For each object X of $\mathcal{B}$, a cell F^0 in $\mathcal{M}$ referred to as the oplax identities constraint.

$$\begin{tikzcd} & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "F^0\Nearrow"] & \\ \mathcal{B}_{X,X} \arrow[rr, "F"'] & & \mathcal{C}_{FX,FX} \end{tikzcd}$$

These are subject to three axioms for an enriched oplax functor, directly generalising (OLF1), (OLF2) and (OLF3) from Section 4.1.

If in the definition of an oplax $\mathcal{M}$-functor, the oplax composition constraint and the oplax identities constraint are isomorphisms in $\mathcal{M}$ we speak of an $\mathcal{M}$-*pseudofunctor*.

Definition 4.3.3. Let $\mathcal{M}$ be a monoidal bicategory, F and $G : \mathcal{B} \longrightarrow \mathcal{C}$ a parallel pair of oplax $\mathcal{M}$-functors such that $FX = GX$ for every object X of $\mathcal{B}$. An $\mathcal{M}$-*icon* α consists of, for each pair of objects X and Y of $\mathcal{B}$, a cell α_{XY} in $\mathcal{M}$ called the oplax components.

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha\Uparrow"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha_{XY}\Uparrow"] & \mathcal{C}_{FX,FY} \end{tikzcd}$$

These are subject to two compatibility conditions with respect to composition and identities, directly generalising (ICON1) and (ICON2) from Section 4.1.

Proposition 4.3.4. *For every monoidal bicategory $\mathcal{M}$ there is a bicategory* $\mathsf{Icon}\,\mathcal{M}$*, whose objects are $\mathcal{M}$-bicategories, arrows are oplax $\mathcal{M}$-functors, and cells are $\mathcal{M}$-icons. Composition, identities, associators, and unitors are calculated locally as the ones in $\mathcal{M}$.*

$$\begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, bend left=60, "H"] \arrow[r, "G" description] \arrow[r, bend right=60, "F"'] \arrow[r, phantom, shift left=3, "\alpha'_{XY}\Uparrow"] \arrow[r, phantom, shift right=3, "\alpha_{XY}\Uparrow"] & \mathcal{C}_{FX,FY} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha_{XY}\Uparrow"] & \mathcal{C}_{FX,FY} \arrow[r, bend left, "G'"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\beta_{XY}\Uparrow"] & \mathcal{C}_{F'FX,F'FY} \end{tikzcd}$$

■

Remark 4.3.5. The full subbicategory of $\mathsf{Icon}\,\mathcal{M}$ on the $\mathcal{M}$-bicategories with exactly one object is isomorphic to $\mathsf{OpMon}\,\mathcal{M}$. The isomorphism is witnessed by the strict fully faithful *suspension* functor,

$$\mathsf{OpMon}\,\mathcal{M} \xrightarrow{\Sigma} \mathsf{Icon}\,\mathcal{M}$$

which takes a monoidale M to the $\mathcal{M}$-bicategory ΣM with exactly one object $\star$, whose $\mathcal{M}$-hom object $\Sigma M_{\star,\star}$ is M, the composition arrow of ΣM is the product arrow of M, the identities arrow of ΣM is the unit arrow of M, and the associator and left and right unitors of ΣM are those of M.

Remark 4.3.6. In a similar way that graphs enriched in Cat are defined, one may define a graph enriched in a monoidal bicategory $\mathcal{M}$ (in fact the monoidal structure is not needed). So an $\mathcal{M}$-*enriched graph* or $\mathcal{M}$-*graph* $\mathcal{B}$ consists of a collection of vertices, and for each pair of vertices X and Y a hom object $\mathcal{B}_{X,Y}$ in $\mathcal{M}$. Similarly a morphism of $\mathcal{M}$-graphs $F : \mathcal{B} \longrightarrow \mathcal{C}$ consists of a function between the collections of vertices, and for each pair of vertices X and Y in $\mathcal{B}$ an arrow $F : \mathcal{B}_{X,Y} \longrightarrow \mathcal{C}_{FX,FY}$ in $\mathcal{M}$. A transformation of $\mathcal{M}$-graphs α between a parallel pair of morphisms of $\mathcal{M}$-graphs F and $G : \mathcal{B} \longrightarrow \mathcal{C}$ such that $FX = GX$ for every vertex X in $\mathcal{B}$ consists of a collection of cells $\alpha_{X,Y}$ for each pair of objects X and Y in $\mathcal{B}$ as below.

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha\Uparrow"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha_{XY}\Uparrow"] & \mathcal{C}_{FX,FY} \end{tikzcd}$$

These compose and have identities in the obvious way, arranging themselves into a bicategory $\mathsf{Grph}\,\mathcal{M}$. Thus, there is a strict forgetful functor U which forgets about everything that needs the monoidal structure of $\mathcal{M}$.

$$U : \mathsf{Icon}\,\mathcal{M} \longrightarrow \mathsf{Grph}\,\mathcal{M}$$

4.3.1 Change of base along a monoidal pseudofunctor

Let $\mathcal{M}$ and $\mathcal{N}$ be two monoidal bicategories and $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ a monoidal pseudofunctor between them. We describe a pseudofunctor $\mathsf{Icon}\,\mathcal{F} : \mathsf{Icon}\,\mathcal{M} \longrightarrow \mathsf{Icon}\,\mathcal{N}$.

Lemma 4.3.7. *For every $\mathcal{M}$-bicategory $\mathcal{B}$ there is an $\mathcal{N}$-bicategory $\mathsf{Icon}\,\mathcal{F}(\mathcal{B})$ whose set of objects is equal to $\mathsf{Ob}\,\mathcal{B}$, whose $\mathcal{N}$-hom objects are the objects $\mathcal{F}(\mathcal{B}_{X,Y})$, whose composition and identities arrows are given by the two arrows below,*

$$\mathcal{F}(\mathcal{B}_{Y,Z})\mathcal{F}(\mathcal{B}_{X,Y}) \xrightarrow{\mathcal{F}_2} \mathcal{F}(\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}) \xrightarrow{\mathcal{F}(m)} \mathcal{F}(\mathcal{B}_{X,Z})$$

$$I \xrightarrow{\mathcal{F}_0} \mathcal{F}(I) \xrightarrow{\mathcal{F}(u)} \mathcal{F}(\mathcal{B}_{XX})$$

and whose associator cells, left unitor cells, and right unitor cells are defined by the three pasting diagrams below, where the isomorphisms with no name are instances of the pseudo-naturality of $\mathcal{F}^2$.

$$\begin{tikzcd}
\mathcal{F}(\mathcal{B}_{Z,W})\mathcal{F}(\mathcal{B}_{Y,Z})\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}_2 1"] \arrow[d, "1\mathcal{F}_2"'] \arrow[dr, phantom, "\omega\cong"] & \mathcal{F}(\mathcal{B}_{Z,W}\mathcal{B}_{Y,Z})\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}(m)1"] \arrow[d, "\mathcal{F}_2"'] \arrow[dr, phantom, "\cong"] & \mathcal{F}(\mathcal{B}_{Y,Z})\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[d, "\mathcal{F}_2"] \\
\mathcal{F}(\mathcal{B}_{Z,W})\mathcal{F}(\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}_2"] \arrow[d, "1\mathcal{F}(m)"'] \arrow[dr, phantom, "\cong"] & \mathcal{F}(\mathcal{B}_{Z,W}\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}(m1)"] \arrow[d, "\mathcal{F}(1m)"'] \arrow[dr, phantom, "\mathcal{F}^{\square}\alpha\cong"] & \mathcal{F}(\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}) \arrow[d, "\mathcal{F}(m)"] \\
\mathcal{F}(\mathcal{B}_{Y,Z})\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}_2"'] & \mathcal{F}(\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}(m)"'] & \mathcal{F}(\mathcal{B}_{X,Y})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}_0 1"] \arrow[dr, "1"'] & \mathcal{F}(I)\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}(u)1"] \arrow[d, "\mathcal{F}_2"] \arrow[dl, phantom, "\gamma\cong"] \arrow[dr, phantom, "\cong"] & \mathcal{F}(\mathcal{B}_{Y,Y})\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[d, "\mathcal{F}_2"] \\
& \mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}(u1)"] \arrow[dr, "1"'] & \mathcal{F}(\mathcal{B}_{Y,Y}\mathcal{B}_{X,Y}) \arrow[d, "\mathcal{F}(m)"] \arrow[dl, phantom, "\mathcal{F}^{\triangledown}\lambda\cong"] \\
& & \mathcal{F}(\mathcal{B}_{Y,Y})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(\mathcal{B}_{X,Y})\mathcal{F}(\mathcal{B}_{X,X}) \arrow[d, "\mathcal{F}_2"'] \arrow[dr, phantom, "\cong"] & \mathcal{F}(\mathcal{B}_{X,Y})\mathcal{F}(I) \arrow[l, "1\mathcal{F}(u)"'] \arrow[d, "\mathcal{F}_2"] \arrow[dr, phantom, "\cong\delta"] & \mathcal{F}(\mathcal{B}_{X,Y}) \arrow[l, "1\mathcal{F}_0"'] \arrow[dl, "1"] \\
\mathcal{F}(\mathcal{B}_{X,Y}\mathcal{B}_{X,X}) \arrow[d, "\mathcal{F}(m)"'] \arrow[dr, phantom, "\cong\mathcal{F}^{\triangledown}\rho"] & \mathcal{F}(\mathcal{B}_{X,Y}) \arrow[l, "\mathcal{F}(1u)"'] \arrow[dl, "1"] & \\
\mathcal{F}(\mathcal{B}_{X,Y}) & &
\end{tikzcd}$$

The two axioms for an $\mathcal{N}$-bicategory follow from the coherence axioms between ω, γ, and δ, and from the two axioms for the $\mathcal{M}$-bicategory $\mathcal{B}$ between α, λ, and ρ appropriately translated into axioms for $\mathcal{F}^{\square}\alpha$, $\mathcal{F}^{\triangledown}\lambda$, and $\mathcal{F}^{\triangledown}\rho$ by the pseudofunctoriality of $\mathcal{F}$. ∎

Lemma 4.3.8. *For every pair of $\mathcal{M}$-bicategories $\mathcal{B}$ and $\mathcal{C}$, and every oplax $\mathcal{M}$-functor $F : \mathcal{B}\longrightarrow\mathcal{C}$ there is an oplax $\mathcal{N}$-functor $\mathsf{Icon}\,\mathcal{F}(F) : \mathsf{Icon}\,\mathcal{F}(\mathcal{B})\longrightarrow\mathsf{Icon}\,\mathcal{F}(\mathcal{C})$ defined on objects in the same way as F, whose action on $\mathcal{N}$-hom objects is the arrow $\mathcal{F}(F) : \mathcal{F}(\mathcal{B}_{X,Y})\longrightarrow\mathcal{F}(\mathcal{C}_{FX,FY})$ in $\mathcal{N}$, and whose oplax composition and oplax identities constraints are given by the pasting diagrams below.*

$$\begin{tikzcd}
\mathcal{F}(\mathcal{B}_{Y,Z})\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}(F)\mathcal{F}(F)"] \arrow[d, "\mathcal{F}_2"'] \arrow[dr, phantom, "\cong"] & \mathcal{F}(\mathcal{C}_{FY,FZ})\mathcal{F}(\mathcal{C}_{FX,FY}) \arrow[d, "\mathcal{F}_2"] \\
\mathcal{F}(\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}) \arrow[r, "\mathcal{F}(FF)"] \arrow[d, "\mathcal{F}(m)"'] \arrow[dr, phantom, "\mathcal{F}^{\square}(F^2)\Nearrow"] & \mathcal{F}(\mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY}) \arrow[d, "\mathcal{F}(m)"] \\
\mathcal{F}(\mathcal{B}_{X,Z}) \arrow[r, "\mathcal{F}(F)"'] & \mathcal{F}(\mathcal{C}_{FX,FZ})
\end{tikzcd}$$

$$\begin{tikzcd}
& I \arrow[d, "\mathcal{F}_0"'] & \\
& \mathcal{F}(I) \arrow[dl, "\mathcal{F}(u)"'] \arrow[dr, "\mathcal{F}(u)"] \arrow[d, phantom, "\mathcal{F}^{\triangledown}(F^0)\Nearrow"] & \\
\mathcal{F}(\mathcal{B}_{X,X}) \arrow[rr, "\mathcal{F}(F)"'] & {} & \mathcal{F}(\mathcal{C}_{FX,FX})
\end{tikzcd}$$

The three axioms for an oplax $\mathcal{N}$-functor follow from the three coherence axioms for F^2 and F^0 appropriately translated into axioms for $\mathcal{F}^{\square}(F^2)$ and $\mathcal{F}^{\triangledown}(F^0)$ by the pseudofunctoriality of $\mathcal{F}$. ∎

Lemma 4.3.9. *For every parallel pair F and $G : \mathcal{B}\longrightarrow\mathcal{C}$ of oplax $\mathcal{M}$-functors such that $FX =$*

GX for every X in $\mathcal{B}$, and for every $\mathcal{M}$-icon α between them,

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha\Uparrow"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Icon}\,\mathcal{F}(\mathcal{B}) \arrow[r, bend left, "\mathsf{Icon}\,\mathcal{F}(G)"] \arrow[r, bend right, "\mathsf{Icon}\,\mathcal{F}(F)"'] \arrow[r, phantom, "\mathsf{Icon}\,\mathcal{F}(\alpha)\Uparrow"] & \mathsf{Icon}\,\mathcal{F}(\mathcal{C}) \end{tikzcd}$$

there is an $\mathcal{N}$-icon $\mathsf{Icon}\,\mathcal{F}(\alpha)$ between $\mathsf{Icon}\,\mathcal{F}(F)$ and $\mathsf{Icon}\,\mathcal{F}(G)$ whose oplax components are the cells

$$\begin{tikzcd} \mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, bend left, "\mathcal{F}(G)"] \arrow[r, bend right, "\mathcal{F}(F)"'] \arrow[r, phantom, "\mathcal{F}(\alpha_{XY})\Uparrow"] & \mathcal{F}(\mathcal{C}_{FX,FY}). \end{tikzcd}$$

The oplax $\mathcal{N}$-functors $\mathsf{Icon}\,\mathcal{F}(F)$ and $\mathsf{Icon}\,\mathcal{F}(G)$ indeed coincide on objects since F and G do. The two axioms for an $\mathcal{N}$-icon follow from those of α appropriately translated by the pseudofunctoriality of $\mathcal{F}$. ∎

For a pair of $\mathcal{M}$-bicategories $\mathcal{B}$ and $\mathcal{C}$, the three lemmas in this subsection define a functor

$$\mathsf{Icon}\,\mathcal{M}(\mathcal{B},\mathcal{C}) \xrightarrow{\mathsf{Icon}\,\mathcal{F}} \mathsf{Icon}\,\mathcal{N}(\mathsf{Icon}\,\mathcal{F}(\mathcal{B}), \mathsf{Icon}\,\mathcal{F}(\mathcal{C}))$$

whose functoriality follows from the fact that $\mathcal{F}$ strictly preserves vertical composition and identities.

Proposition 4.3.10. *For every monoidal pseudofunctor $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ there is a pseudofunctor $\mathsf{Icon}\,\mathcal{F} : \mathsf{Icon}\,\mathcal{M} \longrightarrow \mathsf{Icon}\,\mathcal{N}$ whose pseudofunctoriality follows from that of $\mathcal{F}$.*

Proof.

The pseudofunctoriality constraints are defined as follows. For a triple of $\mathcal{M}$-bicategories $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ there are two invertible natural transformations as shown.

$$\begin{tikzcd} \mathsf{Icon}\,\mathcal{M}(\mathcal{C},\mathcal{D}) \times \mathsf{Icon}\,\mathcal{M}(\mathcal{B},\mathcal{C}) \arrow[r, "\mathsf{Icon}\,\mathcal{F}\times\mathsf{Icon}\,\mathcal{F}"'] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathsf{Icon}\,\mathcal{N}(\mathsf{Icon}\,\mathcal{F}(\mathcal{C}), \mathsf{Icon}\,\mathcal{F}(\mathcal{D})) \times \mathsf{Icon}\,\mathcal{N}(\mathsf{Icon}\,\mathcal{F}(\mathcal{B}), \mathsf{Icon}\,\mathcal{F}(\mathcal{C})) \arrow[d] \\ \mathsf{Icon}\,\mathcal{M}(\mathcal{B},\mathcal{D}) \arrow[r, "\mathsf{Icon}\,\mathcal{F}"'] & \mathsf{Icon}\,\mathcal{N}(\mathsf{Icon}\,\mathcal{F}(\mathcal{B}), \mathsf{Icon}\,\mathcal{F}(\mathcal{D})) \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "\cong"] & \\ \mathsf{Icon}\,\mathcal{M}(\mathcal{B},\mathcal{B}) \arrow[rr, "\mathsf{Icon}\,\mathcal{F}"'] & & \mathsf{Icon}\,\mathcal{N}(\mathsf{Icon}\,\mathcal{F}(\mathcal{B}), \mathsf{Icon}\,\mathcal{F}(\mathcal{B})) \end{tikzcd}$$

These are defined, for a composable pair of oplax $\mathcal{M}$-functors F and F', by $\mathcal{N}$-icons whose source and target indeed coincide

$$\begin{tikzcd} X \arrow[rr, mapsto, bend left] \arrow[dr, mapsto] & & F'FX \\ & FX \arrow[ur, mapsto] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, mapsto, bend left] \arrow[r, mapsto, bend right] & X \end{tikzcd}$$

and whose oplax components are given by pseudofunctoriality isomorphisms of $\mathcal{F}$.

$$\begin{tikzcd} \mathcal{F}(\mathcal{B}_{X,Y}) \arrow[rr, bend left, "\mathcal{F}(F'\circ F)"] \arrow[dr, "\mathcal{F}(F)"'] & \cong & \mathcal{F}(\mathcal{D}_{F'FX,F'FY}) \\ & \mathcal{F}(\mathcal{C}_{FX,FY}) \arrow[ur, "\mathcal{F}(F')"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, bend left, "\mathcal{F}(\mathrm{id})"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\cong"] & \mathcal{F}(\mathcal{B}_{X,Y}) \end{tikzcd}$$

These isomorphisms satisfy the two axioms for an $\mathcal{N}$-icon because of the compatibility axioms between the pseudofunctoriality constraints and the monoidal constraints of $\mathcal{F}$. Now, these $\mathcal{N}$-icons constitute the components of the proposed natural transformations pictured above, and the naturality condition follows from the naturality of the pseudofunctoriality constraints of $\mathcal{F}$. And the two axioms that make $\mathsf{Icon}\,\mathcal{F} : \mathsf{Icon}\,\mathcal{M} \longrightarrow \mathsf{Icon}\,\mathcal{N}$ into a pseudofunctor follow from their respective counterparts for $\mathcal{F}$. ∎

4.3.2 Change of change of base along a monoidal oplax natural transformation

Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal bicategories, $\mathcal{F}$ and $\mathcal{G} : \mathcal{M} \longrightarrow \mathsf{Icon}\,\mathcal{N}$ a parallel pair of monoidal pseudofunctors, and a a monoidal oplax natural transformation between $\mathcal{F}$ and $\mathcal{G}$.

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow a"] & \mathcal{N} \end{tikzcd}$$

We describe an oplax natural transformation $\mathsf{Icon}\,a$ between $\mathsf{Icon}\,\mathcal{F}$ and $\mathsf{Icon}\,\mathcal{G}$.

$$\begin{tikzcd} \mathsf{Icon}\,\mathcal{M} \arrow[r, bend left, "\mathsf{Icon}\,\mathcal{F}"] \arrow[r, bend right, "\mathsf{Icon}\,\mathcal{G}"'] \arrow[r, phantom, "\Downarrow \mathsf{Icon}\,a"] & \mathsf{Icon}\,\mathcal{N} \end{tikzcd}$$

Lemma 4.3.11. *For every $\mathcal{M}$-bicategory $\mathcal{B}$ there is an oplax $\mathcal{N}$-functor*

$$(\mathsf{Icon}\,a)_{\mathcal{B}} : \mathsf{Icon}\,\mathcal{F}(\mathcal{B}) \longrightarrow \mathsf{Icon}\,\mathcal{G}(\mathcal{B})$$

defined on objects as the identity on the objects of $\mathcal{B}$, whose action on $\mathcal{N}$-homs is the arrow

$a_{\mathcal{B}_{XY}} : \mathcal{F}(\mathcal{B}_{X,Y}) \longrightarrow \mathcal{G}(\mathcal{B}_{X,Y})$ *in* $\mathcal{N}$*, and whose oplax composition and oplax identities constraints are given by the pasting diagrams below.*

$$\begin{tikzcd}
\mathcal{F}(\mathcal{B}_{Y,Z})\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "a_{\mathcal{B}_{YZ}} a_{\mathcal{B}_{XY}}"] \arrow[d, "\mathcal{F}_2"'] \arrow[dr, phantom, "\Pi_{XYZ} \Swarrow"] & \mathcal{G}(\mathcal{B}_{Y,Z})\mathcal{G}(\mathcal{B}_{X,Y}) \arrow[d, "\mathcal{G}_2"] \\
\mathcal{F}(\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}) \arrow[r, "a_{(\mathcal{B}_{YZ}\mathcal{B}_{XY})}"] \arrow[d, "\mathcal{F}(m)"'] \arrow[dr, phantom, "a_m \Nearrow"] & \mathcal{G}(\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}) \arrow[d, "\mathcal{G}(m)"] \\
\mathcal{F}(\mathcal{B}_{X,Z}) \arrow[r, "a_{\mathcal{B}_{X,Z}}"'] & \mathcal{G}(\mathcal{B}_{X,Z})
\end{tikzcd}$$

$$\begin{tikzcd}
& I \arrow[dl, "\mathcal{F}_0"'] \arrow[dr, "\mathcal{G}_0"] \arrow[d, phantom, "M \Swarrow"] & \\
\mathcal{F}(I) \arrow[rr, "a_I"'] \arrow[d, "\mathcal{F}(u)"'] \arrow[drr, phantom, "a_u \Nearrow"] & & \mathcal{G}(I) \arrow[d, "\mathcal{F}(u)"] \\
\mathcal{F}(\mathcal{B}_{X,X}) \arrow[rr, "a_{\mathcal{B}_{XX}}"'] & & \mathcal{G}(\mathcal{B}_{X,X})
\end{tikzcd}$$

The three axioms for an oplax $\mathcal{N}$*-functor follow from the oplax naturality coherence of* a *and the monoidal axioms of* a *which involve* Π *and* M*.* ■

Lemma 4.3.12. *For every pair of* $\mathcal{M}$*-bicategories* $\mathcal{B}$ *and* $\mathcal{C}$ *and for every oplax* $\mathcal{M}$*-functor* $F : \mathcal{B} \longrightarrow \mathcal{C}$ *between them there is an* $\mathcal{N}$*-icon* $(\mathsf{Icon}\, a)_F$ *with source and target as shown,*

$$\begin{tikzcd}
\mathsf{Icon}\, \mathcal{F}(\mathcal{B}) \arrow[r, "(\mathsf{Icon}\, a)_{\mathcal{B}}"] \arrow[d, "\mathsf{Icon}\, \mathcal{F}(F)"'] \arrow[dr, phantom, "(\mathsf{Icon}\, a)_F \Nearrow"] & \mathsf{Icon}\, \mathcal{G}(\mathcal{B}) \arrow[d, "\mathsf{Icon}\, \mathcal{G}(F)"] \\
\mathsf{Icon}\, \mathcal{F}(\mathcal{C}) \arrow[r, "(\mathsf{Icon}\, a)_{\mathcal{C}}"'] & \mathsf{Icon}\, \mathcal{G}(\mathcal{C})
\end{tikzcd}$$

which indeed coincide on objects,

$$\begin{tikzcd}
X \arrow[r, mapsto, "(\mathsf{Icon}\, a)_{\mathcal{B}}"] \arrow[d, mapsto, "\mathsf{Icon}\, \mathcal{F}(F)"'] & X \arrow[d, mapsto, "\mathsf{Icon}\, \mathcal{G}(F)"] \\
FX \arrow[r, mapsto, "(\mathsf{Icon}\, a)_{\mathcal{C}}"'] & FX = GX
\end{tikzcd}$$

and whose components on $\mathcal{N}$-hom objects are the cells a_F in $\mathcal{N}$.

$$\begin{tikzcd}
\mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, "a_{\mathcal{B}_{X,Y}}"] \arrow[d, "\mathcal{F}(F)"'] \arrow[dr, phantom, "a_F \nearrow"] & \mathcal{G}(\mathcal{B}_{X,Y}) \arrow[d, "\mathcal{G}(F)"] \\
\mathcal{F}(\mathcal{C}_{FX,FY}) \arrow[r, "a_{\mathcal{C}_{FX,FY}}"'] & \mathcal{G}(\mathcal{C}_{FX,FY})
\end{tikzcd}$$

The two axioms for an $\mathcal{N}$-icon follow from the oplax naturality coherence of a with respect to F^2 and F^0, and from the modification axiom of Π with respect to FF. ■

The two lemmas in this subsection define the components on objects and arrows of an oplax natural transformation $\mathsf{Icon}\, a$.

Proposition 4.3.13. *For every monoidal oplax natural transformation a there is an oplax natural transformation*

$$\begin{tikzcd}
\mathsf{Icon}\,\mathcal{M} \arrow[r, bend left, "\mathsf{Icon}\,\mathcal{F}"] \arrow[r, bend right, "\mathsf{Icon}\,\mathcal{G}"'] \arrow[r, phantom, "\Downarrow \mathsf{Icon}\, a"] & \mathsf{Icon}\,\mathcal{N}
\end{tikzcd}$$

whose oplax constraints $(\mathsf{Icon}\, a)_F$ are natural in F and coherent with respect to the pseudofunctoriality constraints of $\mathsf{Icon}\,\mathcal{F}$ and $\mathsf{Icon}\,\mathcal{G}$ since their corresponding analogues for a_F are. Furthermore, the pseudonaturality of a implies that of $\mathsf{Icon}\, a$.

4.3.3 Change of change of change of base along a monoidal modification

Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal bicategories, $\mathcal{F}$ and $\mathcal{G} : \mathcal{M} \longrightarrow \mathcal{N}$ a parallel pair of monoidal pseudofunctors, a and b a parallel pair of monoidal oplax natural transformations between $\mathcal{F}$ and $\mathcal{G}$, and Ξ a monoidal modification between a and b.

$$\begin{tikzcd}[column sep=huge]
\mathcal{M} \arrow[r, bend left=50, "\mathcal{F}"] \arrow[r, bend right=50, "\mathcal{G}"'] \arrow[r, phantom, "a \Downarrow \overset{\Xi}{\Rrightarrow} \Downarrow b"] & \mathcal{N}
\end{tikzcd}$$

We describe a modification $\mathsf{Icon}\,\Xi$ as shown below.

$$\begin{tikzcd}[column sep=huge]
\mathsf{Icon}\,\mathcal{M} \arrow[r, bend left=40, "\mathsf{Icon}\,\mathcal{F}"] \arrow[r, bend right=40, "\mathsf{Icon}\,\mathcal{G}"'] \arrow[r, phantom, "\mathsf{Icon}\, a \Downarrow \overset{\mathsf{Icon}\,\Xi}{\Rrightarrow} \Downarrow \mathsf{Icon}\, b"] & \mathsf{Icon}\,\mathcal{N}
\end{tikzcd}$$

Lemma 4.3.14. *For every $\mathcal{M}$-bicategory $\mathcal{B}$ there is an $\mathcal{N}$-icon $(\mathsf{Icon}\,\Xi)_{\mathcal{B}}$ with source and target as shown,*

$$\begin{tikzcd}
\mathsf{Icon}\,\mathcal{F}(\mathcal{B}) \arrow[r, bend left, "(\mathsf{Icon}\, a)_{\mathcal{B}}"] \arrow[r, bend right, "(\mathsf{Icon}\, b)_{\mathcal{B}}"'] \arrow[r, phantom, "(\mathsf{Icon}\,\Xi)_{\mathcal{B}} \Uparrow"] & \mathsf{Icon}\,\mathcal{G}(\mathcal{B})
\end{tikzcd}$$

which indeed coincide on objects since $(\mathsf{Icon}\, a)_{\mathcal{B}}$ *and* $(\mathsf{Icon}\, b)_{\mathcal{B}}$ *are the identity on the set of objects of* $\mathcal{B}$*, and whose components on* $\mathcal{N}$*-hom objects are the cells* $\Xi_{\mathcal{B}_{X,Y}}$ *in* $\mathcal{N}$*.*

$$\begin{tikzcd} \mathcal{F}(\mathcal{B}_{X,Y}) \arrow[r, bend left, "a_{\mathcal{B}_{X,Y}}"] \arrow[r, bend right, "b_{\mathcal{B}_{X,Y}}"'] \arrow[r, phantom, "\Xi_{\mathcal{B}_{X,Y}} \Uparrow"] & \mathcal{G}(\mathcal{B}_{X,Y}) \end{tikzcd}$$

The two axioms for an $\mathcal{N}$*-icon follow from the modification axiom for* Ξ *and the coherence between* Ξ *and the monoidal constraints* Π *and* M *of* a *and* b*.* ■

The lemma in this subsection defines the components on objects of a modification $\mathsf{Icon}\,\Xi$.

Proposition 4.3.15. *For every monoidal modification* Ξ *there is a modification* $\mathsf{Icon}\,\Xi$ *as above whose modification axiom follows from that of* Ξ*.* ■

4.3.4 The trifunctor $\mathsf{Icon}(\_)$

There is a tricategory $\mathsf{MonBicat}$ of monoidal bicategories, monoidal pseudofunctors, monoidal pseudonatural transformations, and monoidal modifications. Therefore, we have defined the data for a trifunctor

$$\mathsf{Icon} : \mathsf{MonBicat} \longrightarrow \mathsf{Bicat}$$

in Propositions 4.3.4, 4.3.10, 4.3.13 and 4.3.15. One may check that it preserves horizontal composition and identities of monoidal pseudofunctors because these compose strictly. And it preserves vertical composition and identities of monoidal pseudonatural transformations, as well as transversal composition and identities of monoidal modifications because these are taken component-wise.

Although we defined the "change of change of base" along monoidal *oplax* natural transformations, these do not form part of a tricategory since horizontal composition is not well defined; the middle four interchange law holds up to a *non-invertible* modification. Thus, we are not able to define a trifunctor Icon taking oplax natural transformations into account.

Theorem 4.3.16. *There is a trifunctor* $\mathsf{Icon} : \mathsf{MonBicat} \longrightarrow \mathsf{Bicat}$*.*

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left=50, "\mathcal{F}"] \arrow[r, bend right=50, "\mathcal{G}"'] \arrow[r, phantom, "a \Downarrow \overset{\Xi}{\Rrightarrow} \Downarrow b"] & \mathcal{N} \end{tikzcd} \longmapsto \begin{tikzcd} \mathsf{Icon}\,\mathcal{M} \arrow[r, bend left=50, "\mathsf{Icon}\,\mathcal{F}"] \arrow[r, bend right=50, "\mathsf{Icon}\,\mathcal{G}"'] \arrow[r, phantom, "\mathsf{Icon}\,a \Downarrow \overset{\mathsf{Icon}\,\Xi}{\Rrightarrow} \Downarrow \mathsf{Icon}\,b"] & \mathsf{Icon}\,\mathcal{N} \end{tikzcd}$$

Remark 4.3.17. The careful reader might notice that we did not use the invertibility of the monoidal constraints ω, δ, and γ for monoidal pseudofunctors, and the invertibility of Π and M for oplax natural transformations. One might feel like relaxing these modifications to be not necessarily invertible. The direction that makes everything work is pointing up-right in the diagrams of the previous subsections (except for δ which points down-right). We call the resulting structures: *right skew monoidal pseudofunctors* and *right skew monoidal oplax natural transformations*. Hence we have change of base along right skew monoidal pseudofunctors, and change of change of base along right skew monoidal pseudonatural transformations. This is worth considering since right skew monoidal pseudofunctors $\mathcal{F} : \mathbb{1} \longrightarrow \mathcal{M}$ are in bijection with right skew monoidales in a monoidal bicategory $\mathcal{M}$, and right skew monoidal natural transformations between them are in bijection with opmonoidal arrows between the corresponding right skew monoidales.

One may restrict the trifunctor Icon to the 3-category of monoidal 2-categories, monoidal 2-functors, monoidal 2-natural transformations, and monoidal modifications.

Theorem 4.3.18. *There is a 3-functor* $\mathsf{Icon} : \mathsf{Mon2\text{-}Cat} \longrightarrow \mathsf{2\text{-}Cat}$. ■

Theorem 4.3.19. *There is a trifunctor* $\mathsf{OpMon} : \mathsf{MonBicat} \longrightarrow \mathsf{Bicat}$ *and a trinatural transformation whose components are the fully faithful suspension functors.*

$$\begin{tikzcd} \mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[r, bend right, "\mathsf{Icon}"'] \arrow[r, phantom, "\Downarrow\Sigma"] & \mathsf{Bicat} \end{tikzcd}$$

Proof.
By the definition of change of base along a monoidal pseudofunctor $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{N}$ the set of objects of an $\mathcal{M}$-bicategory $\mathcal{B}$ is equal to the set of objects of $\mathsf{Icon}\,\mathcal{F}(\mathcal{B})$. Hence, one may define $\mathsf{OpMon}\,\mathcal{F}$ as the restriction of $\mathsf{Icon}\,\mathcal{F}$ to the one object enriched bicategories.

$$\begin{tikzcd} \mathsf{OpMon}\,\mathcal{M} \arrow[r, "\mathsf{OpMon}\,\mathcal{F}"] \arrow[d, "\Sigma"'] & \mathsf{OpMon}\,\mathcal{N} \arrow[d, "\Sigma"] \\ \mathsf{Icon}\,\mathcal{M} \arrow[r, "\mathsf{Icon}\,\mathcal{F}"'] & \mathsf{Icon}\,\mathcal{N} \end{tikzcd}$$

Since $\mathsf{OpMon}\,\mathcal{N}$ is a full subbicategory of $\mathsf{Icon}\,\mathcal{N}$, one may define $\mathsf{OpMon}\,a$ for a monoidal pseudonatural transformation a, and $\mathsf{OpMon}\,\Xi$ for a monoidal modification Ξ in the same way as Icon is defined. ■

Remember from Remark 4.3.6 that the data for an $\mathcal{M}$-graph is all the data for an $\mathcal{M}$-bicategory except for that which requires the monoidal structure of $\mathcal{M}$. There is a forgetful functor $U : \mathsf{Icon}\,\mathcal{M} \longrightarrow \mathsf{Grph}\,\mathcal{M}$ that witnesses this fact, from which we can easily conclude the following theorem.

Theorem 4.3.20. *There is a trifunctor* $\mathsf{Grph} : \mathsf{MonBicat} \longrightarrow \mathsf{Bicat}$ *and a trinatural transformation*

$$\begin{tikzcd} \mathsf{MonBicat} \arrow[r, bend left, "\mathsf{Icon}"] \arrow[r, bend right, "\mathsf{Grph}"'] \arrow[r, phantom, "\Downarrow U"] & \mathsf{Bicat} \end{tikzcd}$$

whose components are faithful strict functors that only forget the data that is written in the language of monoidal bicategories. ■

Theorem 4.3.21. *Let* $(\_) : \mathsf{MonBicat} \longrightarrow \mathsf{Bicat}$ *be the trifunctor that takes a monoidal bicategory to its underlying bicategory. There is a trinatural transformation*

$$\begin{tikzcd} \mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[r, bend right, "(\_)"'] \arrow[r, phantom, "\Downarrow U"] & \mathsf{Bicat} \end{tikzcd}$$

whose components are faithful strict functors, and such that the following equation holds.

$$\begin{tikzcd} \mathsf{MonBicat} \arrow[r, "\mathsf{Icon}" description] \arrow[r, bend left=60, "\mathsf{OpMon}"] \arrow[r, bend right=60, "\mathsf{Grph}"'] \arrow[r, phantom, bend left=30, "\Downarrow\Sigma"] \arrow[r, phantom, bend right=30, "\Downarrow U"] & \mathsf{Bicat} \end{tikzcd} = \begin{tikzcd} \mathsf{MonBicat} \arrow[r, "(\_)" description] \arrow[r, bend left=60, "\mathsf{OpMon}"] \arrow[r, bend right=60, "\mathsf{Grph}"'] \arrow[r, phantom, bend left=30, "\Downarrow U"] \arrow[r, phantom, bend right=30, "\Downarrow\Sigma"] & \mathsf{Bicat} \end{tikzcd}$$

Proof.
For a monoidal bicategory $\mathcal{M}$ one has to verify the commutativity of the square below.

$$\begin{tikzcd} \mathsf{OpMon}\,\mathcal{M} \arrow[r, "\Sigma"] \arrow[d, "U"'] & \mathsf{Icon}\,\mathcal{M} \arrow[d, "U"] \\ \mathcal{M} \arrow[r, "\Sigma"'] & \mathsf{Grph}\,\mathcal{M} \end{tikzcd}$$

Both paths take a monoidale M in $\mathcal{M}$ to the $\mathcal{M}$-graph with only one vertex $\star$ and hom object M. An opmonoidal arrow $C : M \longrightarrow N$ in $\mathcal{M}$ goes through both paths to the $\mathcal{M}$-graph morphism given on vertices by the identity function on the singleton $\{\star\}$, and whose action on hom objects is the underlying arrow in $\mathcal{M}$ of C. Similarly for opmonoidal cells in $\mathcal{M}$. ∎

We are now ready to prove the third point made in Remark 4.2.9 which at this level of generality is about monads in $\mathsf{Icon}\,\mathcal{M}$ for a monoidal bicategory $\mathcal{M}$. In fact, we prove that the trifunctors Mnd and Icon commute up to isomorphism.

Theorem 4.3.22. *The following square of trifunctors commutes up to strict trinatural isomorphism.*

$$\begin{tikzcd} \mathsf{MonBicat} \arrow[r, "\mathsf{Icon}"] \arrow[d, "\mathsf{Mnd}"'] \arrow[dr, phantom, "\cong"] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\ \mathsf{MonBicat} \arrow[r, "\mathsf{Icon}"'] & \mathsf{Bicat} \end{tikzcd}$$

In particular monads of $\mathcal{M}$-icons are icons enriched in monads in $\mathcal{M}$.

$$\mathsf{Mnd}(\mathsf{Icon}(\mathcal{M})) \cong \mathsf{Icon}(\mathsf{Mnd}(\mathcal{M}))$$

Proof.
If one unpacks what an $(\mathsf{Mnd}\,\mathcal{M})$-enriched bicategory $\mathcal{B}$ consists of, one gets the following items:

- A set $\mathsf{Ob}\,\mathcal{B}$ of objects.
- For every pair of objects X and Y of $\mathcal{B}$, the "hom monad" $(\mathcal{B}_{X,Y}, T)$ in $\mathcal{M}$.

$$\mathcal{B}_{X,Y} \xrightarrow{T} \mathcal{B}_{X,Y}$$

$$\begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, bend left, "T"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\eta\Uparrow"] & \mathcal{B}_{X,Y} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_{X,Y} \arrow[rr, bend left, "T"] \arrow[dr, "T"'] & \mu\Uparrow & \mathcal{B}_{X,Y} \\ & \mathcal{B}_{X,Y} \arrow[ur, "T"'] & \end{tikzcd}$$

- For every triple of objects X, Y and Z of $\mathcal{B}$, the "composition monad morphism" (m, T^2) in $\mathcal{M}$.

$$\mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \xrightarrow{m} \mathcal{B}_{X,Z} \qquad \begin{tikzcd} \mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \arrow[r, "TT"] \arrow[d, "m"'] \arrow[dr, phantom, "T^2\nearrow"] & \mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \arrow[d, "m"] \\ \mathcal{B}_{X,Z} \arrow[r, "T"'] & \mathcal{B}_{X,Z} \end{tikzcd}$$

- For every object X of $\mathcal{B}$, the "identities monad morphism" (u, T^0) in $\mathcal{M}$.

$$I \xrightarrow{u} \mathcal{B}_{X,X} \qquad \begin{tikzcd} & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "T^0\nearrow"] & \\ \mathcal{B}_{X,X} \arrow[rr, "T"'] & & \mathcal{B}_{X,X} \end{tikzcd}$$

- For every quadruple of objects W, X, Y, and Z of $\mathcal{B}$, the "associator invertible monad transformation" α in $\mathcal{M}$.

$$\begin{tikzcd} \mathcal{B}_{Y,Z}\mathcal{B}_{X,Y}\mathcal{B}_{W,X} \arrow[r, "1m"] \arrow[d, "m1"'] \arrow[dr, phantom, "\alpha\cong"] & \mathcal{B}_{X,Z}\mathcal{B}_{W,X} \arrow[d, "m"] \\ \mathcal{B}_{X,Z}\mathcal{B}_{W,X} \arrow[r, "m"'] & \mathcal{B}_{W,Z} \end{tikzcd}$$

- For every pair of objects X and Y of $\mathcal{B}$, the "left unitor invertible monad transformation" λ in $\mathcal{M}$.

$$\begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, "u1"] \arrow[dr, "1"'] \arrow[dr, phantom, shift left=3, "\lambda\cong"] & \mathcal{B}_{Y,Y}\mathcal{B}_{X,Y} \arrow[d, "m"] \\ & \mathcal{B}_{X,Y} \end{tikzcd}$$

- For every pair of objects X and Y of $\mathcal{B}$, the "right unitor invertible monad transformation" ρ in $\mathcal{M}$.

$$\begin{tikzcd} \mathcal{B}_{X,Y}\mathcal{B}_{X,X} \arrow[d, "m"'] & \mathcal{B}_{X,Y} \arrow[l, "1u"'] \arrow[dl, "1"] \arrow[dl, phantom, shift right=3, "\cong\rho"] \\ \mathcal{B}_{X,Y} & \end{tikzcd}$$

These are subject to the pentagon and triangle axioms for α, λ and ρ. One may rearrange these data and axioms so that the items $(\mathsf{Ob}\,\mathcal{B}, \{\mathcal{B}_{X,Y}\}_{X,Y}, m, u, \alpha, \lambda, \rho)$ assemble into an $\mathcal{M}$-bicategory, (T, T^2, T^0) assemble into an oplax $\mathcal{M}$-functor, η and μ are $\mathcal{M}$-icons, which together constitute a monad in $\mathsf{Icon}\,\mathcal{M}$. This rearrangement is precisely the desired isomorphism between the objects of $\mathsf{Icon}(\mathsf{Mnd}(\mathcal{M}))$ and $\mathsf{Mnd}(\mathsf{Icon}(\mathcal{M}))$. For the rest of the proof we abbreviate these objects by $(\mathcal{B}, T)$.

Let $(\mathcal{B}, T)$ and $(\mathcal{C}, S)$ be two $(\mathsf{Mnd}\,\mathcal{M})$-bicategories. If one unpacks what does an oplax $(\mathsf{Mnd}\,\mathcal{M})$-functor $F : (\mathcal{B}, T) \longrightarrow (\mathcal{C}, S)$ consists of, one gets the following items:

- A function $F : \mathsf{Ob}\,\mathcal{B} \longrightarrow \mathsf{Ob}\,\mathcal{C}$.
- For every pair of objects X and Y in $\mathcal{B}$, the "action on $\mathcal{M}$-hom monads" is a monad morphism (F, φ) in $\mathcal{M}$.

$$\mathcal{B}_{X,Y} \xrightarrow{F} \mathcal{C}_{FX,FY} \qquad \begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, "F"] \arrow[d, "T"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathcal{C}_{FX,FY} \arrow[d, "S"] \\ \mathcal{B}_{X,Y} \arrow[r, "F"'] & \mathcal{C}_{FX,FY} \end{tikzcd}$$

- For every triple of objects X, Y, and Z in $\mathcal{B}$, the "composition constraint monad transformation" F^2 in $\mathcal{M}$.

$$\begin{tikzcd} \mathcal{B}_{Y,Z}\mathcal{B}_{X,Y} \arrow[r, "FF"] \arrow[d, "m"'] \arrow[dr, phantom, "F^2\nearrow"] & \mathcal{C}_{FY,FZ}\mathcal{C}_{FX,FY} \arrow[d, "m"] \\ \mathcal{B}_{X,Z} \arrow[r, "F"'] & \mathcal{C}_{FX,FZ} \end{tikzcd}$$

- For every object X of $\mathcal{B}$, the "identities constraint monad transformation" F^0 in $\mathcal{M}$.

$$\begin{tikzcd} & I \arrow[dl, "u"'] \arrow[dr, "u"] \arrow[d, phantom, "F^0\nearrow"] & \\ \mathcal{B}_{X,Z} \arrow[rr, "F"'] & & \mathcal{C}_{FX,FZ} \end{tikzcd}$$

These are subject to the three axioms for an enriched oplax functor. Again, by rearranging this information (F, F^2, F^0) assembles into an oplax $\mathcal{M}$-functor and φ into an $\mathcal{M}$-icon, which together constitute a monad morphism in $\mathsf{Icon}\,\mathcal{M}$. Hence, exhibiting the desired isomorphism between the arrows of the bicategories $\mathsf{Icon}(\mathsf{Mnd}(\mathcal{M}))$ and $\mathsf{Mnd}(\mathsf{Icon}(\mathcal{M}))$. For the rest of the proof we abbreviate them by (F, φ).

Let (F, φ) and $(G, \psi) : (\mathcal{B}, T) \longrightarrow (\mathcal{C}, S)$ be two parallel oplax $(\mathsf{Mnd}\,\mathcal{M})$-functors such that $FX = GX$ for every object X of $(\mathcal{B}, T)$. An $(\mathsf{Mnd}\,\mathcal{M})$-icon

$$\begin{tikzcd} (\mathcal{B}, T) \arrow[r, bend left, "{(G,\psi)}"] \arrow[r, bend right, "{(F,\varphi)}"'] \arrow[r, phantom, "\alpha\Uparrow"] & (\mathcal{C}, S) \end{tikzcd}$$

consists of a monad transformation α_{XY} in $\mathcal{M}$ for every pair of objects X and Y in $\mathcal{B}$,

$$\begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, bend left, "G"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\alpha_{XY}\Uparrow"] & \mathcal{C}_{FX,FY} \end{tikzcd}$$

which are subject to the two axioms for an enriched icon. And if one is viewing $(\mathcal{B}, T)$, $(\mathcal{C}, S)$, (F, φ) and (G, ψ) as monads and monad morphisms in $\mathsf{Icon}\,\mathcal{M}$ as described above, then the axioms for an enriched icon say that $\{\alpha_{XY}\}_{X,Y\in\mathsf{Ob}\,\mathcal{B}}$ is an $\mathcal{M}$-icon which is a monad transformation in $\mathsf{Icon}\,\mathcal{M}$. Hence, exhibiting the desired isomorphism between the cells of $\mathsf{Icon}(\mathsf{Mnd}(\mathcal{M}))$ and $\mathsf{Mnd}(\mathsf{Icon}(\mathcal{M}))$.

Since one is only interpreting the data and axioms in a different way, this rearrangement of information amounts to a strict functor $\mathsf{Icon}(\mathsf{Mnd}(\mathcal{M}))\longrightarrow\mathsf{Mnd}(\mathsf{Icon}(\mathcal{M}))$. For if one takes a pair of composable oplax $(\mathsf{Mnd}\,\mathcal{M})$-functors,

$$(\mathcal{B}, T) \xrightarrow{(F,\varphi)} (\mathcal{C}, S) \xrightarrow{(F',\varphi')} (\mathcal{D}, R)$$

then the composite $(\mathsf{Mnd}\,\mathcal{M})$-functor has underlying function on objects the composite of F and F' on objects. For each pair of objects X and Y the composite on "$\mathcal{M}$-hom monads" is calculated as the composite of the underlying arrows and the pasting of the cells φ and φ' next to each other.

$$\begin{tikzcd} \mathcal{B}_{X,Y} \arrow[r, "F"] \arrow[d, "T"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathcal{C}_{FX,FY} \arrow[r, "F'"] \arrow[d, "S"] \arrow[dr, phantom, "\swarrow\varphi'"] & \mathcal{D}_{F'FX,F'FY} \arrow[d, "R"] \\ \mathcal{B}_{X,Y} \arrow[r, "F"'] & \mathcal{C}_{FX,FY} \arrow[r, "F'"'] & \mathcal{D}_{F'FX,F'FY} \end{tikzcd}$$

Similarly the "composition constraints" and "identities constraints" monad transformations are composed by pasting the cells F^2 and F'^2 next to each other, as well as the cells F^0 and F'^0. And this is precisely the same way as one composes monad morphism in $\mathsf{Icon}\,\mathcal{M}$. Ergo, to first rearrange their information separately into composable pair of monads morphisms in $\mathsf{Icon}\,\mathcal{M}$ and then perform the composition is the same as doing the composition first and then the rearrangement of information.

Hence, for monoidal bicategories $\mathcal{M}$, the square in the statement commutes up to isomorphism. A similar argument shows the commutativity up to isomorphism for monoidal pseudofunctors, monoidal pseudonatural transformations, and monoidal modifications. ■

A subcollection of the arguments used in the proof above may be used to prove the following theorem. In fact, following Remark 4.3.6, the arguments we need are precisely those that do not require the monoidal structure of the various elements of $\mathsf{MonBicat}$.

Theorem 4.3.23. *The following square of trifunctors commutes up to strict trinatural isomorphism.*

$$\begin{tikzcd} \mathsf{MonBicat} \arrow[r, "\mathsf{Grph}"] \arrow[d, "\mathsf{Mnd}"'] \arrow[dr, phantom, "\swarrow\!\!\cong"] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\ \mathsf{MonBicat} \arrow[r, "\mathsf{Grph}"'] & \mathsf{Bicat} \end{tikzcd}$$

Moreover, the equation below holds.

$$
\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{Icon}"] \arrow[r, bend right, "\mathsf{Grph}"'] \arrow[r, phantom, "\Downarrow U"] \arrow[d, "\mathsf{Mnd}"'] \arrow[dr, phantom, "\cong"] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\
\mathsf{MonBicat} \arrow[r, bend right, "\mathsf{Grph}"'] & \mathsf{Bicat}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{Icon}"] \arrow[d, "\mathsf{Mnd}"'] \arrow[dr, phantom, "\cong"] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{Icon}"] \arrow[r, bend right, "\mathsf{Grph}"'] \arrow[r, phantom, "\Downarrow U"] & \mathsf{Bicat}
\end{tikzcd}
$$

■

We may rephrase Remark 4.2.6 where we define the trifunctors Mnd as the following commutative diagram.

$$
\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, "(\_)"] \arrow[d, "\mathsf{Mnd}"'] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\
\mathsf{MonBicat} \arrow[r, "(\_)"'] & \mathsf{Bicat}
\end{tikzcd}
$$

We use this diagram to state the following theorem, which is a one object version of Theorem 4.3.22.

Theorem 4.3.24. *The following square of trifunctors commutes up to strict trinatural isomorphism.*

$$
\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, "\mathsf{OpMon}"] \arrow[d, "\mathsf{Mnd}"'] \arrow[dr, phantom, "\cong"] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\
\mathsf{MonBicat} \arrow[r, "\mathsf{OpMon}"'] & \mathsf{Bicat}
\end{tikzcd}
$$

Moreover, the equations below hold.

$$
\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[r, bend right, "(\_)"'] \arrow[r, phantom, "\Downarrow U"] \arrow[d, "\mathsf{Mnd}"'] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\
\mathsf{MonBicat} \arrow[r, bend right, "(\_)"'] & \mathsf{Bicat}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[d, "\mathsf{Mnd}"'] \arrow[dr, phantom, "\cong"] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[r, bend right, "(\_)"'] \arrow[r, phantom, "\Downarrow U"] & \mathsf{Bicat}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[r, bend right, "\mathsf{Icon}"'] \arrow[r, phantom, "\Downarrow \Sigma"] \arrow[d, "\mathsf{Mnd}"'] \arrow[dr, phantom, "\cong"] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\
\mathsf{MonBicat} \arrow[r, bend right, "\mathsf{Icon}"'] & \mathsf{Bicat}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[d, "\mathsf{Mnd}"'] \arrow[dr, phantom, "\cong"] & \mathsf{Bicat} \arrow[d, "\mathsf{Mnd}"] \\
\mathsf{MonBicat} \arrow[r, bend left, "\mathsf{OpMon}"] \arrow[r, bend right, "\mathsf{Icon}"'] \arrow[r, phantom, "\Downarrow \Sigma"] & \mathsf{Bicat}
\end{tikzcd}
$$

■

Remark 4.3.25. A special case of the isomorphism and the first equation in Theorem 4.3.24 above appears in [Zaw12, Lemma 3.1], which restricts the domain of OpMon to the 3-category of cartesian monoidal 2-categories and the codomain to the 3-category of 2-categories.

Remark 4.3.26. The key points made in Remark 4.2.9, which are used in our alternative proof of the Moerdijk-McCrudden theorem, are now all proven by Theorems 4.3.19, 4.3.21, and 4.3.24.

Now we may turn to our version of the Moerdijk-McCrudden Theorem for enriched bicategories.

Theorem 4.3.27. *Let $\mathcal{M}$ be a monoidal bicategory. The bicategories $\mathsf{Icon}(\mathcal{M})$ and $\mathsf{Grph}(\mathcal{M})$ have Eilenberg-Moore objects for monads if $\mathcal{M}$ does. In this case, the forgetful functor $U : \mathsf{Icon}(\mathcal{M}) \longrightarrow \mathsf{Grph}(\mathcal{M})$ preserves them, and the suspension functors $\Sigma : \mathsf{OpMon}\,\mathcal{M} \longrightarrow \mathsf{Icon}\,\mathcal{M}$ and $\Sigma : \mathcal{M} \longrightarrow \mathsf{Grph}\,\mathcal{M}$ create them.*

Proof.
Apply the trifunctor $\mathsf{Icon}(\_)$ to the biadjunction that witnesses the existence of Eilenberg-Moore objects for monads in $\mathcal{M}$ to get a biadjunction $\mathsf{Icon\,Inc} \dashv \mathsf{Icon\,EM}$ as shown below.

$$\begin{tikzcd}
\mathsf{Mnd}(\mathcal{M}) \arrow[d, bend left, "\mathsf{EM}"] \arrow[d, phantom, "\dashv"] & & \mathsf{Icon}(\mathsf{Mnd}(\mathcal{M})) \arrow[d, bend left, "\mathsf{Icon\,EM}"] \arrow[d, phantom, "\dashv"] \\
\mathcal{M} \arrow[u, bend left, "\mathsf{Inc}"] & \arrow[r, mapsto, squiggly] & \mathsf{Icon}\,\mathcal{M} \arrow[u, bend left, "\mathsf{Icon\,Inc}"]
\end{tikzcd}$$

The isomorphism of bicategories $\mathsf{Icon}(\mathsf{Mnd}(\mathcal{M})) \cong \mathsf{Mnd}(\mathsf{Icon}(\mathcal{M}))$ in Theorem 4.3.22 makes the following triangle commute strictly.

$$\begin{tikzcd}
\mathsf{Icon}\,\mathcal{M} \arrow[r, "\mathsf{Icon\,Inc}"] \arrow[rr, bend right, "\mathsf{Inc}"'] & \mathsf{Icon}(\mathsf{Mnd}(\mathcal{M})) \cong & \mathsf{Mnd}(\mathsf{Icon}(\mathcal{M}))
\end{tikzcd}$$

Thus one can create the biadjunction shown below, in which the dotted arrow is the composite of the pictured isomorphism followed by $\mathsf{Icon\,EM}$.

$$\begin{tikzcd}
\mathsf{Mnd}(\mathsf{Icon}(\mathcal{M})) \arrow[dd, dotted, bend left=60, "\mathsf{EM}"] \arrow[d, phantom, "\cong" rotate=90] \\
\mathsf{Icon}(\mathsf{Mnd}(\mathcal{M})) \arrow[d, bend left, "\mathsf{Icon\,EM}"] \arrow[d, phantom, "\dashv"] \\
\mathsf{Icon}\,\mathcal{M} \arrow[u, bend left, "\mathsf{Icon\,Inc}"] \arrow[uu, bend left=60, "\mathsf{Inc}"]
\end{tikzcd}$$

The outer arrows form a biadjunction which witnesses the existence of Eilenberg-Moore objects for monads in $\mathsf{Icon}\,\mathcal{M}$. A similar argument shows the existence of Eilenberg-Moore objects for monads in $\mathsf{Grph}\,\mathcal{M}$. The forgetful functor $U : \mathsf{Icon}\,\mathcal{M} \longrightarrow \mathsf{Grph}\,\mathcal{M}$ preserves Eilenberg-Moore objects by the strict trinaturality of $U : \mathsf{Icon} \longrightarrow \mathsf{Grph}$, as one can see from the following commutative diagram,

$$\begin{tikzcd}
\mathsf{Mnd}(\mathsf{Icon}(\mathcal{M})) \arrow[r, "\mathsf{Mnd}\,U"] \arrow[d, phantom, "\cong" rotate] \arrow[ddd, bend right, "\mathsf{EM}"'] & \mathsf{Mnd}(\mathsf{Grph}(\mathcal{M})) \arrow[d, phantom, "\cong" rotate] \arrow[ddd, bend left, "\mathsf{EM}"] \\
\mathsf{Icon}(\mathsf{Mnd}(\mathcal{M})) \arrow[r, "U"'] \arrow[dd, "\mathsf{Icon}\,\mathsf{EM}"'] & \mathsf{Grph}(\mathsf{Mnd}(\mathcal{M})) \arrow[dd, "\mathsf{Grph}\,\mathsf{EM}"] \\
& \\
\mathsf{Icon}\,\mathcal{M} \arrow[r, "U"'] & \mathsf{Grph}\,\mathcal{M}
\end{tikzcd}$$

where the commutativity of the small upper square is an instance of the equation in Theorem 4.3.23. ∎

We finish this chapter by giving an explicit description of the adjunction between the underlying object of a monad in $\mathsf{Icon}\,\mathcal{M}$ and (assuming it exists) an Eilenberg-Moore object. First, note that for every adjunction $F \dashv G$ in $\mathsf{Icon}\,\mathcal{M}$ as shown below, the sets of objects of $\mathcal{B}$ and $\mathcal{C}$ are forced to be isomorphic since the unit and counit are $\mathcal{M}$-icons.

$$\begin{tikzcd}
\mathcal{C} \arrow[d, bend left, "G"] \arrow[d, phantom, "\dashv"] \\
\mathcal{B} \arrow[u, bend left, "F"]
\end{tikzcd}$$

So, without loss of generality, we assume that $\mathcal{B}$ and $\mathcal{C}$ have the same set of objects. Also, by a doctrinal adjunction argument, G is an $\mathcal{M}$-pseudofunctor because F is an oplax $\mathcal{M}$-functor.

The following analogy gives us another point of view on adjunctions in $\mathsf{Icon}\,\mathcal{M}$. A Cat-bicategory $\mathcal{B}$ may be regarded as a collection of monoidal categories $\mathcal{B}(X,X)$ for every object X of $\mathcal{B}$, and a collection of two-sided actegories $\mathcal{B}(X,Y)$ for every pair of objects X and Y of $\mathcal{B}$, i.e. categories with an action of a monoidal category on each side, such that the associative and unit laws hold up to coherent isomorphism. Analogously one may think of an adjunction $F \dashv G$ in $\mathsf{Icon}\,\mathcal{M}$ as a collection of opmonoidal adjunctions in $\mathcal{M}$ for every object X of $\mathcal{B}$ (or $\mathcal{C}$), and a collection of "two sided module adjunctions" for every pair of objects of $\mathcal{B}$.

$$\begin{tikzcd}
\mathcal{C}(X,X) \arrow[d, bend left, "G"] \arrow[d, phantom, "\dashv"] & \mathcal{C}(X,Y) \arrow[d, bend left, "G"] \arrow[d, phantom, "\dashv"] \\
\mathcal{B}(X,X) \arrow[u, bend left, "F"] & \mathcal{B}(X,Y) \arrow[u, bend left, "F"]
\end{tikzcd}$$

Opmonoidal adjunctions and module adjunctions over them are considered in [AC13, 4.1] to study a generalised version of Hopf modules for opmonoidal monads.

The following corollary follows from Theorem 4.3.27 and generalises [McC02, Proposition 2.13].

Corollary 4.3.28. *An adjunction in* $\mathsf{Icon}\,\mathcal{M}$ *is monadic if and only if for every pair of objects* X *and* Y *in* $\mathcal{B}$ *the adjunctions on hom objects are monadic in* $\mathcal{M}$. ∎

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