Operads and embeddings

By

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A thesis submitted to Macquarie University
for the degree of Master of Research
Department of Mathematics
June 2017

MACQUARIE
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SYDNEY · AUSTRALIA

Typeset in $\LaTeX\,2_\varepsilon$.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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Acknowledgements

I would like to thank first and foremost my supervisor, Michael Batanin. His suggestions, explanations and clarifications were priceless and I greatly enjoyed all the moments of research and exchange we had together.

I also want to thank everybody in the Department of Mathematics and Statistics at Macquarie University for creating such a friendly and welcoming environment. I am grateful to all the speakers at the Australian Category Seminar who have allowed me to deepen my knowledge of the world of category theory.

Finally, I do not want to forget to thank my family and my friends, from Belgium and from Australia, who have supported and encouraged me all along this master until the very end.

Abstract

In this thesis, our objective is to present a strategy of a new proof of the weak equivalence $\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m) \sim \Omega^2 Map_{NOp}(\mathcal{D}_1, \mathcal{D}_m)$, where $\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m)$ is the space of tangentially straightened long knots in $\mathbb{R}^m$ (see [1]) and $Map_{NOp}(\mathcal{D}_1, \mathcal{D}_m)$ is the space of operadic morphisms from the little 1-disk operad to the little m-disk operad.

The existing proofs of Turchin [2] and Dwyer-Hess [1] are based on homotopy theory. We develop a more categorical proof which uses the theory of internal algebra classifiers [3] and explains conceptually the 'raison d'être' of such a delooping. It also allows us to employ powerful categorical/combinatorial techniques developed in [3] for proving and generalising of this sort of results. Our proof should admit a generalisation to higher dimensions, known as Dwyer-Hess conjecture.

Contents

1
Introduction

A long embedding $\mathbb{R}^l \hookrightarrow \mathbb{R}^m$ is an embedding which agrees with the standard embedding outside the unit cube [1]. The space of long embeddings, written $Emb(\mathbb{R}^l, \mathbb{R}^m)$, is therefore a generalisation of the space of knots, which is the case $l = 1$ and $m = 3$.

In this thesis, we will only consider the case of long knots, that is the spaces $Emb(\mathbb{R}^l, \mathbb{R}^m)$ when $l = 1$.

Sinha showed in [4] that such spaces could be expressed as the totalization of cosimplicial spaces involving configuration spaces. More precisely, there is a weak equivalence

$$Emb(\mathbb{R}^1, \mathbb{R}^m) \sim \widetilde{Tot}(\mathcal{C}^\bullet) \tag{1.1}$$

where $\mathcal{C}^\bullet$ is the cosimplicial space which sends a non-negative integer n to a Fulton-MacPherson completion [5] of the configuration space with n points.

In [6], Sinha established further, using the cosimplicial model of [4], the weak equivalence

$$\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m) \sim \widetilde{Tot}(\mathcal{K}) \tag{1.2}$$

where $\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m)$ is the fiber of the map $Emb(\mathbb{R}^1, \mathbb{R}^m) \to Imm(\mathbb{R}^1, \mathbb{R}^m)$ and $\widetilde{Tot}(\mathcal{K})$ is the totalization of the Kontsevich operad $\mathcal{K}$ [7].

This result can be put in relation with the McClure-Smith solution [8] of Deligne's conjecture. Indeed, Deligne's conjecture implies that the totalization of a multiplicative non-symmetric operad admits an action of an E_2 operad, that is, an operad weakly equivalent to the little 2-disk operad. By classical May's recognition principle [9], such an action often means that the space itself is a double loop space.

In the case of totalization of a multiplicative non-symmetric operad, one could then wonder what are the conditions which guarantee the existence of a double delooping and what this explicit double delooping might be. Turchin proved in [2] that if a multiplicative non-symmetric operad $\mathcal{O}$ is reduced, that is, if $\mathcal{O}_0 = \mathcal{O}_1 = 1$, then there is a weak equivalence

$$\widetilde{Tot}(\mathcal{O}) \sim \Omega^2 Map_{NOp}(Ass, \mathcal{O}) \tag{1.3}$$

where Map_{NOp} is the homotopy mapping space between non-symmetric operads and Ass is the non-symmetric version of the associative operad.

Dwyer and Hess proved in [1] the more general fact that the weak equivalence 1.3 holds as long as $\mathcal{O}_0$ and $\mathcal{O}_1$ are contractible.

The results of Sinha, Turchin and Dwyer-Hess lead then to the following important statement: for $m \geq 4$, there is a weak equivalence of spaces

$$\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m) \sim \Omega^2 Map_{Op}(\mathcal{D}_1, \mathcal{D}_m) \tag{1.4}$$

where $\mathcal{D}_k$ is an operad equivalent to the little k-disk operad.

Dwyer and Hess conjectured also that an analogous statement holds for higher dimensions as well. This conjecture has been proved by Boavida and Weiss in [10]. More precisely,

Theorem 1.0.1 (Boavida-Weiss)**.** *If $m \geq l+3$, there is a weak equivalence*

$$\overline{Emb}(\mathbb{R}^l, \mathbb{R}^m) \sim \Omega^{l+1} Map_{Op}(\mathcal{D}_l, \mathcal{D}_m). \tag{1.5}$$

Such results are very useful to understand the topology of embedding spaces. For example, in [11], a number of results about the rational homotopy type of the embedding spaces was obtained due to the existence of such a delooping and a connection between rational mapping spaces of E_n-operads and Kontsevich's (hairy) graph complex.

The proofs of the weak equivalence 1.3 from Turchin [2] and Dwyer-Hess [1] are both based on homotopy theory but of different flavours. Turchin uses some very explicit cofibrant resolutions for operads, bimodules and weak bimodules and then constructs all necessary higher homotopies by hands. Dwyer and Hess use abstract homotopy theory of Quillen. Unfortunately, both proofs are very technical and do not provide a conceptual explanation of the result. Consequently both proofs are hard to generalise to higher dimensional situation if one wants to prove the Dwyer-Hess conjecture 1.5.

In this thesis we will elaborate a strategy of a more categorical proof which uses the theory of internal algebra classifiers developed by Batanin and Berger in [3, 12]. In a sense, our approach is a combination of both Turchin's and Dwyer-Hess's approaches. The theory of classifiers allows to construct some very explicit cofibrant resolutions of algebras in a spirit of Turchin and abstract homotopy theory allows to complete the proofs à la Dwyer-Hess.

Our approach also reveals the algebraic or, better to say higher categorical, meaning of the explicit delooping of Turchin-Dwyer-Hess. As a baby case one can prove by hands that given two operadic morphisms $Ass \to \mathcal{O}$ in a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ one can construct an Ass-bimodule using first morphism to define left action of Ass on $\mathcal{O}$ and second morphism to define right action of Ass. Now, suppose that $\mathbb{C}$ is a groupoid and $\mathcal{O}_1 = I$. Then the functor above has an inverse, that is any bimodule over Ass is obtained from two operadic morphisms $Ass \to \mathcal{O}$.

The idea of the proof we present here is that the Turchin-Dwyer-Hess delooping is essentially a statement above where $\mathbb{C}$ is an ω-groupoid. Of course, in this case the inverse functor reconstructs two operadic morphisms as well as an operadic structure on $\mathcal{O}$ up to higher homotopies only.

Our thesis is constructed as follows. In the first chapter, we will remind the reader of the existing results, from the work of Sinha [4, 6], where we will present the sketch of the proof of the weak equivalences 1.1 and 1.2, to the delooping theorems of Turchin [2] and Dwyer-Hess [1].

In the second chapter, we will introduce the theory of internal algebra classifiers [3, 12].

In the last chapter, we will present the elements of a proof of the weak equivalence 1.3, using the theory introduced in the second chapter.

2

Presentation of the existing results

2.1 Embedding spaces

In this section, we will introduce embedding spaces, which are the objects we want to study.

2.1.1 First definitions

We define the space of *long embeddings* [1] :

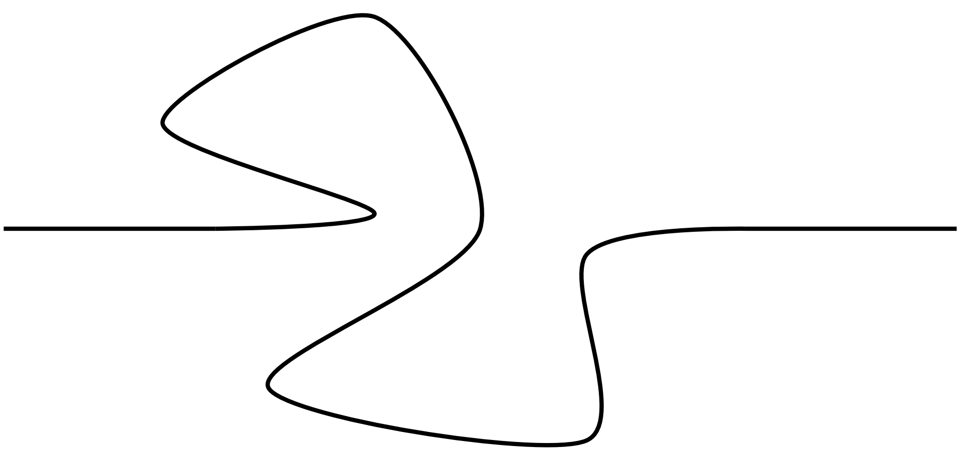
Definition 2.1.1. An embedding

$$f : \mathbb{R}^l \to \mathbb{R}^m$$

is called *long embedding* if it agrees outside a compact with the standard inclusion of $\mathbb{R}^l$ into $\mathbb{R}^m$, that is the map defined by

$$(x_1, \dots, x_l) \in \mathbb{R}^l \mapsto (x_1, \dots, x_l, 0, \dots, 0) \in \mathbb{R}^m$$

Figure 2.1.1: A long embedding of $\mathbb{R}^1$ into $\mathbb{R}^2$



Definition 2.1.2. We write

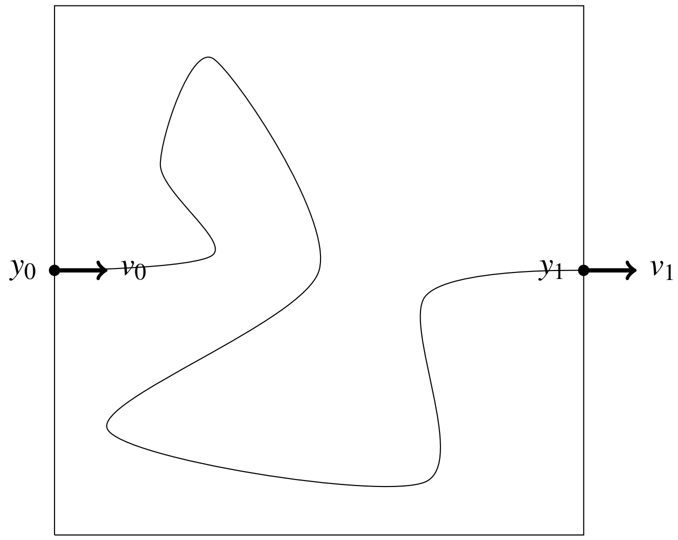
$$Emb(\mathbb{R}^l, \mathbb{R}^m)$$

for the set of long embeddings from $\mathbb{R}^l$ to $\mathbb{R}^m$.

Remark 2.1.3. The set $Emb(\mathbb{R}^l, \mathbb{R}^m)$ can be equipped with a structure of topological space, called Whitney topology [4]. Moreover, this space has a canonical object which is the standard inclusion everywhere. $Emb(\mathbb{R}^l, \mathbb{R}^m)$ is therefore a pointed topological space.

Remark 2.1.4. In [4], Sinha defines the space $Emb(I, I^m)$, where $I = [-1, 1]$, as the space of embeddings from I to I^m with the boundaries of I sent to fixed boundary points y_0 and y_1 of I^m, with fixed tangent vectors v_0 and v_1. This space is homotopy equivalent to $Emb(\mathbb{R}^1, \mathbb{R}^m)$.

Figure 2.1.2: An element of $Emb(I, I^2)$



Remark 2.1.5. The space $Emb(\mathbb{R}^l, \mathbb{R}^m)$ is homotopy equivalent to the space of embeddings of S^l into S^m.

We can already make some remarks about $Emb(\mathbb{R}^l, \mathbb{R}^m)$ in particular cases.

Remark 2.1.6. $Emb(\mathbb{R}^1, \mathbb{R}^2)$ is contractible.

Remark 2.1.7. $Emb(\mathbb{R}^1, \mathbb{R}^m)$ is path connected for $m \geq 4$.

In general, however, $Emb(\mathbb{R}^l, \mathbb{R}^m)$ may be very complicated. For example, $Emb(\mathbb{R}^1, \mathbb{R}^3)$ is the space of knots, which is not path connected.

2.2 First Sinha's paper

In this section, we will present a model for $Emb(\mathbb{R}^1, \mathbb{R}^m)$ using Goodwillie calculus [13]. It will be a presentation of Sinha's paper [4].

2.2.1 Homotopy limits

Before starting the Goodwillie calculus, we will quickly remind the notion of homotopy limit of a functor. Let $\mathbb{C}$ be a small category.

For any object $c \in \mathbb{C}$, the comma category $\mathbb{C}/c$ is the category whose

- objects are morphisms in $\mathbb{C}$ with c as codomain
- morphisms are commutative diagrams

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr] & & b \arrow[dl] \\ & c & \end{tikzcd}$$

We get a functor

$$\mathbb{C}/- : \mathbb{C} \to Cat$$

that we can compose with the functor $|N| : Cat \to Top$ which consists in taking the geometric realisation of the nerve.

Recall that the category of functors $\mathbb{C} \to Top$ is topologically enriched where the enriched hom-functor is given by the space of natural transformations $Nat(-, -)$.

Definition 2.2.1. The homotopy limit of a functor $F : \mathbb{C} \to Top$, written

$$\text{holim } F$$

is the space of natural transformations

$$Nat(|N| \circ (\mathbb{C}/-), F)$$

Here is an important lemma about homotopy limits [14] :

Lemma 2.2.2. *Suppose that we have a category $\mathbb{C}$ and two functors*

$$F, G : \mathbb{C} \to Top$$

If there is a natural transformation $F \Rightarrow G$ which is a weak equivalence pointwise, then the induced map between homotopy limits is a weak equivalence.

We will also need the notion of left cofinality :

Definition 2.2.3. A functor $F : \mathbb{C} \to \mathbb{D}$ is (homotopically) *left cofinal* if for any functor $G : \mathbb{D} \to Top$, the natural morphism

$$\text{holim } GF \to \text{holim } G$$

is a weak equivalence.

2.2.2 Approximation of a functor

In this subsection, we will define the n-th approximation of $Emb(\mathbb{R}^l, \mathbb{R}^m)$, written

$$T_n Emb(\mathbb{R}^l, \mathbb{R}^m)$$

We follow what is done in [6] and [15].

We begin by introducing the category of open subsets of $\mathbb{R}^l$:

Definition 2.2.4. We write

$$\mathcal{U}(\mathbb{R}^l)$$

for the category whose

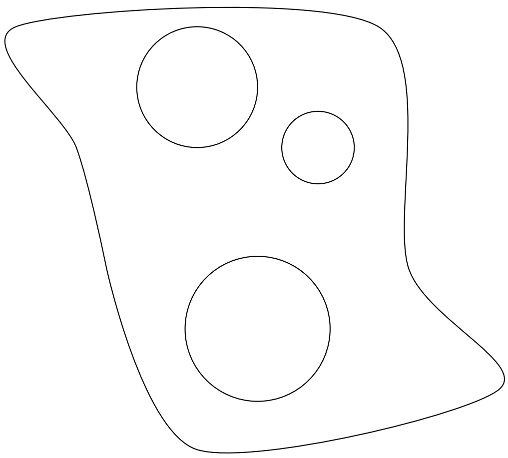
- objects are open subsets of $\mathbb{R}^l$
- morphisms are the inclusions of subsets

Definition 2.2.5. For $W \in \mathcal{U}(\mathbb{R}^l)$, we write

$$\mathcal{U}_n(W)$$

for the subcategory of $\mathcal{U}(\mathbb{R}^l)$ where objects are disjoint unions of at most n open disks in W.

Figure 2.2.1: An object of $\mathcal{U}_3(W)$



Definition 2.2.6. The n-th approximation of a contravariant functor

$$F : \mathcal{U}(\mathbb{R}^l) \to Top$$

is the contravariant functor

$$T_n F : \mathcal{U}(\mathbb{R}^l) \to Top$$

which sends $W \in \mathcal{U}(\mathbb{R}^l)$ to the homotopy limit of F restricted to $\mathcal{U}_n(W)$.

Definition 2.2.7. Let

$$F : \mathcal{U}(\mathbb{R}^l) \to Top$$

be a contravariant functor and $W \in \mathcal{U}(\mathbb{R}^l)$. The following sequence, obtained for all $n \geq 1$ by restriction from $\mathcal{U}_n(W)$ to $\mathcal{U}_{n-1}(W)$, is called *Taylor tower* :

$$T_0 F(W) \leftarrow T_1 F(W) \leftarrow T_2 F(W) \leftarrow \dots$$

Example 2.2.8. $Emb(\mathbb{R}^l, \mathbb{R}^m)$ can be extended to a contravariant functor

$$Emb(-, \mathbb{R}^m) : \mathcal{U}(\mathbb{R}^l) \to Top$$

which sends

- an object $U \subset \mathbb{R}^l$ to the space $Emb(U, \mathbb{R}^m)$ of long embeddings from U to $\mathbb{R}^m$
- the inclusion $U \subset V$ to the restriction map $Emb(V, \mathbb{R}^m) \to Emb(U, \mathbb{R}^m)$.

The interest of this approximation $T_n Emb(\mathbb{R}^l, \mathbb{R}^m)$ comes from the following theorem of Goodwillie calculus [16, Corollary 2.5] :

Theorem 2.2.9. *If $m \geq l+3$, then $Emb(\mathbb{R}^l, \mathbb{R}^m)$ is weakly equivalent to the limit of the Taylor tower*

$$T_0 Emb(\mathbb{R}^l, \mathbb{R}^m) \leftarrow T_1 Emb(\mathbb{R}^l, \mathbb{R}^m) \leftarrow T_2 Emb(\mathbb{R}^l, \mathbb{R}^m) \leftarrow \dots$$

2.2.3 Configuration spaces

In this subsection, we introduce the configuration spaces and we build contravariant functors involving these configuration spaces.

Let us remind that the configuration space of n points in $\mathbb{R}^m$ is defined as

$$C_n(\mathbb{R}^m) = \{(x_1, \dots, x_n) \in (\mathbb{R}^m)^n,\ x_i \neq x_j \text{ if } i \neq j\}$$

We will work with a compactification of the configuration spaces, this is why we need the following definition :

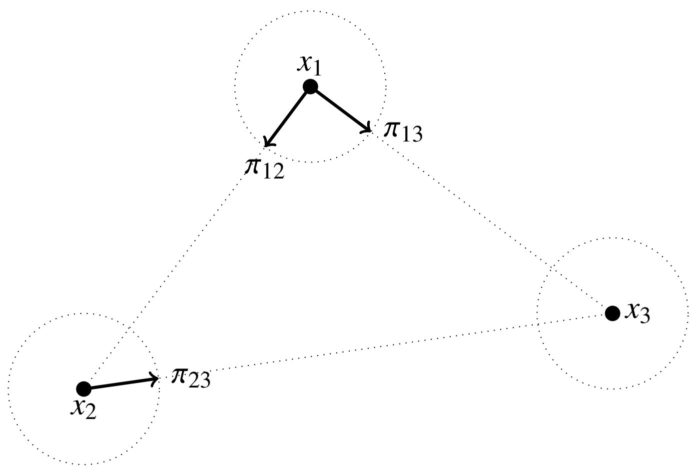
Definition 2.2.10. We define

$$\pi : C_n(\mathbb{R}^m) \to \left(S^{m-1}\right)^{\binom{n}{2}}$$

by

$$(x_1, \dots, x_n) \in C_n(\mathbb{R}^m) \mapsto \left(\pi_{ij} := \frac{x_j - x_i}{|x_j - x_i|}\right)_{(i,j) \in \binom{n}{2}}$$

Figure 2.2.2: The morphism π



Definition 2.2.11. We write

$$C_n[\mathbb{R}^m]$$

the closure of the image of $C_n(\mathbb{R}^m)$ under π.

Finally, we equip these configuration spaces with unit vectors :

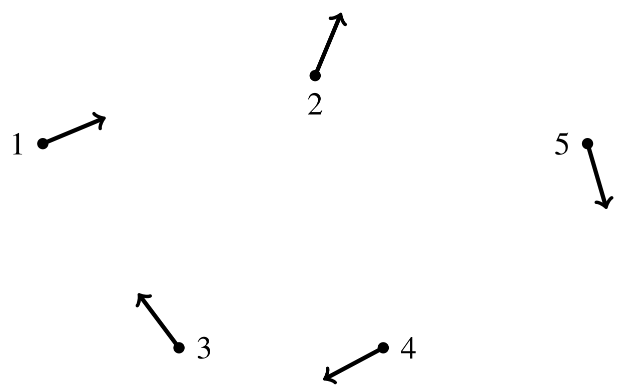
Definition 2.2.12. We write

$$C'_n(\mathbb{R}^m) = C_n(\mathbb{R}^m) \times \left(S^{m-1}\right)^n$$

and

$$C'_n[\mathbb{R}^m] = C_n[\mathbb{R}^m] \times \left(S^{m-1}\right)^n$$

Figure 2.2.3: An element of $C'_5(\mathbb{R}^2)$



2.2.4 Cosimplicial space involving configuration spaces

Let us remind that a cosimplicial object in a category $\mathbb{C}$ is a functor

$$X^\bullet : \Delta \to \mathbb{C}$$

where Δ is the *simplex category*, whose objects are non-negative integers and morphisms are order-preserving maps.

The following proposition is proved in [4, Corollary 4.22] :

Proposition 2.2.13. *There is a cosimplicial space*

$$\mathcal{C}' : \Delta \to Top$$

which sends a non-negative integer i to $C'_i[\mathbb{R}^m]$.

Sketch of the proof. Sinha works with slightly different configuration spaces $C'_i[\mathbb{R}^m, \partial]$ which are the compactification of configuration spaces of $i+2$ points where the first and the last points are fixed.

To a morphism $\sigma : i \to j$ in Δ, Sinha associates a boundary-preserving and order-preserving morphism $\sigma^* : j+1 \to i+1$. The morphism $C'_i[\mathbb{R}^m, \partial] \to C'_j[\mathbb{R}^m, \partial]$ is then induced by σ^*.

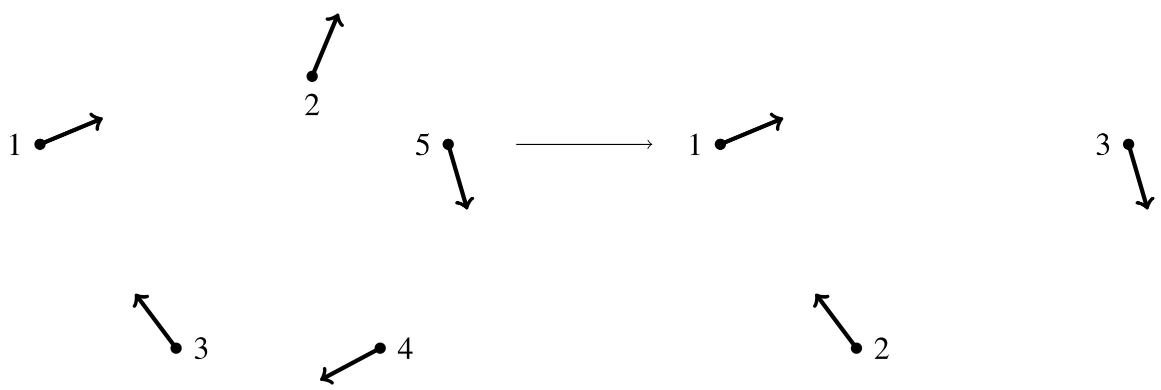
The cosimplicial space $\mathcal{C}'$ can be constructed similarly. □

For example, the morphism

$$\begin{array}{ccc} \{0,1,2,3,4,5\} & \to & \{0,1,2,3\} \\ 0,1,2,3,4,5 & \mapsto & 0,1,1,2,2,3 \end{array}$$

will be sent to the morphism represented is the figure 2.2.4.

Figure 2.2.4: Morphism $C'_5[\mathbb{R}^2] \to C'_3[\mathbb{R}^2]$



Definition 2.2.14. For $n \geq 0$, we write

$$\mathcal{C}'_n : \Delta_n \to Top$$

for the restriction of $\mathcal{C}'$ to Δ_n, where Δ_n is the subcategory of Δ whose objects are non-negative integers $i \leq n$.

2.2.5 Model for embedding spaces

From now on, we will only work with spaces of knots, that is with $Emb(\mathbb{R}^l, \mathbb{R}^m)$ in the case $l = 1$.

The objective of this subsection is to prove the following theorem [4, Theorem 5.4] :

Theorem 2.2.15. *There is a weak equivalence*

$$T_n Emb(\mathbb{R}^1, \mathbb{R}^m) \sim holim\, \mathcal{C}'_n$$

From theorem 2.2.15 and theorem 2.2.9, we deduce immediately the following theorem [4, Theorem 5.5] :

Theorem 2.2.16. *For $m \geq 4$, there is a weak equivalence*

$$Emb(\mathbb{R}^1, \mathbb{R}^m) \sim holim\, \mathcal{C}'$$

To prove this theorem, we need to define functors from $\mathcal{U}_n(\mathbb{R})$ to Δ_n :

Definition 2.2.17. We define the contravariant functors

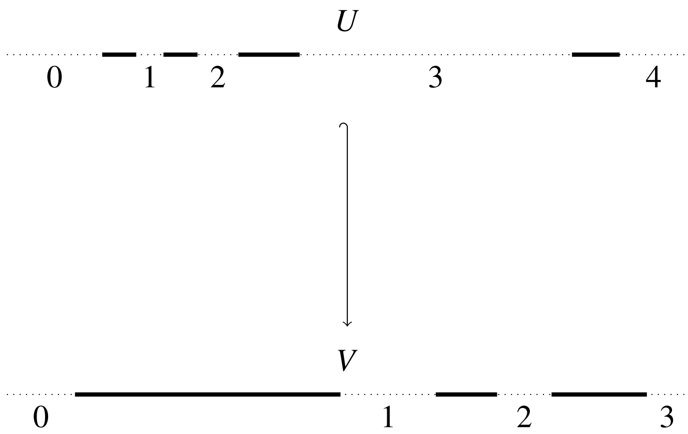
$$F_n : \mathcal{U}_n(\mathbb{R}) \to \Delta_n$$

which send

- an object $U \subset \mathbb{R}$ to the number of connected components of U
- an inclusion $U \subset V$ to the boundary-preserving map induced by the canonical numbering of the connected components of the complement of U and of V

For example, the inclusion described in the following figure

Figure 2.2.5: Inclusion $U \subset V$



is sent to

$$\begin{array}{ccc} \{0,1,2,3\} & \to & \{0,1,2,3,4\} \\ 0,1,2,3 & \mapsto & 0,3,3,4 \end{array}$$

The following lemma and its proof are inspired by [4, Proposition 5.15].

Lemma 2.2.18. *There is a weak equivalence*

$$T_n Emb(\mathbb{R}^1, \mathbb{R}^m) \sim holim\, \mathcal{C}'_n F_n$$

Sketch of the proof. First remind that by definition,

$$T_n Emb(\mathbb{R}^1, \mathbb{R}^m) = \text{holim } Emb(-, \mathbb{R}^m)$$

We would like to prove that there is a natural transformation from $Emb(-, \mathbb{R}^m)$ to $\mathcal{C}'_n F_n$ which is a weak equivalence pointwise. In this case, we get the conclusion from the lemma 2.2.2.

More precisely, we want to find a weak equivalence

$$Emb(U, \mathbb{R}^m) \to C'_i[\mathbb{R}^m]$$

for any $U \in \mathcal{U}_n(\mathbb{R})$ with i connected components.

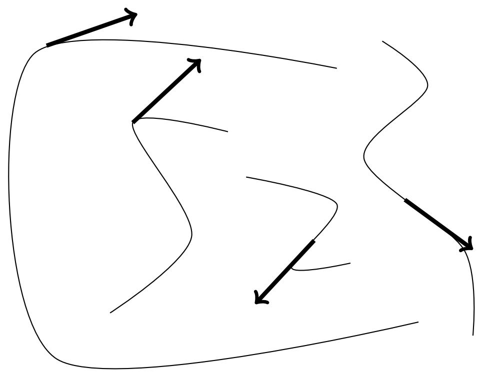
This weak equivalence is given by the evaluation map

$$f \in Emb(U, \mathbb{R}^m) \mapsto \pi\left(f(t_1), \ldots, f(t_i)\right), f'(t_1), \ldots, f'(t_i) \in C'_i[\mathbb{R}^m]$$

where $(t_1, \ldots, t_i)$ are points of each of the i connected components of U, for example their midpoint.

Unfornatunately these evaluation maps do not form a natural transformation $Emb(-, \mathbb{R}^m) \Rightarrow \mathcal{C}'_n F_n$, but this problem can by bypassed easily. For example, in [4, Proposition 5.15], Sinha constructs an intermediate space between $Emb(U, \mathbb{R}^m)$ and $C'_i[\mathbb{R}^m]$.

Figure 2.2.6: The evaluation map



□

Lemma 2.2.19. *The functors $F_n : \mathcal{U}_n(\mathbb{R}) \to \Delta_n$ are left cofinal, therefore there is a weak equivalence*

$$holim\, \mathcal{C}'_n F_n \sim holim\, \mathcal{C}'_n$$

Sketch of the proof. To prove that the functors F_n are cofinal, we just have to show that the nerve of the comma category F_n/d is contractible, which is obvious. □

Proof of theorem 2.2.15. We just have to combine the weak equivalences of the two previous lemmas. □

2.2.6 Fiber of embedding spaces

The theorem 2.2.16 of the previous section gives a good model to study. It is more convenient, however, to work with the configuration spaces $C_n[\mathbb{R}^m]$ instead of $C'_n[\mathbb{R}^m]$.

This is why, in this subsection, following what is done in [6], we will study the fiber of the map

$$Emb(\mathbb{R}^1, \mathbb{R}^m) \to Imm(\mathbb{R}^1, \mathbb{R}^m)$$

Definition 2.2.20. We write

$$\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m)$$

for the fiber over the standard inclusion

$$Emb(\mathbb{R}^1, \mathbb{R}^m) \to Imm(\mathbb{R}^1, \mathbb{R}^m)$$

We can construct a cosimplicial space as in the proposition 2.2.13 :

Proposition 2.2.21. *There is a cosimplicial space*

$$\mathcal{C} : \Delta \to Top$$

which sends an non-negative integer i to $C_i[\mathbb{R}^m]$.

Sketch of the proof. $\mathcal{C}$ can be constructed similarly to $\mathcal{C}'$ in the proposition 2.2.13. □

The following theorem, similar to the theorem 2.2.16, is proved in [6, Theorem 6.4] :

Theorem 2.2.22. *For $m \geq 4$, there is a weak equivalence*

$$\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m) \sim holim\, \mathcal{C}$$

Sketch of the proof. This theorem can be proved by using the same arguments as in the proof of the theorem 2.2.16, since the theorem 2.2.9 is also true for $\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m)$.

The weak equivalence of lemma 2.2.18 becomes the weak equivalence

$$\overline{Emb}(U, \mathbb{R}^m) \to C_i[\mathbb{R}^m]$$

induced by $Emb(U, \mathbb{R}^m) \to C_i'[\mathbb{R}^m]$ and $Imm(U, \mathbb{R}^m) \to (S^{m-1})^i$:

$$\begin{tikzcd}
\overline{Emb}(U, \mathbb{R}^m) \arrow[r] \arrow[d, dotted, "\sim"] & Emb(U, \mathbb{R}^m) \arrow[r] \arrow[d, "\sim"] & Imm(U, \mathbb{R}^m) \arrow[d, "\sim"] \\
C_i[\mathbb{R}^m] \arrow[r] & C_i'[\mathbb{R}^m] \arrow[r] & (S^{m-1})^i
\end{tikzcd}$$

□

2.3 Categorical digression

Before going further, we will remind in this section some elements of category theory.

2.3.1 Non-symmetric operads and operadic morphisms

In this subsection, we introduce the definitions of non-symmetric operads and operadic morphisms.

Let us first remind the definition of non-symmetric operad :

Definition 2.3.1. A *non-symmetric operad* $\mathcal{O}$ in a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ is given by

- an object $\mathcal{O}_n$ in $\mathbb{C}$ for all $n \geq 0$
- a morphism $e : I \to \mathcal{O}_1$ called unit
- morphisms
$$m : \mathcal{O}_k \otimes \mathcal{O}_{n_1} \otimes \ldots \otimes \mathcal{O}_{n_k} \to \mathcal{O}_{n_1+\ldots+n_k}$$
called multiplication

such that the following diagrams commute

$$\begin{tikzcd}
\mathcal{O}_k \otimes I \otimes \ldots \otimes I \arrow[rr, equal] \arrow[dr, "1\otimes e\otimes\ldots\otimes e"'] & & \mathcal{O}_k \\
& \mathcal{O}_k \otimes \mathcal{O}_1 \otimes \ldots \otimes \mathcal{O}_1 \arrow[ur, "m"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \otimes \mathcal{O}_n \arrow[rr, equal] \arrow[dr, "e\otimes 1"'] & & \mathcal{O}_n \\
& \mathcal{O}_1 \otimes \mathcal{O}_n \arrow[ur, "m"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{O}_k \otimes \mathcal{O}_{l_1} \otimes \ldots \otimes \mathcal{O}_{l_k} \otimes \mathcal{O}_{n_{1,1}} \otimes \ldots \otimes \mathcal{O}_{n_{k,l_k}} \arrow[r, "m\otimes 1\otimes\ldots\otimes 1"] \arrow[d, equal] & \mathcal{O}_{l_1+\ldots+l_k} \otimes \mathcal{O}_{n_{1,1}} \otimes \ldots \otimes \mathcal{O}_{n_{k,l_k}} \arrow[dd, "m"] \\
\mathcal{O}_k \otimes \mathcal{O}_{l_1} \otimes \mathcal{O}_{n_{1,1}} \otimes \ldots \otimes \mathcal{O}_{n_{1,l_1}} \otimes \ldots \otimes \mathcal{O}_{l_k} \otimes \mathcal{O}_{n_{k,1}} \otimes \ldots \otimes \mathcal{O}_{n_{k,l_k}} \arrow[d, "1\otimes m\otimes\ldots\otimes m"'] & \\
\mathcal{O}_k \otimes \mathcal{O}_{n_{1,1}+\ldots+n_{1,l_1}} \otimes \ldots \otimes \mathcal{O}_{n_{k,1}+\ldots+n_{k,l_k}} \arrow[r, "m"'] & \mathcal{O}_{n_{1,1}+\ldots+n_{k,l_k}}
\end{tikzcd}$$

Remark 2.3.2. For a sake of brevity, we will often call non-symmetric operads just operads. We hope it will not lead to a confusion.

Definition 2.3.3. An *operadic morphism* between two non-symmetric operads $\mathcal{A}$ and $\mathcal{B}$ is given, for all $n \geq 0$, by a morphism

$$\varphi_n : \mathcal{A}_n \to \mathcal{B}_n$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}_k \otimes \mathcal{A}_{n_1} \otimes \ldots \otimes \mathcal{A}_{n_k} \arrow[r, "m"] \arrow[d, "\varphi_k\otimes\varphi_{n_1}\otimes\ldots\otimes\varphi_{n_k}"'] & \mathcal{A}_{n_1+\ldots+n_k} \arrow[d, "\varphi_{n_1+\ldots+n_k}"] \\
\mathcal{B}_k \otimes \mathcal{B}_{n_1} \otimes \ldots \otimes \mathcal{B}_{n_k} \arrow[r, "m"'] & \mathcal{B}_{n_1+\ldots+n_k}
\end{tikzcd}$$

2.3.2 Multiplicative operads, bimodules and weak bimodules

In this subsection, we introduce the definitions of multiplicative non-symmetric operads, bimodules and weak bimodules.

The most trivial example of non-symmetric operad is the associative operad :

Definition 2.3.4. The *associative operad Ass* in a symmetric monoidal category $\mathbb{C}$ is given for all $n \geq 0$ by

$$Ass_n = I$$

where I is the tensor unit in $\mathbb{C}$.

Remark 2.3.5. In the literature, Ass is often used to denote a symmetrised version of our associative operad.

Definition 2.3.6. A *multiplicative non-symmetric operad* is a non-symmetric operad $\mathcal{O}$ together with an operadic morphism

$$Ass \to \mathcal{O}$$

Definition 2.3.7. Let $\mathcal{A}$ be a non-symmetric operad in a symmetric monoidal category $\mathbb{C}$. A *bimodule* over $\mathcal{A}$ is given by

- an object $\mathcal{B}_n$ in $\mathbb{C}$ for all $n \geq 0$
- morphisms
$$\mathcal{A}_k \otimes \mathcal{B}_{n_1} \otimes \ldots \otimes \mathcal{B}_{n_k} \to \mathcal{B}_{n_1+\ldots+n_k}$$
called left actions
- morphisms
$$\mathcal{B}_k \otimes \mathcal{A}_{n_1} \otimes \ldots \otimes \mathcal{A}_{n_k} \to \mathcal{B}_{n_1+\ldots+n_k}$$
called right actions

satisfying axioms which are obvious analogue of axioms for non-symmetric operad.

Before introducing the notion of weak bimodule, let us remind that a non-symmetric operad can also be defined in terms of $\circ_i$-operations [17, Definition 11]. These $\circ_i$-operations are obtained, for $i = 1, \ldots, k$, as the composite :

$$\mathcal{O}_k \otimes \mathcal{O}_n = \mathcal{O}_k \otimes I \otimes \ldots \otimes \mathcal{O}_n \otimes \ldots \otimes I \xrightarrow{1\otimes e\otimes\ldots\otimes 1\otimes\ldots\otimes e} \mathcal{O}_k \otimes \mathcal{O}_1 \otimes \ldots \otimes \mathcal{O}_n \otimes \ldots \otimes \mathcal{O}_1 \xrightarrow{m} \mathcal{O}_{k+n-1}$$

Definition 2.3.8. Let $\mathcal{A}$ be a non-symmetric operad in a symmetric monoidal category $\mathbb{C}$. A *weak bimodule $\mathcal{W}$* over $\mathcal{A}$ is given by

- an object $\mathcal{W}_n$ in $\mathbb{C}$ for all $n \geq 0$
- for $i = 1, \ldots, k$, morphisms
$$\circ_i : \mathcal{A}_k \otimes \mathcal{W}_n \to \mathcal{W}_{k+n-1}$$
called left actions

- for $i = 1, \ldots, k$, morphisms
$$\bullet_i : \mathcal{W}_k \otimes \mathcal{A}_n \to \mathcal{W}_{k+n-1}$$
called right actions

satisfying axioms which are obvious analogue of axioms for non-symmetric operads in terms of $\circ_i$-operation [17, Definition 11].

Remark 2.3.9. We can define morphisms of bimodules and morphisms of weak bimodules similarly to morphisms of non-symmetric operads.

2.3.3 Forgetful functors from the category of multiplicative operads

In this subsection, we show that there are forgetful functors from the category of multiplicative non-symmetric operads to the category of bimodules over Ass and to the category of weak bimodules over Ass.

Proposition 2.3.10. *A multiplicative non-symmetric operad gives rise to a bimodule over Ass.*

Proof. Let $\mathcal{O}$ be a multiplicative non-symmetric operad with multiplication
$$m : \mathcal{O}_k \otimes \mathcal{O}_{n_1} \otimes \ldots \otimes \mathcal{O}_{n_k} \to \mathcal{O}_{n_1+\ldots+n_k}$$
and multiplicative structure
$$\varphi_n : Ass_n \to \mathcal{O}_n$$

Then the bimodule is given by
$$\mathcal{B}_n = \mathcal{O}_n \text{ for all } n \geq 0$$
and the left and right actions are given by
$$Ass_k \otimes \mathcal{B}_{n_1} \otimes \ldots \otimes \mathcal{B}_{n_k} \xrightarrow{\varphi_k \otimes 1 \otimes \ldots \otimes 1} \mathcal{B}_k \otimes \mathcal{B}_{n_1} \otimes \ldots \otimes \mathcal{B}_{n_k} \xrightarrow{m} \mathcal{B}_{n_1+\ldots+n_k}$$
and
$$\mathcal{B}_k \otimes Ass_{n_1} \otimes \ldots \otimes Ass_{n_k} \xrightarrow{1 \otimes \varphi_{n_1} \otimes \ldots \otimes \varphi_{n_k}} \mathcal{B}_k \otimes \mathcal{B}_{n_1} \otimes \ldots \otimes \mathcal{B}_{n_k} \xrightarrow{m} \mathcal{B}_{n_1+\ldots+n_k}$$
□

Proposition 2.3.11. *A multiplicative non-symmetric operad gives rise to a weak bimodule over Ass.*

Proof. Let $\mathcal{O}$ be a multiplicative non-symmetric operad. Recall that, as it was already mentioned in the previous subsection, such an operad can be defined in terms of $\circ_i$-operations for $i = 1, \ldots, k$ [17, Definition 11]
$$\circ_i : \mathcal{O}_k \otimes \mathcal{O}_n \to \mathcal{O}_{k+n-1}$$
Moreover, we write
$$\varphi_n : Ass_n \to \mathcal{O}_n$$
for the morphisms given by the multiplicative structure.

Then the weak bimodule is given by

$$\mathcal{W}_n = \mathcal{O}_n \text{ for all } n \geq 0$$

and the left and right actions are given, for $i = 1, \ldots, k$, by

$$Ass_k \otimes \mathcal{W}_n \xrightarrow{\varphi_k \otimes 1} \mathcal{W}_k \otimes \mathcal{W}_n \xrightarrow{\circ_i} \mathcal{W}_{k+n-1}$$

and

$$\mathcal{W}_k \otimes Ass_n \xrightarrow{1 \otimes \varphi_n} \mathcal{W}_k \otimes \mathcal{W}_n \xrightarrow{\circ_i} \mathcal{W}_{k+n-1}$$

□

Remark 2.3.12. Let us denote by $NOp(\mathbb{C})$, $MultOp(\mathbb{C})$, $Bimod(\mathbb{C})$ and $WBimod(\mathbb{C})$ the categories of non-symmetric operads, multiplicative non-symmetric operads, bimodules over Ass and weak bimodules over Ass correspondingly. The two previous propositions give us two functors :

$$\Psi^* : MultOp(\mathbb{C}) \to Bimod(\mathbb{C})$$
$$\Upsilon^* : MultOp(\mathbb{C}) \to WBimod(\mathbb{C})$$

We also have a functor

$$\Phi^* : MultOp(\mathbb{C}) \to NOp(\mathbb{C})$$

which forgets the multiplicative structure.

2.3.4 Weak bimodules over Ass and cosimplicial objects

In this subsection, we establish the isomorphism between the category of weak bimodules over Ass and the category of cosimplicial objects in a symmetric monoidal category $\mathbb{C}$.

The following proposition comes from [18, Lemma 4.2] but the proof is inspired by [8, Section 3] :

Proposition 2.3.13. *The category of weak bimodules over Ass is isomorphic to the category of cosimplicial objects in $\mathbb{C}$.*

Sketch of the proof. Let $\mathcal{W}$ be a weak bimodule with left and right actions given, for $i = 1, \ldots, k$, by

$$\circ_i : Ass_k \otimes \mathcal{W}_n \to \mathcal{W}_{k+n-1}$$

and

$$\bullet_i : \mathcal{W}_k \otimes Ass_n \to \mathcal{W}_{k+n-1}$$

Then the cosimplicial object $X^\bullet$ is given by

$$X^n = \mathcal{W}_n \text{ for all } n \geq 0$$

and

$$d^i : X^n \to X^{n+1}$$

is given by

$$d^i = \begin{cases} X^n = Ass_2 \otimes X^n \xrightarrow{\circ_1} X^{n+1} & \text{if } i = 0 \\ X^n = X^n \otimes Ass_2 \xrightarrow{\bullet_i} X^{n+1} & \text{if } 1 \leq i \leq n \\ X^n = Ass_2 \otimes X^n \xrightarrow{\circ_2} X^{n+1} & \text{if } i = n+1 \end{cases}$$

Finally,

$$s^i : X^n \to X^{n-1}$$

is given by

$$X^n = X^n \otimes Ass_0 \xrightarrow{\bullet_i} X^{n-1}$$

It is not hard to see that this gives us a functor from the category of weak bimodules over Ass to the category of cosimplicial objects in $\mathbb{C}$ and that this functor is an isomorphism of categories. □

2.4 Second Sinha's paper

In this section, we will present the main theorem of Sinha's paper [6, Theorem 6.9], which is the weak equivalence

$$\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m) \sim \widetilde{Tot}(\mathcal{K})$$

2.4.1 Choose-two operad

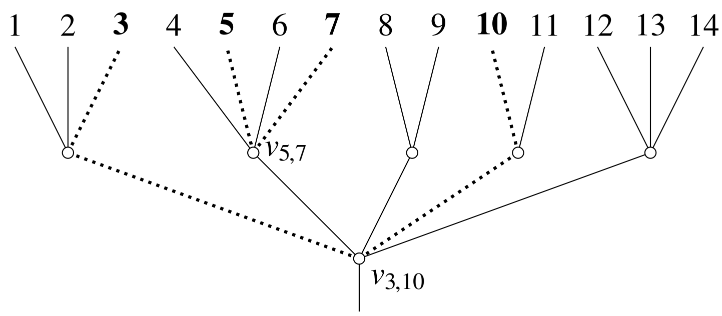
In this subsection, we introduce the non-symmetric version of the choose-two operad. This presentation is inspired by [6, Section 4]. The symmetric version of this operad is presented in [19, Section 3.2].

We want to define the multiplication of the choose-two operad. To the non-negative integers $(k; n_1, \ldots, n_k)$ we associate the tree where the root has k branches and the i-th branch of this root has n_i leaves (see figure 2.4.1 for the case where $(k; n_1, \ldots, n_k) = (5; 3, 4, 2, 2, 3)$).

We call the *root path* of a leaf the shortest path from this leaf to the root. We write $\binom{n}{2}$ for the set of pairs of elements $i, j \in \{1, \ldots, n\}$ with $i < j$. If $(i, j) \in \binom{n}{2}$, we write $v_{i,j}$ for the first vertex at which the root paths of the i-th leaf and the j-th leaf coincide. Also, we write $|v_{i,j}|$ for the number of incoming edges of $v_{i,j}$. Finally, we write $(J_i, J_j) \in \binom{|v_{i,j}|}{2}$ if the i-th leaf and the j-th leaf lies over the J_i-th and the J_j-th incoming edge of $v_{i,j}$ respectively.

For example, in the following picture, $(J_3, J_{10}) = (1, 4) \in \binom{5}{2}$ and $(J_5, J_7) = (2, 4) \in \binom{4}{2}$:

Figure 2.4.1: Multiplication of the choose-two operad



The category $FSet$ of pointed sets is a symmetric monoidal category with tensor product given by pointed union written $\vee$ and tensor unit given by the singleton set 1. We write $FSet^{op}$ for the dual of this category.

For a set S, we also write S_+ for the disjoint union of S with a point.

We can finally define the choose-two operad :

Definition 2.4.1. The *choose-two operad* $\mathcal{B}$ in $(FSet^{op}, \vee, 1)$ is given for all $n \geq 0$ by

$$\mathcal{B}_n = \binom{n}{2}_+$$

and

$$m : \mathcal{B}_k \vee \mathcal{B}_{n_1} \vee \ldots \vee \mathcal{B}_{n_k} \to \mathcal{B}_{n_1+\ldots+n_k}$$

is defined by $m(+) = +$ and for $(i, j) \in \binom{n_1+\ldots+n_k}{2}$,

$$m(i, j) = (J_i, J_j) \in \binom{|v_{i,j}|}{2}$$

2.4.2 Kontsevich operad

Definition 2.4.2. We write

$$\left(S^{m-1}\right)^{\mathcal{B}}$$

for the non-symmetric operad in Top given by $(S^{m-1})^{\binom{n}{2}}$ for all $n \geq 0$ and

$$m : \left(S^{m-1}\right)^{\binom{k}{2}} \times \left(S^{m-1}\right)^{\binom{n_1}{2}} \times \ldots \times \left(S^{m-1}\right)^{\binom{n_k}{2}} \to \left(S^{m-1}\right)^{\binom{n_1+\ldots+n_k}{2}}$$

is induced by the multiplication of the choose-two operad.

The following proposition is in [6, Theorem 4.5] :

Proposition 2.4.3. *The operad $(S^{m-1})^{\mathcal{B}}$ described in definition 2.4.2 restricts on $C_n[\mathbb{R}^m]$.*

We can therefore define a non-symmetric operad called Kontsevich operad [7] :

Definition 2.4.4. The *Kontsevich operad $\mathcal{K}$* is the operad given by the previous proposition.

The Kontsevich operad can be equipped with multiplicative structure [6] :

Proposition 2.4.5. *The Kontsevich operad is multiplicative. Its multiplicative structure*

$$Ass \to \mathcal{K}$$

is given by the basepoint

$$(\boldsymbol{v}, \ldots, \boldsymbol{v}) \in C_n[\mathbb{R}^m]$$

where $\boldsymbol{v}$ is a fixed point in S^{m-1}.

2.4.3 Sinha's theorem

In this subsection, we will present the main result of Sinha's paper [6, Theorem 6.9].

Before introducing this result, let us remind the definition of totalization :

Definition 2.4.6. The totalization

$$\widetilde{Tot}(X^\bullet)$$

of a cosimplicial space $X^\bullet$ is the homotopy limit of this cosimplicial space.

The main result of Sinha, [6, Theorem 6.9], can be stated as follows :

Theorem 2.4.7. *For $m \geq 4$, there is weak equivalence :*

$$\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m) \sim \widetilde{Tot}(\mathcal{K}^\bullet)$$

where $\mathcal{K}^\bullet$ is the cosimplicial space obtained from the Kontsevich operad (see proposition 2.3.13).

Proof of theorem 2.4.7. The theorem 2.2.22 gives us the weak equivalence :

$$\overline{Emb}(\mathbb{R}^1, \mathbb{R}^m) \sim \text{holim } \mathcal{C}$$

It is easy to see that $\mathcal{C}$ is actually the cosimplicial space obtained from the Kontsevich operad, that is $\mathcal{C} = \mathcal{K}^\bullet$. The conclusion is obvious. □

2.5 Delooping theorems

In this section, we present the delooping theorems from Turchin [2] and Dwyer-Hess [1].

2.5.1 Model structures

We will first introduce some notations. We write $SSet$ for the category of simplicial sets.

We write NOp for the category of non-symmetric operads in $SSet$, $MultOp$ for the category of multiplicative non-symmetric operads in $SSet$, $Bimod$ for the category of bimodules over Ass in $SSet$ and $WBimod$ for the category of weak bimodules over Ass in $SSet$.

We equip NOp with projective model structure [3, 20, 21]. That is, $f : A \to B$ is a weak equivalence (resp. fibration) if and only if $Uf : UA \to UB$ is a weak equivalence (resp. fibration), where

$$U : NOp \to SSet^{\mathbb{N}}$$

is the forgetful functor.

The categories $MultOp$, $Bimod$ and $WBimod$ can be equipped with projective model structures similarly to NOp.

Let $\mathbb{M}$ be a model category and $\mathcal{A}, \mathcal{B} \in \mathbb{M}$ be two objects. We write

$$Map_{\mathbb{M}}(\mathcal{A}, \mathcal{B})$$

for the *homotopy mapping space* from $\mathcal{A}$ to $\mathcal{B}$ in the model category $\mathbb{M}$.

Recall that we have three forgetful functors

$$\Phi^* : MultOp \to Nop$$
$$\Psi^* : MultOp \to Bimod$$
$$\Upsilon^* : MultOp \to WBimod$$

If $\mathcal{O}$ is a multiplicative non-symmetric operad, then the spaces $Map_{NOp}(Ass, \Phi^*\mathcal{O})$, $Map_{Bimod}(Ass, \Psi^*\mathcal{O})$ and $Map_{WBimod}(Ass, \Upsilon^*\mathcal{O})$ are pointed with the canonical map.

2.5.2 Turchin - Dwyer-Hess theorems

The main theorem of [2, Theorems 6.2 and 7.2] is the following :

Theorem 2.5.1. *Let $\mathcal{O}$ be a multiplicative non-symmetric operad. If $\mathcal{O}_1 = 1$, then there is a weak equivalence*

$$\Omega Map_{NOp}(Ass, \Phi^*\mathcal{O}) \sim Map_{Bimod}(Ass, \Psi^*\mathcal{O})$$

Moreover, if $\mathcal{O}_0 = 1$, then there is a weak equivalence

$$\Omega Map_{Bimod}(Ass, \Psi^*\mathcal{O}) \sim Map_{WBimod}(Ass, \Upsilon^*\mathcal{O})$$

By combining these two weak equivalences, we get the following corollary.

Corollary 2.5.2. *Let $\mathcal{O}$ be a multiplicative non-symmetric operad. If $\mathcal{O}$ is* reduced*, that is if $\mathcal{O}_0 = \mathcal{O}_1 = 1$, then there is a weak equivalence*

$$\Omega^2 Map_{NOp}(Ass, \Phi^*\mathcal{O}) \sim Map_{WBimod}(Ass, \Upsilon^*\mathcal{O})$$

In the paper of Dwyer-Hess [1, Theorems 1.9 and 1.12], the following theorems are proved:

Theorem 2.5.3. *Let $\mathcal{O}$ be a multiplicative non-symmetric operad. There are a fibration sequences*

$$\Omega Map_{NOp}(Ass, \Phi^*\mathcal{O}) \to Map_{Bimod}(Ass, \Psi^*\mathcal{O}) \to \mathcal{O}_1$$

and

$$\Omega Map_{Bimod}(Ass, \Psi^*\mathcal{O}) \to Map_{WBimod}(Ass, \Upsilon^*\mathcal{O}) \to \mathcal{O}_0$$

Remark 2.5.4. Dwyer and Hess have actually a more general result about monoids in a monoidal model category [1, Theorem 1.7] for which this theorem is a consequence.

Again, we can deduce an immediate corollary about double delooping [1, Theorem 1.1] :

Corollary 2.5.5. *Let $\mathcal{O}$ be a multiplicative non-symmetric operad. If $\mathcal{O}_0$ and $\mathcal{O}_1$ are contractible, then there is a weak equivalence*

$$\Omega^2 Map_{NOp}(Ass, \Phi^*\mathcal{O}) \sim Map_{WBimod}(Ass, \Upsilon^*\mathcal{O})$$

Remark 2.5.6. In fact, both Sinha [4, 6] and Turchin [2] use operads in topological spaces whereas Dwyer and Hess [1] work with simplicial operads. We adopt the last approach. Geometric realisation/singular complex functors obviously establish an equivalence between the two approaches.

3

Introduction to classifiers

3.1 Monads, algebras and lax morphisms

In this section, we will introduce the notion of internal algebra classifier. This section presents some of the ideas developed by Batanin in [12] and Batanin-Berger in [3].

3.1.1 Monads and algebras

In this subsection, we remind the reader of the definitions of monad and algebra.

Recall that a monad on a category $\mathbb{C}$ is given by a functor $T : \mathbb{C} \to \mathbb{C}$ and two natural transformations $\mu : T^2 \Rightarrow T$ and $\eta : id_{\mathbb{C}} \Rightarrow T$ called multiplication and unit respectively, satisfying associativity and identity axioms.

Definition 3.1.1. An *algebra* over a monad T on a category $\mathbb{C}$ is given by

- an object $A \in \mathbb{C}$
- a morphism $\xi_A : TA \to A$

such that the following diagram commute

$$\begin{tikzcd}
T^2A \arrow[r, "T\xi_A"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\xi_A"] \\
TA \arrow[r, "\xi_A"'] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, equal, "1_A"'] & TA \arrow[d, "\xi_A"] \\
& A
\end{tikzcd}$$

Definition 3.1.2. Let T be monad. A *morphism of algebras* between (A, ξ_A) and (B, ξ_B) is given by a morphism $f : A \to B$ such that the following diagram commutes

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\xi_A"'] & TB \arrow[d, "\xi_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Definition 3.1.3. Let T be a monad on a category $\mathbb{C}$. We write

$$Alg_T(\mathbb{C})$$

for the category of algebras over T.

Remark 3.1.4. If the category $\mathbb{C}$ is implicitly given, we will simply write Alg_T instead of $Alg_T(\mathbb{C})$.

3.1.2 2-monads and their lax morphisms

There is a 2-dimensional generalisation of the classical theory of monads [3]. There are different versions of it [22] but we only need strict 2-monads. These are simply monads enriched over Cat. The definition of category of algebras is just a Cat-enriched version of classical definition but since the resulting category is Cat-enriched, we have a 2-category of algebras for a 2-monad T. A new phenomenon in here is that we can extend the usual definition of morphism between algebras (now called strict T-algebras morphisms).

Definition 3.1.5. Let T be a cartesian monad on a 2-category. A *lax morphism* between two algebras (A, ξ_A) and (B, ξ_B) over T is given by

- a morphism $f : A \to B$
- a 2-cell $\phi : \xi_B \cdot Tf \Rightarrow f \cdot \xi_A$

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\swarrow\phi"] & TB \arrow[d, "\xi_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

such that

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\
TA \arrow[r, "Tf"] \arrow[d, "\xi_A"'] \arrow[dr, phantom, "\Swarrow\phi"] & TB \arrow[d, "\xi_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dd, "1_A"'] & B \arrow[dd, "1_B"] \\
& \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & T^2A \arrow[dll, "\mu_A"'] \arrow[drr, "T^2f"] & & \\
TA \arrow[drr, "Tf"'] \arrow[dd, "\xi_A"'] & & & & T^2B \arrow[dll, "\mu_B"] \arrow[dd, "T\xi_B"] \\
\arrow[r, phantom, "\Swarrow\phi"] & & TB \arrow[dd, "\xi_B"] & & \\
A \arrow[drr, "f"'] & & & & TB \arrow[dll, "\xi_B"] \\
& & B & &
\end{tikzcd}
=
\begin{tikzcd}
& & T^2A \arrow[dll, "\mu_A"'] \arrow[drr, "T^2f"] \arrow[dd, "T\xi_A"] & & \\
TA \arrow[dd, "\xi_A"'] & & & \arrow[r, phantom, "\Swarrow T\phi"] & T^2B \arrow[dd, "T\xi_B"] \\
& & TA \arrow[dll, "\xi_A"] \arrow[drr, "Tf"'] & & \\
A \arrow[drr, "f"'] & & \arrow[d, phantom, "\Leftarrow\phi"] & & TB \arrow[dll, "\xi_B"] \\
& & B & &
\end{tikzcd}$$

3.2 Internal algebra classifiers

3.2.1 Internal categories

Let $\mathbb{C}$ be a category with pullbacks. We will define internal categories in $\mathbb{C}$ as 2-truncated simplicial objects

$$\begin{tikzcd}
C_0 \arrow[r, "i" description] & C_1 \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & C_1 \times_{C_0} C_1 \arrow[l, shift right=2, "p_1"'] \arrow[l, "m" description] \arrow[l, shift left=2, "p_2"]
\end{tikzcd}$$

with the additional property that the following square commutes :

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "1\times_{C_0} m"] \arrow[d, "m\times_{C_0} 1"'] & C_1 \times_{C_0} C_1 \arrow[d, "m"] \\
C_1 \times_{C_0} C_1 \arrow[r, "m"'] & C_1
\end{tikzcd}$$

One can also define internal functors and internal natural transformations [3].

Definition 3.2.1. Let $\mathbb{C}$ be a category with pullbacks. We write

$$Cat(\mathbb{C})$$

for the 2-category whose

- objects are internal categories in $\mathbb{C}$
- morphisms are internal functors
- 2-cells are internal natural transformations

Remark 3.2.2. It is well known that internal categories in $\mathbb{C}$ can be defined equivalently as simplicial objects in $\mathbb{C}$ which satisfy Segal's conditions.

Definition 3.2.3. We say that a monad T on a category with pullbacks is *cartesian* if

- it preserves pullbacks
- the multiplication and unit are cartesian natural transformations, that is all naturality squares involved in these tranformations are pullbacks

Proposition 3.2.4. *[3, 12] A cartesian monad T on a category $\mathbb{C}$ with pullbacks induces a 2-monad on $Cat(\mathbb{C})$.*

Proof. We apply T termwise to an internal category in $\mathbb{C}$. Since T preserves pullbacks, the resulting simplicial object is again an internal category in $\mathbb{C}$. It is obvious also that this correspondence can be equipped with the 2-monad structure extending the monad T. □

By slightly abusing notations, we will call this 2-monad T again. We will also call algebras of T in $Cat(\mathbb{C})$ categorical algebras of T.

3.2.2 Absolute internal algebra classifiers

Definition 3.2.5. Let T be a monad on a 2-category $\mathbb{C}$ with terminal object and A be a T-algebra. An *internal T-algebra a* in A is given by a lax morphism of T-algebra

$$a : 1 \xrightarrow{lax} A$$

where 1 is the terminal T-algebra.

The following theorem is [3, Theorem 5.4] :

Theorem 3.2.6. *Let T be a cartesian monad on $Cat(\mathbb{C})$. There is a categorical T-algebra T^T such that for all categorical T-algebra A, there is an isomorphism between the category of internal T-algebras in A and the category of strict T-algebras morphisms $T^T \to A$:*

$$\frac{1 \xrightarrow{lax} A}{T^T \to A}$$

Moreover the underlying internal category T^T is given by

$$\begin{tikzcd} T1 & T^2 1 \arrow[l, shift right=2, "\mu_1"'] \arrow[l, shift left=2, "T!"] & T^3 1 \arrow[l, shift right=2, "\mu_{T1}"'] \arrow[l, "T\mu_1" description] \arrow[l, shift left=2, "T^2!"] \\ \end{tikzcd}$$

$$T1 \xrightarrow{T\eta_1} T^2 1$$

Definition 3.2.7. The T-algebra T^T is called the *absolute internal algebra classifier* of T.

Example 3.2.8. The *free monoid monad* [3]

$$\mathbf{Mon} : Set \to Set$$

which is defined by

$$\mathbf{Mon}(X) = \coprod_{n \in \mathbb{N}} X^n$$

gives a cartesian monad on Set. The multiplication of this monad is the concatenation. The category of algebras of this monad is the category of monoids.

This monad induces a 2-monad on $Cat(Set) = Cat$ (see proposition 3.2.4). Algebras over this 2-monad are strict monoidal categories. Internal algebras are monoids in these strict monoidal categories.

The absolute classifier $\mathbf{Mon}^{\mathbf{Mon}}$ is the *augmented simplex category* Δ_+, that is the simplex category Δ, whose objects are non-negative integers and morphisms are order-preserving functions, augmented with an initial object [23].

3.2.3 Relative internal algebra classifiers

A cartesian morphism between two cartesian monads is a morphism between monads where all natural transformations involved are cartesian. Notice that these two monads can be monads on two different categories, so that a part of the morphism structure is a functor between these categories.

Definition 3.2.9. Let $\Phi : S \to T$ be a cartesian monad morphism between two cartesian monads. We write

$$\Phi^* : Alg_T \to Alg_S$$

for the restriction functor between categories of algebras.

Definition 3.2.10. Let $\Phi : S \to T$ be a monad morphism and A be a T-algebra. An *internal S-algebra a* in A is given by a lax morphism of S-algebra

$$a : 1 \xrightarrow{lax} \Phi^* A$$

where 1 is the terminal S-algebra.

The following theorem is [3, Theorem 5.10] :

Theorem 3.2.11. *Let $\Phi : S \to T$ be a cartesian monad morphism. There is a categorical T-algebra T^S such that for all categorical T-algebra A, there is an isomorphism between the category of internal S-algebras in A and the category of strict T-algebras morphisms $T^S \to A$:*

$$\frac{1 \xrightarrow{lax} \Phi^* A}{T^S \to A}$$

Moreover the underlying internal category T^S is given by

$$\begin{tikzcd} T1 \arrow[r, "T\eta_1" description] & TS1 \arrow[l, shift right=2, "\mu_1 \cdot T\Phi_1"'] \arrow[l, shift left=2, "T!"'] & TS^2 1 \arrow[l, "T\mu_1" description] \arrow[l, shift right=2, "\mu_{S1} \cdot T\Phi_{S1}"'] \arrow[l, shift left=2, "TS!"'] \end{tikzcd}$$

Example 3.2.12. The monad
$$\mathbf{Id}_+ : Set \to Set$$
which is defined by
$$\mathbf{Id}_+(X) = 1 \coprod X$$
gives a cartesian monad on Set. The category of algebras of this monad is the category of pointed sets.

This monad induces a 2-monad on $Cat(Set) = Cat$ (see proposition 3.2.4).

There is a canonical cartesian monad morphism
$$\mathbf{Id}_+ \to \mathbf{Mon}$$
where **Mon** is the monad of the example 3.2.8.

The classifier $\mathbf{Mon}^{\mathbf{Id}_+}$ is the category Δ_+^{inj} which is the subcategory of Δ_+ (see example 3.2.8) where morphisms are injective.

3.2.4 Polynomial monads

In this subsection, we introduce the definition of polynomial monads, following what is done in [3, Section 6]. We will see later, and particularly in the subsection 4.1.2, that polynomial monads are easy to describe.

Definition 3.2.13. A *polynomial* is a diagram in Set of the form
$$J \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

Remark 3.2.14. A polynomial induces a functor
$$P : Set/J \to Set/I$$
which is defined as the composite
$$Set/J \xrightarrow{s_*} Set/E \xrightarrow{p^*} Set/B \xrightarrow{t_!} Set/I$$
where
$$s_*(X)_e = X_{s(e)}$$
and
$$p^*(X)_b = \prod_{e \in p^{-1}(b)} X_e$$
and
$$t_!(X)_i = \coprod_{b \in t^{-1}(i)} X_b$$

Definition 3.2.15. A monad
$$T : Set/I \to Set/I$$
is *polynomial* if its functor part is induced by a polynomial and unit and multiplication are cartesian natural transformations.

Definition 3.2.16. A polynomial monad is *finitary* if the generating polynomial is of finite type, that is $p^{-1}(b)$ is a finite set for any $b \in B$.

From now on we will consider only finitary polynomial monads.

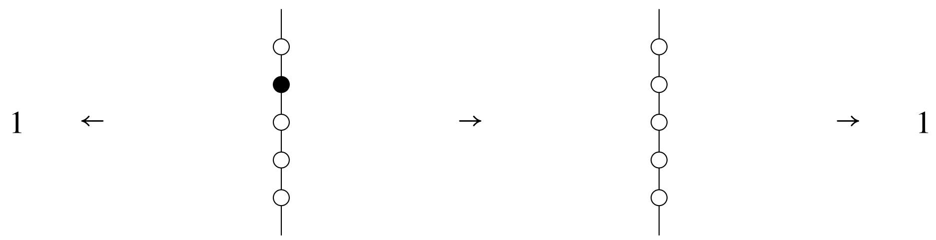
Example 3.2.17. The free monoid monad **Mon** of the example 3.2.8 is a finitary polynomial monad. Indeed, this monad induced by the polynomial

$$1 \xleftarrow{\ s\ } MonTr^* \xrightarrow{\ p\ } MonTr \xrightarrow{\ t\ } 1$$

where

- $MonTr^*$ is the set of linear trees with one marked vertex
- $MonTr$ is the set of linear trees
- p forgets the marking

Figure 3.2.1: Representation of the polynomial



Example 3.2.18. The monads for non-symmetric operads, multiplicative non-symmetric operads, bimodules and weak bimodules over Ass are finitary polynomial monads. We will give their explicit description in the subsection 4.1.2.

Definition 3.2.19. A *morphism of polynomials* is given by a diagram of the form

$$\begin{tikzcd}
I' \arrow[d] & E' \arrow[l, "s'"'] \arrow[r, "p'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[r, "t'"] \arrow[d] & I' \arrow[d] \\
I & E \arrow[l, "s"] \arrow[r, "p"'] & B \arrow[r, "t"'] & I
\end{tikzcd}$$

where the horizontal lines are polynomials and the middle square is a pullback.

Definition 3.2.20. A *morphism of polynomial monads* is a morphism of the corresponding polynomials compatible with the multiplications and units in the obvious sense.

Proposition 3.2.21 ([24])**.** *The category of finitary polynomial monads over I is equivalent to the category of symmetric I-coloured operads in Set. In particular, the monad generated by a finitary polynomial monad is cartesian.*

Remark 3.2.22. Since polynomial monads are cartesian and every morphism of polynomial monads induces a cartesian morphism of monads, the theory of internal algebra classifiers (see theorems 3.2.6 and 3.2.11) is applicable to this class of monads. In the Batanin-Berger paper [3], an explicit description of the classifiers in terms of polynomial monad morphism is given.

Remark 3.2.23. It was shown in [3, Proposition 6.9], that finitary polynomial monads may have algebras in an arbitrary symmetric monoidal category $\mathbb{C}$. From now on, we will write $Alg_T\mathbb{C}$ for this category of algebras if the category $\mathbb{C}$ has to be specified. Sometimes we will omit $\mathbb{C}$ from the notation for the sake of brevity if the category in question is clear from the context.

3.2.5 Important results about classifiers

The following results about classifiers are contained or can be deduced from [3].

This first theorem can be deduced from [3, Theorem 6.17] :

Theorem 3.2.24. *Let $\Phi : S \to T$ be a polynomial monad morphism and $\mathbb{C}$ be a cocomplete symmetric monoidal category. Then the restriction functor*

$$\Phi^* : Alg_T\mathbb{C} \to Alg_S\mathbb{C}$$

has a left adjoint

$$\Phi_! : Alg_S\mathbb{C} \to Alg_T\mathbb{C}$$

Moreover, if $Alg_S\mathbb{C}$ and $Alg_T\mathbb{C}$ admit projective model structures then this adjunction is a Quillen adjunction.

Remark 3.2.25. It was observed in [3] that for $\mathbb{C} = SSet$ or Top the projective model structures on $Alg_T\mathbb{C}$ exists for any finitary polynomial monad T.

Proposition 3.2.26. *For a polynomial monad morphism $\Phi : S \to T$, there is a natural isomorphism of categorical T-algebras*

$$\Phi_!(S^S) \simeq T^S$$

Sketch of the proof. The category Cat is a cocomplete symmetric monoidal category. So $\Phi_!$ exists and the isomorphism above can be established by comparing the universal properties of $\Phi_!(S^S)$ and T^S. □

The following theorem is the homotopy analogue of proposition 3.2.26. It can be deduced from [3, Theorem 8.2] :

Theorem 3.2.27. *For any polynomial monad morphism $\Phi : S \to T$, the simplicial T-algebra $N(T^S)$ is cofibrant. Moreover, there is a natural weak equivalence of simplicial T-algebras*

$$\Phi_! N(S^S) \sim N(T^S)$$

where N is the nerve functor applied componentwise to the corresponding categorical algebras.

We can deduce from this theorem the following corollary :

Corollary 3.2.28. *Let T be a polynomial monad. Then the nerve $N(T^T)$ is a cofibrant replacement of the terminal T-algebras in the model category of simplicial T-algebras.*

Sketch of the proof. If I is the set of colours of T then the T-algebras T^T is an I-collection of categories. For each $i \in I$, $(T^T)_i$ has a terminal object given by the canonical lax morphism $(1)_i \xrightarrow{lax} (T^T)_i$ [3]. □

The following fact has been proved by Michael Batanin as a part of a theorem about characterisation of aspherical morphisms between polynomial monads :

Theorem 3.2.29. *For a commutative diagram of polynomial monads*

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "h"'] & & T \arrow[dl, "g"] \\ & R & \end{tikzcd}$$

$N(T^S)$ is contractible if and only if $N(R^f) : N(R^S) \to N(R^T)$ is a weak equivalence of simplicial R-algebras for any commutative triangle as above.

Sketch of the proof. Assume that $N(T^S)$ is contractible. Then the canonical map $T^f : T^S \to T^T$ induces a weak equivalence between nerves because of the corollary 3.2.28.

The functor $g_!$ is a left Quillen functor and, thanks to theorem 3.2.27, $N(T^S)$ and $N(T^T)$ are cofibrant. This implies that

$$g_! N(T^S) \to g_! N(T^T)$$

is a weak equivalence.

Thanks to theorem 3.2.27, we have on the left

$$g_! N(T^S) \sim g_! f_! N(S^S) \simeq h_! N(S^S) \sim N(R^S)$$

and on the right

$$g_! N(T^T) \sim N(R^T)$$

Conversely, suppose $N(R^f)$ is a weak equivalence for any morphism $g : T \to R$. Take $g = id$. So, we have that $N(T^S) \to N(T^T)$ is a weak equivalence. Since $N(T^T)$ is contractible, we finish the proof. $\square$

4

New proof of the delooping theorems

4.1 First delooping using classifiers

We will concentrate in this chapter on a proof of the first delooping of the Dwyer-Hess theorem 2.5.3. The second delooping can be proved similarly. We will also comment on how the first delooping of the Turchin's theorem 2.5.1 admits a similar treatment.

4.1.1 Classifiers and mapping spaces

Let us introduce some notations.

Definition 4.1.1. Let T be a polynomial monad. The category of simplicial algebras over T, $Alg_T = Alg_T(SSet)$, has a natural enrichment over $SSet$, and, for $X, Y \in Alg_T$, we write

$$SSet_{Alg_T}(X, Y)$$

for the associate simplicial set.

Remark 4.1.2. [21] The model category of simplicial algebras of T is also a simplicial model category, so that for $X, Y \in Alg_T$, we have

$$Map_{Alg_T}(X, Y) \sim SSet_{Alg_T}(cof(X), fib(Y))$$

where cof and fib are cofibrant and fibrant replacements respectively.

The following weak equivalence will be used later :

Theorem 4.1.3. *Let S, T be polynomial monads, $X \in Alg_T$ and $\Phi : S \to T$ be a cartesian morphism of monads. Then there is a weak equivalence*

$$Map_{Alg_S}(1, \Phi^* X) \sim SSet_{Alg_T}(N(T^S), fibX)$$

where 1 is the terminal object in Alg_S.

Proof. Thanks to theorem 3.2.24, there is an adjunction

$$\Phi_! : Alg_S \leftrightarrows Alg_T : \Phi^*$$

We have successively

$$\begin{aligned}
Map_{Alg_S}(1, \Phi^* X) &\sim SSet_{Alg_S}(cof(1), fib(\Phi^* X)) && \text{remark 4.1.2} \\
&\sim SSet_{Alg_S}(cof(1), \Phi^* fibX) && \\
&\sim SSet_{Alg_S}(N(S^S), \Phi^* fibX) && \text{corollary 3.2.28} \\
&\simeq SSet_{Alg_T}(\Phi_! N(S^S), fibX) && \text{adjunction} \\
&\sim SSet_{Alg_T}(N(T^S), fibX) && \text{theorem 3.2.27}
\end{aligned}$$

□

4.1.2 Polynomial representation of basic monads

Definition 4.1.4. We write $NOpTr$ for the set of planar trees with white vertices and $NOpTr^*$ for the set of planar trees with white vertices and one marked vertex.

We write

$$\mathbf{NOp}$$

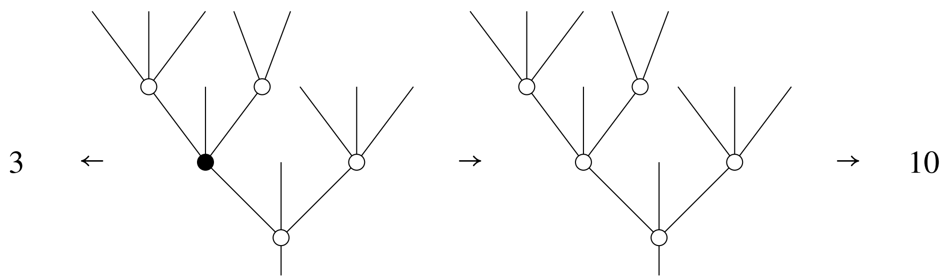
for the monad induced by the polynomial

$$\mathbb{N} \xleftarrow{s} NOpTr^* \xrightarrow{p} NOpTr \xrightarrow{t} \mathbb{N}$$

where

- s counts the number of incoming edges of the marked vertex
- p forgets the marking
- t counts the number of leaves
- multiplication in this monad is induced by insertion of a tree inside a vertex of another tree

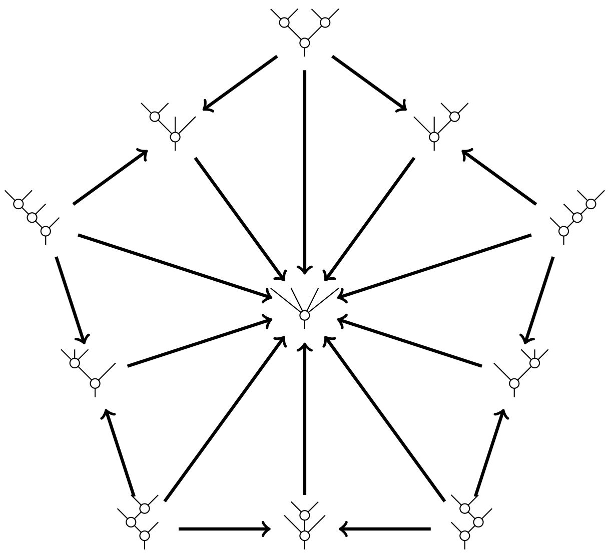
Figure 4.1.1: Representation of the polynomial



Remark 4.1.5. The category of algebras of **NOp** is the category of non-symmetric operads. The classifier $\mathbf{NOp}^{\mathbf{NOp}}$ is the classical categorical operad of trees.

Remark 4.1.6. If in the definition of the polynomial above, we use planar trees whose vertices have valencies at least 3, we obtain a monad for the reduced version of the non-symmetric operads. The corresponding absolute classifier is the collection of categories $(\pentagon_n)_{n\in\mathbb{N}}$ described in Turchin's paper [2].

Figure 4.1.2: The category $\pentagon_4$ [2, Figure 4]



Definition 4.1.7. We write

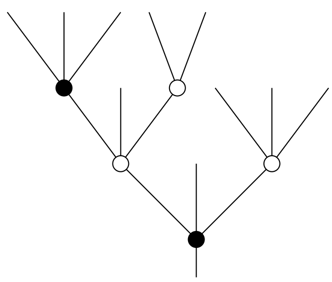
$$NOpTr_+$$

for the set of planar trees with black and white vertices, where there can not be two adjacent black vertices and

$$NOpTr_+^*$$

if one white vertex is marked.

Figure 4.1.3: A tree in $NOpTr_+$



Definition 4.1.8. We write

$$\mathbf{NOp}_+$$

for the monad induced by the polynomial

$$\mathbb{N} \xleftarrow{s} NOpTr_+^* \xrightarrow{p} NOpTr_+ \xrightarrow{t} \mathbb{N}$$

where

- s counts the number of incoming edges of the marked vertex
- p forgets the marking
- t counts the number of leaves
- multiplication in this monad is induced by insertion of a tree inside a white vertex of another tree and contraction of edges which connect two black vertices

Remark 4.1.9. The category of algebras of $\mathbf{NOp}_+$ is the category $MultOp$ of multiplicative non-symmetric operads. The description of this polynomial monad as a coloured Σ-free operad is given in [19]. This operad is denoted $\mathcal{L}_{(2)}$ in [19] as it is identified with the second filtration stage of the *lattice path operad*.

Definition 4.1.10. We write

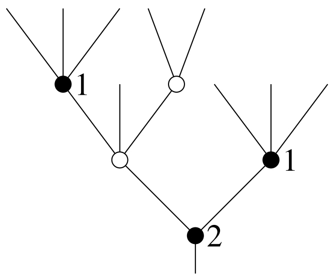
$$NOpTr_{++}$$

for the set of planar trees with white vertices and two types of black vertices, where there can not be two adjacent black vertices of the same type and

$$NOpTr_{++}^*$$

if one white vertex is marked.

Figure 4.1.4: A tree in $NOpTr_{++}$



Definition 4.1.11. We write

$$\mathbf{NOp}_{++}$$

for the monad induced by the polynomial

$$\mathbb{N} \xleftarrow{s} NOpTr_{++}^* \xrightarrow{p} NOpTr_{++} \xrightarrow{t} \mathbb{N}$$

where

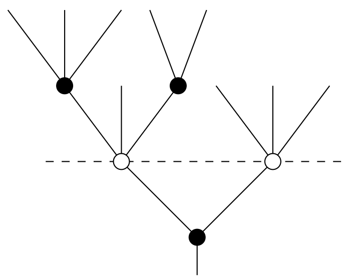
- s counts the number of incoming edges of the marked vertex
- p forgets the marking
- t counts the number of leaves
- multiplication in this monad is induced by insertion of a tree inside a white vertex of another tree and contraction of edges which connect two black vertices of the same type

Remark 4.1.12. The category of algebras of $\mathbf{NOp}_{++}$ is the category of double multiplicative non-symmetric operads, that is, non-symmetric operads $\mathcal{O}$ equipped with two operadic morphisms $\alpha, \beta : Ass \to \mathcal{O}$.

Since the description of these polynomial monads are very similar, we will omit some explanations for the rest of our monads list.

Definition 4.1.13. We write
$$BimodTr$$
for the set of planar trees with black and white vertices, where there can not be two adjacent black vertices and white vertices are aligned on the same level.

Figure 4.1.5: A tree in $BimodTr$



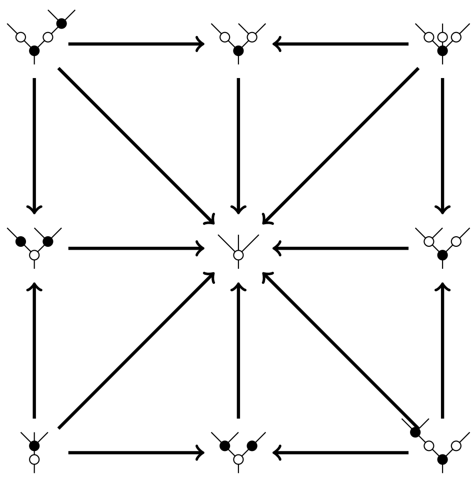
Definition 4.1.14. We write
$$\mathbf{Bimod}$$
for the monad induced by the polynomial

$$\mathbb{N} \xleftarrow{\ s\ } BimodTr^* \xrightarrow{\ p\ } BimodTr \xrightarrow{\ t\ } \mathbb{N}$$

Remark 4.1.15. The category of algebras of **Bimod** is the category of bimodules over Ass.

Remark 4.1.16. If in the definition of the polynomial above, we use planar trees whose black and white vertices have valencies at least 2, we obtain a monad for the reduced version of bimodules. The corresponding absolute classifier is the collection of categories $(\Box_n)_{n\in\mathbb{N}}$ described in Turchin's paper [2].

Figure 4.1.6: The category $\Box_3$ [2, Figure 3]

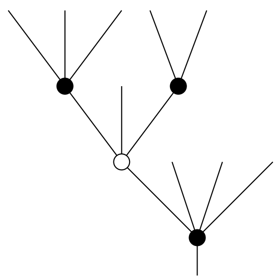


Definition 4.1.17. We write

$$WBimodTr$$

for the set of planar trees with black and white vertices, where there can not be two adjacent black vertices and there is only one white vertex.

Figure 4.1.7: A tree in $WBimodTr$



Definition 4.1.18. We write

$$\mathbf{WBimod}$$

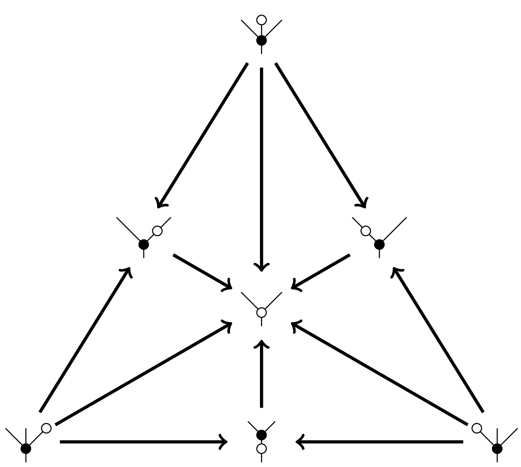
for the monad induced by the polynomial

$$\mathbb{N} \xleftarrow{s} WBimodTr^* \xrightarrow{p} WBimodTr \xrightarrow{t} \mathbb{N}$$

Remark 4.1.19. The category of algebras of **WBimod** is the category of weak bimodules over Ass.

Remark 4.1.20. If in the definition of the polynomial above, we use planar trees whose black vertices have valencies at least 3, we obtain a monad for the reduced version of weak bimodules. The corresponding absolute classifier is the collection of categories $(\triangle_n)_{n\in\mathbb{N}}$ described in Turchin's paper [2].

Figure 4.1.8: The category $\triangle_2$ [2, Figure 2]



Proposition 4.1.21. *There are morphisms of polynomial monads*

$$\Phi : \boldsymbol{NOp} \to \boldsymbol{NOp}_+$$
$$\Psi : \boldsymbol{Bimod} \to \boldsymbol{NOp}_+$$
$$\Upsilon : \boldsymbol{WBimod} \to \boldsymbol{NOp}_+$$

such that the forgetful functors

$$\Phi^* : MultOp \to NOp$$
$$\Psi^* : MultOp \to Bimod$$
$$\Upsilon^* : MultOp \to WBimod$$

are restriction functors along these morphisms.

Proof. These morphisms are induced by the inclusion of $NOpTr$, $BimodTr$ and $WBimodTr$ into $NOpTr_+$. □

We will also need several other polynomial monads :

Definition 4.1.22. We write

$$BimodTr_+$$

for the set of planar trees with black and white vertices in $BimodTr$, where leaves can be added to the lowest black vertex, plus trees with only one black vertex and no white vertices.

Definition 4.1.23. We write

$$\mathbf{Bimod}_+$$

for the monad induced by the polynomial

$$\mathbb{N} \xleftarrow{s} BimodTr_+^* \xrightarrow{p} BimodTr_+ \xrightarrow{t} \mathbb{N}$$

Remark 4.1.24. The algebras of $\mathbf{Bimod}_+$ are bimodules $\mathcal{B}$ over Ass equiped with an additional morphism $1 \to \mathcal{B}_1$. We call them *pointed bimodules*.

Let also **Id** be the identity monad on Set. This monad is a polynomial monad induced by

$$1 \xleftarrow{s} 1 \xrightarrow{p} 1 \xrightarrow{t} 1$$

Finally, we consider the polynomial monad $\mathbf{Id}_+$ on Set which adds a point to a set (see example 3.2.12). So algebras of $\mathbf{Id}_+$ are pointed sets. The polynomial for this monad is

$$1 \xleftarrow{s} 1 \xrightarrow{p} 1 \coprod 1 \xrightarrow{t} 1$$

where p is the inclusion of 1 (a single point) to the two elements set $1 \coprod 1$.

4.1.3 Turchin's theorem in the language of classifiers

Lemma 4.1.25. *The following commutative square is a homotopy pushout of cofibrant multiplicative operads whose legs are cofibrations :*

$$\begin{tikzcd} N\boldsymbol{NOp}_+^{\boldsymbol{Nop}_{++}} \arrow[dr, phantom, "\lrcorner", very near start] & N\boldsymbol{NOp}_+^{\boldsymbol{Nop}_+} \arrow[l] \\ N\boldsymbol{NOp}_+^{\boldsymbol{Nop}_+} \arrow[u] & N\boldsymbol{NOp}_+^{\boldsymbol{Nop}} \arrow[l] \arrow[u] \end{tikzcd} \tag{4.1}$$

Proof. First we observe from universal property that there is a pushout of categorical multiplicative non-symmetric operads :

$$\begin{tikzcd} \mathbf{NOp}_+^{\mathbf{Nop}_{++}} \arrow[dr, phantom, "\lrcorner"] & \mathbf{NOp}_+^{\mathbf{Nop}_+} \arrow[l] \\ \mathbf{NOp}_+^{\mathbf{Nop}_+} \arrow[u] & \mathbf{NOp}_+^{\mathbf{Nop}} \arrow[l] \arrow[u] \end{tikzcd}$$

To finish the proof we apply [3, Theorem 8.2] again. □

Theorem 4.1.26. *For any multiplicative non-symmetric operad $\mathcal{O}$ in SSet, there are two weak equivalences*

$$\Omega Map_{NOp}(Ass, \Phi^*\mathcal{O}) \sim \Omega SSet_{\boldsymbol{NOp}_+}(N\boldsymbol{NOp}_+^{\boldsymbol{NOp}}, fib\mathcal{O}) \sim SSet_{\boldsymbol{NOp}_+}(N\boldsymbol{NOp}_+^{\boldsymbol{NOp}_{++}}, fib\mathcal{O})$$

Sketch of the proof. The first equivalence is obtained by an application of the theorem 4.1.3 to the morphism of polynomial monads

$$\Phi : \mathbf{NOp} \to \mathbf{NOp}_+$$

For the second equivalence, we apply the contravariant functor

$$SSet_{\mathbf{NOp}_+}(-, fib\mathcal{O})$$

to the pushout 4.1, to get the following homotopy pullback

$$\begin{tikzcd} SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{NOp}_{++}}, fib\mathcal{O}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{NOp}_+}, fib\mathcal{O}\right) \arrow[d] \\ SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{NOp}_+}, fib\mathcal{O}\right) \arrow[r] & SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{NOp}}, fib\mathcal{O}\right) \end{tikzcd}$$

Thanks to the theorem 4.1.3,

$$SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{NOp}_+}, fib\mathcal{O}\right) \sim Map_{NOp_+}(Ass, \mathcal{O})$$

which is contractible since Ass is a zero object (both initial and terminal) in NOp_+. Hence we have the second weak equivalence

$$\Omega SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{NOp}}, fib\mathcal{O}\right) \sim SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{NOp}_{++}}, fib\mathcal{O}\right)$$

□

Theorem 4.1.27. *For any multiplicative non-symmetric operad $\mathcal{O}$ in SSet, there is a weak equivalence*

$$Map_{Bimod}(Ass, \Psi^*\mathcal{O}) \sim SSet_{\boldsymbol{NOp}_+}\left(N\boldsymbol{NOp}_+^{\boldsymbol{Bimod}}, fib\mathcal{O}\right)$$

Proof. It is a direct application of the theorem 4.1.3. □

At this point, we can already apply the results obtained to prove the first delooping of Turchin's theorem 2.5.1. We only need to adapt the argument to the reduced versions of non-symmetric operads and bimodules as used by Turchin [2]. To simplify the exposition, we use the same notations for polynomial monads involved as in the previous sections. We warn the reader that we do it only in the rest of this section.

Theorem 4.1.28 (Turchin)**.** *For any multiplicative non-symmetric operad $\mathcal{O}$ in SSet such that $\mathcal{O}_1 = 1$, there is weak equivalence*

$$\Omega Map_{NOp}(Ass, \Phi^*\mathcal{O}) \sim Map_{Bimod}(Ass, \Psi^*\mathcal{O})$$

Sketch of the proof. Thanks to the two previous theorems, we only need to establish that the morphism of monads $\mathbf{Bimod} \to \mathbf{NOp}_{++}$ induces a weak equivalence

$$N\mathbf{NOp}_{+}^{\mathbf{Bimod}} \to N\mathbf{NOp}_{+}^{\mathbf{NOp}_{++}} \tag{4.2}$$

There is the following commutative triangle of polynomial monads :

$$\begin{tikzcd} \mathbf{Bimod} \arrow[rr] \arrow[dr] & & \mathbf{NOp}_{++} \arrow[dl] \\ & \mathbf{NOp}_{+} & \end{tikzcd}$$

Thanks to the theorem 3.2.29, we only need to prove the contractibility of the nerve of $\mathbf{NOp}_{++}^{\mathbf{Bimod}}$ to establish the weak equivalence 4.2.

The classifier $\mathbf{NOp}_{++}^{\mathbf{Bimod}}$ can be computed explicitely using the machinery from the Batanin-Berger paper [3]. One can prove that each component of it is a finite poset which has a nice cover by contractible subsets such that all intersections of these subsets are contractible. We show an example of this poset (for trees with 3 leaves) on the picture 4.1.5. □

4.1.4 Dwyer-Hess's theorem in the language of classifiers

The Dwyer-Hess's theorem requires some more preparations.

Recall that in the subsection 4.1.2, two monads on *Set* were introduced : **Id** and $\mathbf{Id}_+$. The unit of the monad $\mathbf{Id}_+$ is a morphism of polynomial monads $\epsilon : \mathbf{Id} \to \mathbf{Id}_+$. We also have a morphism of polynomial monads $\gamma_1 : \mathbf{Id} \to \mathbf{Bimod}$ which sends 1 to the linear tree with a single white vertex in the component of degree 1.

Lemma 4.1.29. *The pushout of γ_1 and ϵ in the category of polynomial monads is the monad $\boldsymbol{Bimod}_+$.*

Proof. This statement is obvious if we study the algebras of this pushout. □

There are also two morphisms of monads $\mathbf{Bimod} \to \mathbf{NOp}_{++}$ and $\mathbf{Id}_+ \to \mathbf{NOp}_{++}$ which make the square with γ_1 and ϵ commutative. So, this generates a morphism of polynomial monads $\mathbf{Bimod}_+ \to \mathbf{NOp}_{++}$. Observe, that all morphisms constructed are morphisms of polynomial monads over the monad $\mathbf{NOp}_+$.

Proposition 4.1.30. *The morphisms described above generate the following pushout of classifiers:*

$$\begin{tikzcd}
\boldsymbol{NOp}_+^{\boldsymbol{Bimod}_+} \arrow[dr, phantom, "\lrcorner" very near start] & \boldsymbol{NOp}_+^{\boldsymbol{Id}_+} \arrow[l] \\
\boldsymbol{NOp}_+^{\boldsymbol{Bimod}} \arrow[u] & \boldsymbol{NOp}_+^{\boldsymbol{Id}} \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.3}$$

and a morphism of classifiers

$$\sigma : \boldsymbol{NOp}_+^{\boldsymbol{Bimod}_+} \to \boldsymbol{NOp}_+^{\boldsymbol{NOp}_{++}}$$

After application of the nerve functor the square 4.3 gives a homotopy pushout of simplicial multiplicative operads

$$\begin{tikzcd}
N\boldsymbol{NOp}_+^{\boldsymbol{Bimod}_+} \arrow[dr, phantom, "\lrcorner" very near start] & N\boldsymbol{NOp}_+^{\boldsymbol{Id}_+} \arrow[l] \\
N\boldsymbol{NOp}_+^{\boldsymbol{Bimod}} \arrow[u] & N\boldsymbol{NOp}_+^{\boldsymbol{Id}} \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.4}$$

whose legs are cofibrations in the category of simpicial multiplicative operads.

Proof. The fact that this is a pushout can be checked directly by universal properties of the objects involved. The fact that after application of nerve this produces a homotopy pushout can be again deduced from [3, Theorem 8.2]. □

Theorem 4.1.31. *For any multiplicative non-symmetric operad $\mathcal{O}$ in $SSet$, there is a fibration sequence*

$$SSet_{\boldsymbol{NOp}_+}\left(N\boldsymbol{NOp}_+^{\boldsymbol{Bimod}_+}, fib\mathcal{O}\right) \to SSet_{\boldsymbol{NOp}_+}\left(N\boldsymbol{NOp}_+^{\boldsymbol{Bimod}}, fib\mathcal{O}\right) \to \mathcal{O}_1$$

Proof. By applying the contravariant functor

$$SSet_{\mathbf{NOp}_+}(-, fib\mathcal{O})$$

to the pushout 4.4, we get the following homotopy pullback

$$\begin{tikzcd}
SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{Bimod}_+}, fib\mathcal{O}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{Id}_+}, fib\mathcal{O}\right) \arrow[d] \\
SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{Bimod}}, fib\mathcal{O}\right) \arrow[r] & SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{Id}}, fib\mathcal{O}\right)
\end{tikzcd}$$

Observe that

$$SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{Id}_+}, fib\mathcal{O}\right) \to SSet_{\mathbf{NOp}_+}\left(N\mathbf{NOp}_+^{\mathbf{Id}}, fib\mathcal{O}\right)$$

is homotopy equivalent to the path-fibration over $\mathcal{O}_1$.

Indeed, let $\beta : \mathbf{Id} \to \mathbf{NOp}_+$ be the morphism of polynomial monad which sends 1 to the linear tree with a single white vertex. Thanks to theorem 4.1.3, we have

$$SSet_{\mathbf{NOp}_+}(N\mathbf{NOp}_+^{\mathbf{Id}}, fib\mathcal{O}) \sim Map_{Id}(1, \beta^*\mathcal{O}) \sim Map_{Id}(1, \mathcal{O}_1) \sim \mathcal{O}_1$$

Similarly, if $\alpha : \mathbf{Id}_+ \to \mathbf{NOp}_+$ is a morphism which sends one copy of 1 to the linear tree with a single white vertex and another copy of 1 to the linear tree with a single black vertex, then

$$SSet_{\mathbf{NOp}_+}(N\mathbf{NOp}_+^{\mathbf{Id}_+}, fib\mathcal{O}) \sim SSet_{\mathbf{Id}_+}(N\mathbf{Id}_+^{\mathbf{Id}_+}, fib(\alpha^*\mathcal{O})).$$

The category of algebras of $\mathbf{Id}_+$ is the category of pointed simplicial sets. The space $\alpha^*\mathcal{O}$ is the space $\mathcal{O}_1$ with the unit of $\mathcal{O}$ as its based point. Finally, it is not hard to see by direct verification of universal property that the classifier $\mathbf{Id}_+^{\mathbf{Id}_+}$ is just a pointed category with two objects 0 (a point) and 1, and one nontrivial arrow $0 \to 1$. The nerve of this category is a pointed simplicial interval and the result follows. □

Theorem 4.1.32 (Dwyer-Hess)**.** *For any multiplicative non-symmetric operad $\mathcal{O}$ in SSet, there is a fibration sequence*

$$\Omega Map_{NOp}(Ass, \Phi^*\mathcal{O}) \to Map_{Bimod}(Ass, \Psi^*\mathcal{O}) \to \mathcal{O}_1$$

Sketch of a proof. The theorem 4.1.31 in combination with the theorems 4.1.26 and 4.1.27 shows that the first Dwyer-Hess fibration sequence will be established if we manage to prove that

$$N(\sigma) : N\mathbf{NOp}_+^{\mathbf{Bimod}_+} \to N\mathbf{NOp}_+^{\mathbf{NOp}_{++}}$$

is a weak equivalence.

Similarly to the argument in the proof of Turchin's theorem we have the following commutative triangle of polynomial monads :

$$\begin{tikzcd} \mathbf{Bimod}_+ \arrow[rr] \arrow[dr] & & \mathbf{NOp}_{++} \arrow[dl] \\ & \mathbf{NOp}_+ & \end{tikzcd}$$

Applying theorem 3.2.29 again, we see that we only need to prove the contractibility of the nerve of

$$\mathbf{NOp}_{++}^{\mathbf{Bimod}_+}$$

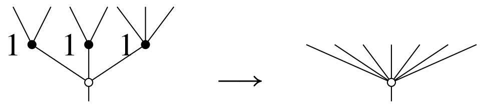
to complete the proof. As we said it before such a classifier admits an explicit combinatorial description. This time it is not a finite poset but as we are going to show it does contain a contractible poset as a deformation retract. The details will be published in a future but we give some indications how it goes in the next subsection. □

4.1.5 Description of the category $\mathbf{NOp}_{++}^{\mathbf{Bimod}_+}$

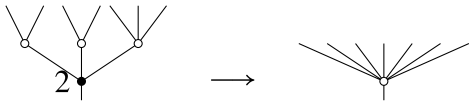
Using a machinery developed in [3] we describe the classifier

$$\mathbf{NOp}_{++}^{\mathbf{Bimod}_+}$$

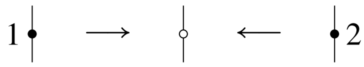
- the objects of this category are elements of the set $NOpTr_{++}$ of trees with white vertices and two types of black vertices, where there can not be two adjacent black vertices of the same type
- the morphisms are generated by
  - contractions to a white vertex of edges where the upper vertices are black of first type and the lower vertex is white



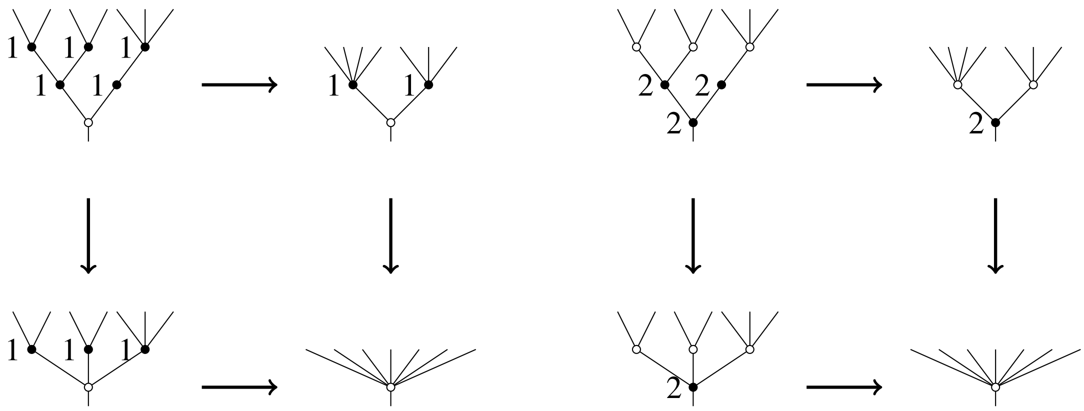
  - contractions to a white vertex of edges where the upper vertices are white and the lower vertex is black of second type

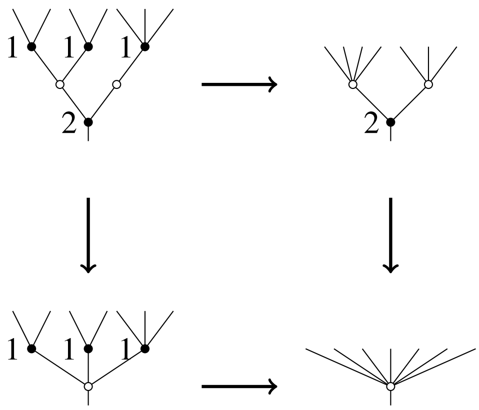


  - transformation of a unary black vertex of type 1 or 2 to a unary white vertex.



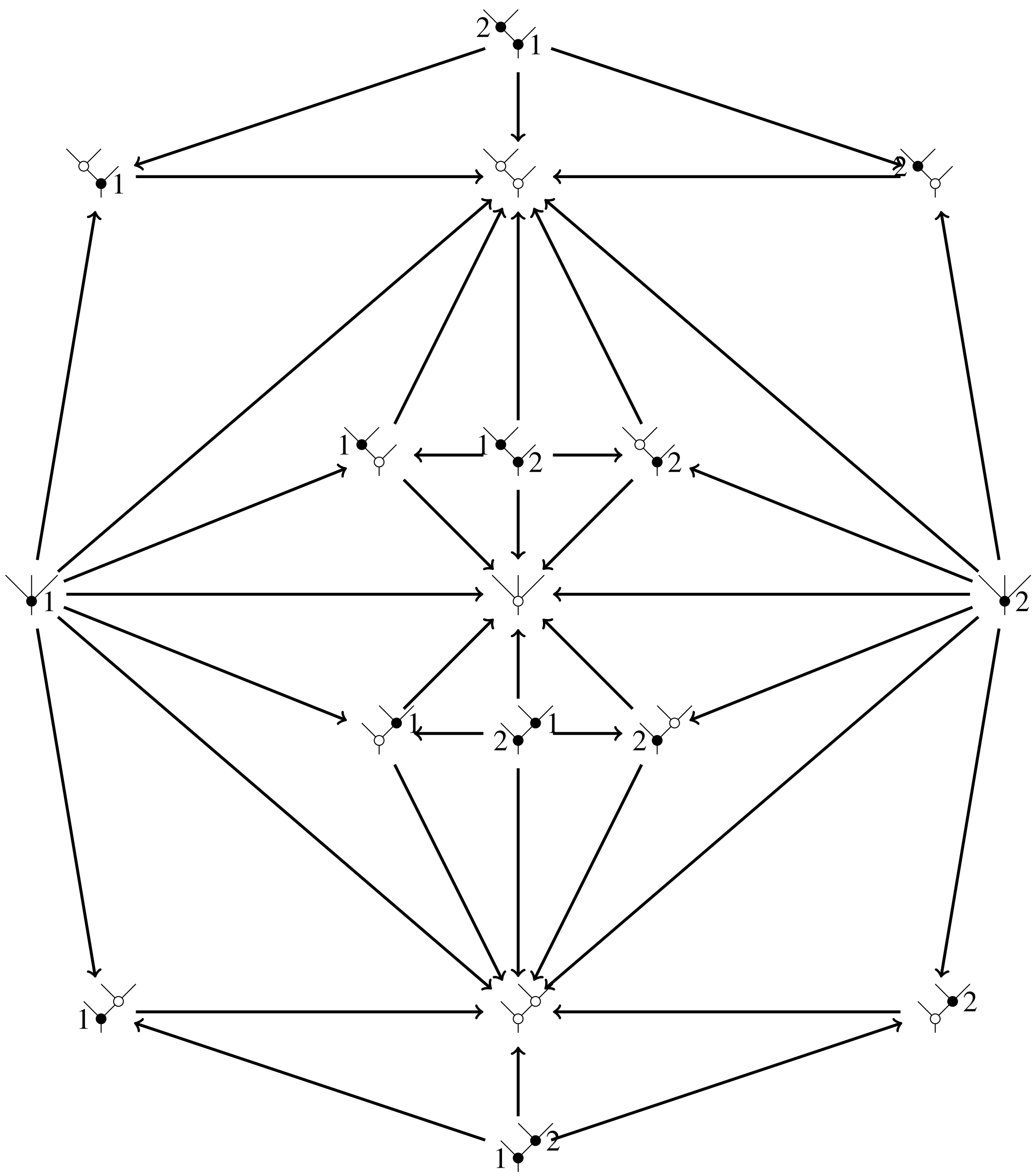
- the relations are generated by the relations in the category of bimodules over Ass, which means that the squares like below commute :





In the reduced case as in the Turchin paper [2], we get the following category containing the trees with 3 leaves. In general, it is expected that the whole picture can be contracted to the reduced case by using degeneracies.

Figure 4.1.9: The category in the reduced case and when $n = 3$



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