2-derivators

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Statement of Originality

This work has not previously been submitted for a degree or diploma at any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

Nicola Di Vittorio, November 2020

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Abstract

Introduced independently by Grothendieck and Heller in the 1980s, derivators provide a formal way to study homotopy theories by working in some quotient category such as the homotopy category of a model category. One of the advantages of derivator theory is that they enable a calculus of homotopy Kan extensions that relies almost entirely on ordinary category theory (with a bit of 2-category theory). They can also be seen as an approximation of $(\infty, 1)$-categories, a concept which has been realized using a range of combinatorial and homotopy theoretic models. Quasi-categories are presumably the best developed between such models, and their theory has been established in the 2000s by Joyal and Lurie. In 2015 Riehl and Verity introduced ∞-cosmoi, which are particular $(\infty, 2)$-categories where one can develop $(\infty, 1)$-category theory in a synthetic way. They noticed that much of the theory of ∞-cosmoi can be developed inside a quotient, the homotopy 2-category.

Inspired by this philosophy, we introduce a set of axioms that mirror key properties of the ∞-cosmological approach to ∞-category theory and demonstrate they hold in a variety of models, including common models related to ∞-category theory. We also prove that these axioms are stable under a particular shift operation.

Contents

Introduction

The *categorification* of ordinary mathematical structures usually comes with new axioms expressing some extra coherences that used to hold trivially. As an example, let us take one of the most familiar algebraic structures: monoids. It is common knowledge that a monoid is a *set* X equipped with an internal binary operation $m\colon X \times X \to X$ which is associative and has a unit. These axioms are expressed using equations, hence for instance it is enough to test associativity between three generic elements. Categorification prescribes to replace sets X with categories $\mathcal{C}$ and functions $X \to Y$ with functors $\mathcal{C} \to \mathcal{D}$. Already in the case $\mathcal{C} = \mathbf{Set}$ equipped with the functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that sends a pair of sets to their cartesian product, we notice that associativity and unitality no longer hold strictly but instead "weakly", that is up to isomorphism (induced by the universal property of products). This example sits inside the theory of *monoidal categories*, whose coherences ensure not so much that the (tensor) product of an arbitrary number of objects is well defined (up to isomorphism) regardless of the bracketing[1] but that there is exactly one way to go from one bracketing to another. This follows from the pentagon and triangle conditions via the celebrated Mac Lane's Coherence Theorem (see Chapter VII of [Mac98]). In this case a very small amount of coherences was required, but we can't always get away so easily. In the realm of homotopy theory, each diagram expressing a coherence law is required to commute up to homotopy equivalence which, in turn, has to satisfy further coherences and so forth. The natural notion of monoid in this setting is embodied by A_∞-spaces, first defined by Stasheff in [Sta63], in which the

[1] This is already true thanks to the associators and unitors.

coherences for associativity are expressed using associahedra.

It is not surprising that this philosophy led to the development of *homotopy coherent mathematics*, having higher category theory as a cornerstone. Homotopy coherent mathematics is the mathematical study of homotopy coherent structures in a way that keeps track of all the coherences. A lot of work has been done in particular to build the theory of $(\infty, 1)$-categories, also known as ∞-categories. Roughly speaking, ∞-categories are categories having n-dimensional morphisms (for every n) such that every n-morphism with $n > 1$ is invertible. According to the point of view described before, everything here should be thought as "weak": composition of n-morphisms, associativity, unitality and invertibility are defined up to higher coherences. Trying to handle this infinite amount of data, people came up with different models implementing the intuitive notion of ∞-category. The most developed model is likely the one provided by quasi-categories, that were first named "weak Kan complexes" by Boardman and Vogt in [BV06]. The theory of quasi-categories has been developed further mainly by Joyal in [Joy02], [JT07], [Joy08] and Lurie in his books [Lur09] and [Lur17]. However, the literature is full of other models each of which has its particular features and usefulness. Luckily, connections between these models have been established and even better, as proved by Toën in [Toë05], all models of ∞-categories define fibrant objects of Quillen equivalent model categories. This implies in particular that all of their homotopy categories are equivalent, in other words "they share the same homotopy theory".

At this point, we are naturally led to ask ourselves what happens if we go "one level up". Namely we would like to know whether these models share the same category theory. The category theory of ∞-categories (limits and colimits, adjunctions, Yoneda lemma etc.) has been formulated in great detail by Lurie using the language of quasi-categories. These are a very convenient model, and yet there are situations in which the most natural model is another one. Complete Segal spaces appear in the internal category theory of ∞-categories, for one thing. And what's more, from iterated complete Segal spaces and Segal n-categories one can build models of (∞, n)-categories. Among the authors following this approach there is Barwick, see in particular his Ph.D. thesis [Bar05]. A significant number of results established by Lurie in the context of quasi-categories have been translated into the language of complete Segal spaces in [Ras18]. Nonetheless, this translation from a model to another is a time-consuming process. Generally speaking, quasi-categories do not capture all the richness of fibered versions of ∞-categories. In order to formulate the category theory of higher categories and their slices in a systematic and organic way, in [RV15] Riehl and Verity started a program (that evolved in the soon-to-be-published book [RV18b]) which can be thought as *model-independent* or *synthetic* higher category theory. The word "synthetic" shall be understood as the opposite of *analytic*, i.e. using the combinatorics of a particular model. A central object of investigation in this program is the concept of ∞-*cosmos*. This is a finitely complete $(\infty, 2)$-category equipped with a class of *isofibrations*, maps allowing to strictify homotopical constructions, just like their counterparts in ordinary category theory (e.g. pseudopullbacks of isofibrations are equivalent to strict pullbacks as proved in [JS93]).

The objects of ∞-cosmoi are called ∞-categories and morphisms (i.e. vertices of the functor spaces) are called ∞-functors. The natural notion of morphism of ∞-cosmoi is represented by *cosmological functors*, simplicial functors that preserve the classes of isofibrations and (simplicial) limits involved in the definition above. In a traditional bottom-up approach one would redevelop category theory inside each model of ∞-category and then prove that it is equivalent to that of quasi-categories. In the program of Riehl and Verity, on the contrary, a key idea is to work inside a quotient of a generic ∞-cosmos: its *homotopy* 2-*category* $h_*\mathcal{K}$, a particular case of Definition 1.2.6. It turns out that this 2-category retains enough information to develop synthetically a great deal of the category theory we are interested in. For instance, it has weak 2-limits, such as *weak comma objects*, that can be used to encode suitable universal properties defining e.g. (co)limits and adjunctions, as well as to prove theorems relating them within the homotopy 2-category. The next step they make is to show that usual models for ∞-categories assemble into ∞-cosmoi. Furthermore, for every ∞-cosmos $\mathcal{K}$ and object $B \in \mathcal{K}$ we can define a slice ∞-cosmos $\mathcal{K}_{/B}$ of isofibrations over B as well as its sub-∞-cosmoi of Cartesian and coCartesian fibrations over B. Moreover, even some models of (∞, n)-category are shown to form an ∞-cosmos. Therefore results proved in this abstract framework apply to all of them. In addition, results of Riehl and Verity guarantee that the homotopy 2-category of an ∞-cosmos can be endowed with a virtual double category structure which supports a virtual equipment. Every ∞-categorical notion that can be encoded as an equivalence-invariant proposition inside this equipment is model invariant as a consequence of Theorem 11.3.3 of [RV18b]. In other words, ∞-cosmoi allow a model-independent study of higher categories.

Another central theme in ∞-cosmology revolves around the concept of internalization. This is a matter that emerges every time we have to deal with the process of quotienting out information from some kind of higher structure, which can be for instance the categorification or the homotopy coherent version of an ordinary one. Much of the theory of ∞-cosmoi is indeed set in their associated homotopy 2-categories. So, for example, adjunctions of ∞-categories are defined as adjunctions inside the homotopy 2-category of the ∞-cosmos in which they live. In other words, they are 2-functors out of the free-living adjunction 2-category $\mathbf{Adj}$ introduced by Schanuel and Street in [SS86]. Riehl and Verity prove in their article [RV16] that every (ordinary) adjunction T in the homotopy 2-category $\mathrm{h}\mathcal{K}$ of the ∞-cosmos $\mathcal{K}$ can be promoted to a homotopy coherent adjunction $\mathbf{Adj} \to \mathcal{K}$ in a essentially unique way, meaning that the cospan

$$\begin{tikzcd} & \mathcal{K} \arrow[d, "Q"] \\ \mathbf{Adj} \arrow[r, "T"'] \arrow[ur, dashed] & h_*\mathcal{K} \end{tikzcd}$$

lifts along the quotient simplicial functor Q, and the lifts assemble into a contractible Kan complex. Here we are regarding the homotopy 2-category as a simplicial category via the usual nerve embedding, and moreover in [RV16] it is shown that the simplicial category parametrizing homotopy coherent adjunctions arises in the same way from Schanuel and Street's free-living adjunction. Put another way, we are able to pull back an external notion (defined *after* we take the quotient), which is equationally defined and comprises a low amount of data, to an internal and fully coherent setting where there is an infinite amount of data to keep track of. This is what we mean by internalization. Along these lines, internalization permits to prove statements about a (richer but more difficult) internal setting from an (easier but less rich) external one. The principal link between internal and external notions in ∞-cosmology is provided by *smothering* 2*-functors*.

Definition. *A functor* $F\colon \mathcal{A} \to \mathcal{B}$ *is* smothering *if it is surjective on objects, full and conservative. A* 2*-functor is* smothering *if it is surjective on objects and it is smothering at the level of hom-categories.*

An example of smothering 2-functor is given by the 2-functor $h_*(\mathcal{K}_{/B}) \to (h_*\mathcal{K})_{/B}$. Smothering 2-functors can be used to pull back notions defined by means of *weak* universal properties to an internal world in which these are strict universal properties. A functor that is *essentially* surjective on objects, full and conservative is often called *weakly smothering*.

The usage of 2-category theory to formalize parts of homotopy theory is also at the basis of the theory of *derivators*. Derivators were introduced independently by Grothendieck and Heller in the context of abstract homotopy theory. To be precise, Grothendieck first mentioned derivators in his unpublished manuscript [Gro83] and further continued to develop this theory in the other unpublished manuscript [Gro90]. Heller instead studied derivators in his book [Hel88], where they are called "homotopy theories". Modern references on the subject include [Mal01], [Mal07] and [Gro13]. One main aim of derivators is to provide an enhancement of model categories permitting to characterize homotopy (co)limits through a nice universal property. This goal is reached by investigating the relations between homotopy coherent and homotopy commutative diagrams at a high level of generality. As a concrete example let us take a (combinatorial) model category $\mathcal{M}$. In this case, an *homotopy coherent diagram* is an object of $\mathrm{Ho}(\mathcal{M}^I)$, while a *homotopy commutative* or *incoherent diagram* is an object of $\mathrm{Ho}(\mathcal{M})^I$. Notice that in the former case we consider the whole model category of diagrams (and then take the homotopy category) while in the latter we just take diagrams in the homotopy category. The universal property of the localization gives us a family of functors $\mathsf{Ho}(\mathcal{M}^I) \to \mathsf{Ho}(\mathcal{M})^I$ parametrized by small categories.

The first ingredient in derivator theory is the notion of *prederivator*, namely a 2-functor $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$, where **Dia** is a suitable sub-2-category of **Cat** that contains finite posets and is closed under certain (co)limits. Here **Cat** denotes the 2-category of small categories and **CAT** denotes the 2-category of (possibly large) categories. An important tool in derivator theory are *exact squares*, which generalize pointwise left and right Kan extensions. A *derivator* will be then a prederivator satisfying some goodness conditions. Some of these axioms provide a weak version of those defining smothering functors in the context of ∞-cosmology. Another connection between derivators and ∞-cosmoi can be found in [RV17], where the authors give a description of pointwise Kan extensions for certain ∞-cosmoi adapting the theory of exact squares to this framework. But, as we mentioned, much of the work done by Riehl and Verity in order to internalize various notions makes use of smothering 2-functors rather than smothering functors. This increase of dimensions solicits the development of a suitable theory of 2-derivators which will provide an axiomatization of ∞-cosmology. The basic idea is to encode $(\infty, 2)$-categorical limits in 2-derivator theory just like $(\infty, 1)$-categorical ones are encoded by ordinary derivators. In this thesis we will discuss some basic aspects of this theory and prove that we can reinterpret the models in which we are interested (e.g. ∞-cosmoi) in this framework.

Chapter 1

Background

1.1 Derivators

Derivators were introduced by Grothendieck in [Gro83] (and further developed in [Gro90]) and Heller in [Hel88]. Modern references include [Mal01], [Mal07] and [Gro13].

Definition 1.1.1. Let **Dia** be a full sub-2-category of **Cat** satisfying the following axioms.

(Dia 0) **Dia** contains finite posets,

(Dia 1) **Dia** is stable under finite coproducts and pullbacks,

(Dia 2) if $\mathcal{A} \in \mathbf{Dia}$ and $a \in \mathcal{A}$ then the slice $\mathcal{A}/a$ is in **Dia** as well,

(Dia 3) if $\mathcal{A} \in \mathbf{Dia}$ then $\mathcal{A}^{\mathrm{op}} \in \mathbf{Dia}$

Remark 1.1.2.

- (Dia 0) implies in particular that the terminal category $\mathbb{1}$ is in **Dia**,
- the underlying category of **Dia** is finitely complete since it has pullbacks and terminal object,

- for every functor $u\colon \mathcal{A} \to \mathcal{B}$ and every $b \in \mathcal{B}$ in **Dia**, the comma category (u/b) is in **Dia**. Indeed, it can be obtained as the following pullback

$$\begin{tikzcd} (u/b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "u"] \\ \mathcal{B}/b \arrow[r, "forget"'] & \mathcal{B} \end{tikzcd}$$

  which is in **Dia** by axioms (Dia 1) and (Dia 2),

- $(b/u) \in$ **Dia** by the previous point and (Dia 3).

Definition 1.1.3. A prederivator is a 2-functor $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$.

$\mathbb{D}(\mathbb{1})$ is called the *base* of $\mathbb{D}$ or the *fiber over* $\mathbb{1}$, $\mathbb{D}(I)$ is the category of *coherent* diagrams of shape I or the *fiber over* I and $\mathbb{D}(\mathbb{1})^I$ is the category of *incoherent* diagrams of shape I.

Example 1.1.4. The very first example of prederivator is the contravariant hom 2-functor $よ_C := \mathbf{Dia}(-, C)$ for some $C \in \mathbf{Dia}$, often referred to as *represented prederivator* or the *prederivator represented by* C.

Inspired by this example, we'll use the notation $-^* := \mathbb{D}(-)$, where the placeholder can be filled with a general 1-cell or 2-cell in **Dia**.

Example 1.1.5. If $\mathcal{M}$ is a cofibrantly generated model category and J is a category, we can equip the functor category $[J, \mathcal{M}]$ with the *projective model structure* (in which fibrations and weak equivalences are defined pointwise). The assignment

$$\mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$$
$$J \mapsto \mathsf{Ho}([J, \mathcal{M}])$$

defines then a prederivator, called *the prederivator associated to the model category* $\mathcal{M}$.

Definition 1.1.6. The diagram

$$\begin{tikzcd} A \arrow[r, "f^*"] \arrow[d, "g^*"'] \arrow[dr, phantom, "\swArrow_{\alpha}"] & B \arrow[d, "k^*"] \\ C \arrow[r, "h^*"'] & D \end{tikzcd}$$

in which there are the adjoint pairs $f_! \dashv f^*$ and $h_! \dashv h^*$ is said to be *left Beck-Chevalley* if the mate

$$\begin{tikzcd} & B \arrow[r, "k^*"] \arrow[d, phantom, "\Searrow\eta"' near end] & D \arrow[r, "h_!"] \arrow[dr, phantom, "\Searrow\epsilon"] & C \arrow[dl, equal, bend left] \\ B \arrow[r, "f_!"'] \arrow[ur, equal, bend left] & A \arrow[u, "f^*"'] \arrow[r, "g^*"'] \arrow[ur, phantom, "\Searrow\alpha"] & C \arrow[u, "h^*"'] & \end{tikzcd}$$

is an isomorphism. There exists an analogous right Beck-Chevalley condition, which is dual to the one above.

Remark 1.1.7. Recall that whenever $\mathcal{C}$ is a complete and cocomplete category, both left and right Kan extensions of $X \in [J, \mathcal{C}]$ along $u\colon J \to I$ exist and can be computed in a pointwise fashion as

$$(u_! X)(k) := k^* u_! X \cong \operatorname*{colim}_{u/k} \mathrm{pr}^* X \qquad \text{and} \qquad (u_* X)(k) := k^* u_* X \cong \lim_{k/u} \mathrm{pr}^* X,$$

where $k\colon \mathbb{1} \to I$ is an object of I and the two "pr" stand for the suitable forgetful functors from the comma categories. We can rephrase these formulas in terms of a Beck-Chevalley condition. For instance, the left Kan extension formula is equivalent to saying that

$$\begin{tikzcd} \mathcal{C}^I \arrow[r, "u^*"] \arrow[d, "k^*"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{C}^J \arrow[d, "\mathrm{pr}^*"] \\ \mathcal{C} \arrow[r, "\Delta"'] & \mathcal{C}^{(u/k)} \end{tikzcd}$$

is left Beck-Chevalley.

Definition 1.1.8. Given a prederivator $\mathbb{D}$ and $I \in \mathbf{Dia}$, we define the *shifted prederivator*

$$\begin{aligned} \mathbb{D}^I \colon \mathbf{Dia}^{\mathrm{op}} &\to \mathbf{CAT} \\ J &\mapsto \mathbb{D}(I \times J) \end{aligned}$$

obtained as the composition of the 2-functors $\mathbb{D}$ and $I \times -$. This prederivator has base $\mathbb{D}^I(\mathbb{1}) = \mathbb{D}(I \times \mathbb{1}) \cong \mathbb{D}(I)$.

The purpose of Definition 1.1.8 is to work on the fiber over $\mathbb{1}$ and then infer information on other fibers by shifting.

One of the main points of (pre)derivator theory is to connect coherent and incoherent diagrams and try to recover from the latter as much information on the former as possible. This is achieved through the underlying diagram functors.

Definition 1.1.9. Given $I \in \mathbf{Dia}$, the *underlying diagram functor*

$$\begin{aligned} \mathrm{dia}_I \colon \mathbb{D}(I) &\to \mathbb{D}(\mathbb{1})^I \\ X &\mapsto \mathrm{dia}_I X \end{aligned}$$

assigns to every coherent diagram an incoherent one defined as follows:

$$\begin{aligned} \mathrm{dia}_I X \colon I &\longrightarrow \mathbb{D}(\mathbb{1}) \\ i &\longmapsto X_i := i^* X \\ (i \xrightarrow{\alpha} j) &\mapsto (X_i \xrightarrow{X_\alpha} X_j) := i^* X \xrightarrow{\alpha^*_X} j^* X \end{aligned}$$

where we see an object $i \in I$ as a functor $\mathbb{1} \xrightarrow{i} I$ and a morphism $i \to j$ as a natural transformation between the respective functors.

Definition 1.1.10. A prederivator $\mathbb{D} \colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$ is called a *derivator* if the following axioms hold.

(Der 1) $\mathbb{D}(\varnothing) = \mathbb{1}$ and $\mathbb{D}(\coprod_{a \in A} I_a) \xrightarrow{\sim} \prod_{a \in A} \mathbb{D}(I_a)$ is an equivalence of categories[1],

(Der 2) $\mathrm{dia}_I \colon \mathbb{D}(I) \to \mathbb{D}(\mathbb{1})^I$ is conservative for every I,

(Der 3) the image u^* through $\mathbb{D}$ of every $u \colon J \to K$ in $\mathbf{Dia}$ has both left and right adjoints, called *homotopy left and right Kan extensions* along u,

(Der 4) homotopy Kan extensions are pointwise, i.e. the images under $\mathbb{D}$ of

[1] Notice that by the axioms on **Dia** this makes sense only if A is a finite set, but if we take **Dia** to be **Cat** one usually assumes this even for infinite coproducts.

$$
\begin{tikzcd}
(u/k) \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{pt}"'] \arrow[dr, phantom, "\swArrow"] & J \arrow[d, "u"] \\
\mathbb{1} \arrow[r, "k"'] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
(k/u) \arrow[r, "\mathrm{pt}"] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "\swArrow"] & \mathbb{1} \arrow[d, "k"] \\
J \arrow[r, "u"'] & K
\end{tikzcd}
$$

are (left and right) Beck-Chevalley squares, where "pt" is the unique functor to the terminal category.

Proposition 1.1.11. *Let $\mathcal{C}$ be a complete and cocomplete category. Then the represented prederivator $よ_{\mathcal{C}}$ is a derivator.*

Proof. The first two axioms hold since contravariant homs send coproducts into products and a natural transformation is a natural isomorphism if and only if each of its components is an isomorphism. The last two axioms are the usual results on (pointwise) Kan extensions in **Cat** since $\mathcal{C}$ is complete and cocomplete. □

The following result, that is proved e.g. in [Gro13], allows to generalize results about a derivator $\mathbb{D}$ from the base category $\mathbb{D}(\mathbb{1})$ to any category of the form $\mathbb{D}(I)$.

Theorem 1.1.12. Let $\mathbb{D}$ be a derivator and $I \in$ **Dia**. Then the shifted prederivator $\mathbb{D}^I : \mathbf{Dia}^{\mathrm{op}} \to \mathbf{CAT}$ is a derivator.

An interesting subclass of derivators satisfies another axiom. Such derivators are known as *strong derivators*.

Axiom. (Der 5) Let $\mathbb{2}$ be the walking arrow. The functor $\mathrm{dia}_{\mathbb{2}} : \mathbb{D}^I(\mathbb{2}) \to (\mathbb{D}^I(\mathbb{1}))^{\mathbb{2}}$ is full and essentially surjective for every $I \in$ **Dia**.

Remark 1.1.13. Since $\mathbb{D}^I$ is a derivator whenever $\mathbb{D}$ is, the functor involved in (Der 5) is conservative. That is, strong derivators are the derivators for which the functor $\mathrm{dia}_{\mathbb{2}} : \mathbb{D}^I(\mathbb{2}) \to (\mathbb{D}^I(\mathbb{1}))^{\mathbb{2}}$ is weakly smothering for every $I \in$ **Dia**.

Unsurprisingly this good property is enjoyed by a large class of derivators, in particular by those associated to nice model categories as showed in [Gro13].

Proposition 1.1.14. *Let $\mathcal{M}$ be a combinatorial model category. Then the prederivator associated to $\mathcal{M}$ is a strong derivator.*

1.2 ∞-cosmoi

Recall that Δ is the full subcategory of **Cat** spanned by the finite non-empty ordinals and $\mathbf{sSet} \coloneqq \mathbf{Set}^{\Delta^{\mathrm{op}}}$ is the (presheaf) category of simplicial sets.

Definition 1.2.1. A simplicial set X is said to be a *quasi-category* if for every $0 < k < n$ and $n \geq 2$ the horn $\Lambda^n_k \to X$ has a filler $\Delta^n \to X$, i.e.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

commutes. A *Kan complex* is a quasi-category in which this filling condition holds also for $k = 0, n$.

The full subcategory **qCat** of **sSet** spanned by quasi-categories is cartesian closed (for instance, closure is an immediate consequence of Corollary 15.2.3 in [Rie14]) hence it is a good base of enrichment.

Definition 1.2.2. A morphism $f\colon A \to B$ in **qCat** is an *isofibration* if the following lifting problems admit a solution for every $0 < k < n$ and $n \geq 2$.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

The following theorem, due to André Joyal, is also proved by Lurie in [Lur09].

Theorem 1.2.3. *There exists a model structure on* **sSet***, called the* Joyal model structure*, having monomorphisms as cofibrations and such that an object is fibrant if and only if it is a quasi-category. Furthermore, the class of fibrations between fibrant objects coincides with the class of isofibrations between quasi-categories.*

Definition 1.2.4. An ∞-*cosmos* $\mathcal{K}$ is a category enriched over quasi-categories, equipped with a class of maps called *isofibrations* (denoted by $\twoheadrightarrow$) which satisfies the following axioms:

(i) $\mathcal{K}$ has a terminal object, small products, pullbacks of isofibrations, limits of countable towers of isofibrations, and cotensors with all simplicial sets[2],

(ii) the class of isofibrations contains all isomorphisms and any map to the terminal object; is closed under composition, product, pullback, inverse limits of towers, and Leibniz cotensors with monomorphisms of simplicial sets; and for every isofibration $f\colon A \twoheadrightarrow B$ and $X \in \mathcal{K}$ we have that $f_*\colon \mathcal{K}(X, A) \twoheadrightarrow \mathcal{K}(X, B)$ is an isofibration of quasi-categories.

The name "isofibration" chosen to denote this class of maps inside an ∞-cosmos is not fortuitous. In fact, as shown in Proposition 1.2.10. of [RV18b], **qCat** defines an ∞-cosmos in which isofibrations are exactly those defined in Definition 1.2.2.

Remark 1.2.5. The nerve functor $N\colon \mathbf{Cat} \to \mathbf{sSet}$ has a left adjoint $h\colon \mathbf{sSet} \to \mathbf{Cat}$ sending a simplicial set to its homotopy category (also known as fundamental category). They are both strong monoidal with respect to the cartesian closed structure on each of these categories, so in particular h induces a 2-functor $h_*\colon \mathbf{sSet}\text{-}\mathbf{Cat} \to 2\text{-}\mathbf{Cat}$ which is called *the homotopy 2-category 2-functor*.

Definition 1.2.6. The homotopy 2-category of a **sSet**-category $\mathcal{M}$ has 0-cells being the objects of $\mathcal{M}$, 1-cells being the 0-arrows of $\mathcal{M}$ and 2-cells being homotopy classes of formal

[2] These are *enriched limits*, with the base of enrichment being **sSet**.

composites of 1-arrows. When we restrict to **qCat**-enriched categories we have a simpler description of these formal composites, that reduce to a single arrow. That is, the 2-cells are just the homotopy classes of 1-arrows.

1.3 Enriched profunctors and the collage construction

In this section we introduce collages, which will be useful later on to compute weighted 2-(co)limits by means of Kan extensions. In the following, $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is always a complete and cocomplete symmetric monoidal closed category. For a detailed introduction to enriched category theory we refer to the classic [Kel05]. Let us just say that for every such $\mathcal{V}$, the category of $\mathcal{V}$-categories inherits a tensor product: given a pair of $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ we define a $\mathcal{V}$-category $\mathcal{A} \otimes \mathcal{B}$ having as set of objects the product $\operatorname{Ob} \mathcal{A} \times \operatorname{Ob} \mathcal{B}$ and such that $(\mathcal{A} \otimes \mathcal{B})((a,b),(a',b')) = \mathcal{A}(a,a') \otimes \mathcal{B}(b,b')$. The composition is defined using the ones of $\mathcal{A}$ and $\mathcal{B}$ as well as the symmetry of $\mathcal{V}$, and the identity is obtained via the tensor product

$$I \cong I \otimes I \xrightarrow{\mathrm{id}_a \otimes \mathrm{id}_b} (\mathcal{A} \otimes \mathcal{B})((a,b),(a,b)) = \mathcal{A}(a,a) \otimes \mathcal{B}(b,b).$$

Furthermore, the notion of opposite of a category makes sense also in the enriched context. As usual, the opposite $\mathcal{V}$-category $\mathcal{A}^{\mathrm{op}}$ has the same objects of $\mathcal{A}$ but has objects of morphisms $\mathcal{A}^{\mathrm{op}}(a,a') = \mathcal{A}(a',a)$. The composition is defined using the composition of $\mathcal{A}$ and, once again, the symmetry of $\mathcal{V}$. The units are the same as in $\mathcal{A}$. We now turn to an important construction that makes sense for general $\mathcal{V}$-categories, even though we will use it mainly for simplicial categories.

Definition 1.3.1. An enriched profunctor or $\mathcal{V}$-profunctor $\mathcal{W} \colon \mathcal{A} \nrightarrow \mathcal{B}$ between the $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ is a $\mathcal{V}$-functor $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$.

Definition 1.3.2. Given a $\mathcal{V}$-profunctor $\mathcal{W} \colon \mathcal{A} \nrightarrow \mathcal{B}$, we define its collage $\operatorname{coll}(\mathcal{W})$ to be the

$\mathcal{V}$-category having as objects the coproduct $\operatorname{Ob}\mathcal{A} \sqcup \operatorname{Ob}\mathcal{B}$ and such that

$$\operatorname{coll}(\mathcal{W})(x,y) = \begin{cases} \mathcal{A}(x,y), & \text{if } x,y \in \mathcal{A} \\ \mathcal{B}(x,y), & \text{if } x,y \in \mathcal{B} \\ \mathcal{W}(x,y), & \text{if } x \in \mathcal{B} \text{ and } y \in \mathcal{A} \\ \varnothing, & \text{otherwise} \end{cases}$$

where $\varnothing$ is the initial object in $\mathcal{V}$ and composition comes from the ones in $\mathcal{A}$ and $\mathcal{B}$ and from the functoriality of $\mathcal{W}$.

Remark 1.3.3. The unit $\mathcal{V}$-category $\mathcal{I}$ is defined to be the $\mathcal{V}$-category with one object $*$ and such that $\mathcal{I}(*,*) = I$, which is the unit object of the monoidal category $\mathcal{V}$. Clearly, $\mathcal{I}^{\mathrm{op}} = \mathcal{I}$ and $\mathcal{A} \otimes \mathcal{I} \cong \mathcal{I} \otimes \mathcal{A} \cong \mathcal{A}$ for every $\mathcal{V}$-category $\mathcal{A}$. A profunctor $\mathcal{W}\colon \mathcal{A} \nrightarrow \mathcal{I}$ is then just a $\mathcal{V}$-functor $\mathcal{A} \to \mathcal{V}$, namely a weight. Specializing Definition 1.3.2 to this case we recover the notion of collage of a weight, described e.g. in [RV18b, Definition 6.2.8].

From the definition we deduce that there exist inclusions $\mathcal{A} \hookrightarrow \operatorname{coll}(\mathcal{W})$ and $\mathcal{B} \hookrightarrow \operatorname{coll}(\mathcal{W})$, that is we have a specific cospan $\mathcal{A} \hookrightarrow \operatorname{coll}(\mathcal{W}) \hookleftarrow \mathcal{B}$ in the 2-category of $\mathcal{V}$-categories. In most cases, cospans coming from collages of profunctors are exactly the two-sided codiscrete cofibrations in the 2-category in which they live. This motivates an alternative description of the collage, that will be useful in the following pages.

Definition 1.3.4. Given a cospan $\mathcal{A} \xrightarrow{f} \mathcal{C} \xleftarrow{g} \mathcal{B}$ in $\mathcal{V}$-**Cat** we define the collage $\operatorname{coll}(f,g)$ to be the $\mathcal{V}$-category with set of objects the coproduct $\operatorname{Ob}\mathcal{A} \sqcup \operatorname{Ob}\mathcal{B}$, hom-objects $\mathcal{A}(a,a')$ and $\mathcal{B}(b,b')$ between elements respectively both in $\mathcal{A}$ and both in $\mathcal{B}$, $\mathcal{C}(fa,gb)$ from an element of $\mathcal{A}$ to an element of $\mathcal{B}$ and $\varnothing$ from an element of $\mathcal{B}$ to an element of $\mathcal{A}$.

For instance, Definition 1.3.2 specializes immediately to the last one for the representable bifunctor $\mathcal{C}(f-,g-)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathcal{V}$. In general, there is a $\mathcal{V}$-functor $\pi_{\mathcal{C}}\colon \operatorname{coll}(f,g) \to \mathcal{C}$ acting on objects as the coproduct $f \sqcup g$, on hom-objects as f in the full subcategory $\mathcal{A}$, as g in the full subcategory $\mathcal{B}$, as the identity on hom-objects from an element of $\mathcal{A}$ to an element of $\mathcal{B}$ and as the unique morphism $\varnothing = \mathrm{coll}(f,g)(b,a) \to \mathcal{C}(\pi_{\mathcal{C}}b, \pi_{\mathcal{C}}a) = \mathcal{C}(gb, fa)$ from $b \in \mathcal{B}$ to $a \in \mathcal{A}$.

Collages of weights are especially useful to compute weighted colimits as Kan extensions. A proof of the following characterization can be found in [RV18a, §7.2] for $\mathcal{V} = \mathbf{sSet}$, but the same result holds for every *nice* $\mathcal{V}$, e.g. $\mathcal{V} = \mathbf{Cat}$.

Proposition 1.3.5. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a $\mathcal{V}$-functor and $W\colon \mathcal{A} \to \mathcal{V}$ a weight. The weighted limit $\lim^W F$ exists if and only if the pointwise right Kan extension of F along the inclusion $\iota\colon \mathcal{A} \hookrightarrow \mathrm{coll}(W)$ exists. In this case it can be computed as $\lim^W F \cong \mathrm{Ran}_\iota(\bullet)$, where $\bullet$ is the extra point in the collage. Dually, we can compute weighted colimits as* left *Kan extensions.*

1.4 Accessible model structures

Accessible model structures on enriched categories, especially those enriched in the Joyal model structure are the most important source of examples in the theory of 2-derivators. First of all, we notice that in the enriched context one can reproduce the theory of accessible and locally presentable categories[3]. The main reference for this section is [Kel82]. A locally presentable symmetric monoidal closed category $(\mathcal{V}, \otimes, I)$ is usually called a *locally presentable base* when the generating family of presentable objects contains I and is closed under the tensor product. For a fixed regular cardinal α, α-filtered colimits in an enriched category $\mathcal{A}$ are defined as *conical* colimits in $\mathcal{A}$ of underlying diagrams $\mathcal{J} \to \mathcal{A}_0$, with $\mathcal{J}$ a small α-filtered category. Replacing the unenriched definition of presentable object with this enriched analogue we get the following definition.

Definition 1.4.1. A $\mathcal{V}$-category $\mathcal{A}$ is locally $\mathcal{V}$-presentable if it admits all conical colimits and a (strongly) generating family of presentable objects in the enriched sense.

[3]When we say "locally presentable" we always mean "locally α-presentable for some regular cardinal α".

Definition 1.4.2. A $\mathcal{V}$-functor F that preserves α-filtered colimits is called α-accessible. We call a $\mathcal{V}$-functor accessible if it is α-accessible for some regular cardinal α.

In [Kel82], Kelly also proves a characterization of locally $\mathcal{V}$-presentable $\mathcal{V}$-categories in terms of reflective subcategories of enriched presheaves categories.

Proposition 1.4.3. *A $\mathcal{V}$-category $\mathcal{A}$ is locally $\mathcal{V}$-presentable if and only if it is a full reflective (enriched) subcategory of some $[\mathcal{B}, \mathcal{V}]$ with $\mathcal{B}$ small and the inclusion $\mathcal{A} \hookrightarrow [\mathcal{B}, \mathcal{V}]$ accessible.*

Locally $\mathcal{V}$-presentable $\mathcal{V}$-categories are especially useful when we deal with $\mathcal{V}$-categories that are also model categories. In order to make the enrichment and the model structure interact nicely, we have to add some extra compatibility axioms.

Definition 1.4.4. A *monoidal model category* is a symmetric monoidal closed category $(\mathcal{V}, \otimes, I)$ endowed with a model structure subject to the following compatibility conditions:

1. the pushout-product
$$(X \otimes Y') \coprod_{X \otimes X'} (Y \otimes X') \to Y \otimes Y'$$
of a pair of cofibrations $f\colon X \to Y$ and $f'\colon X' \to Y'$ is itself a cofibration. Furthermore it is a trivial cofibration whenever f or f' is,

2. the map $CI \otimes X \to I \otimes X \cong X$, induced by the cofibrant replacement $CI \to I$ of the unit object I, is a weak equivalence whenever X is cofibrant.

Definition 1.4.5. Let $\mathcal{V}$ be a monoidal model category. A *$\mathcal{V}$-enriched model category* or *$\mathcal{V}$-model category* is a $\mathcal{V}$-category $\mathcal{M}$ such that

1. the underlying category $\mathcal{M}_0$ is a model category,

2. $\mathcal{M}$ is tensored and cotensored over $\mathcal{V}$,

3. for every cofibration $i\colon A \to B$ and fibration $p\colon X \to Y$ in $\mathcal{M}_0$, the pullback-hom
$$(i^*, p_*)\colon \mathcal{M}(B, X) \to \mathcal{M}(A, X) \times_{\mathcal{M}(A,Y)} \mathcal{M}(B, Y)$$
is a fibration in $\mathcal{V}$, which is trivial whenever i or p is a weak equivalence.

Remark 1.4.6. By adjunction, axiom 3. of Definition 1.4.5 is equivalent to the axiom

3'. for every cofibration $i\colon A \to B$ in $\mathcal{M}_0$ and cofibration $j\colon K \to L$ in $\mathcal{V}$, the pushout-product
$$(A \otimes L) \coprod_{A\otimes K} (B \otimes K) \to B \otimes L$$
is a cofibration in $\mathcal{M}_0$, which is trivial whenever i or j is a weak equivalence.

Example 1.4.7. The Quillen model structure on simplicial sets is a closed symmetric monoidal model category. Model categories enriched over it are usually called *simplicial model categories*. In this context, the axiom $3'$ is often referred as *Quillen's axiom SM7*.

Example 1.4.8. The Joyal model structure is a monoidal model category with the tensor product being the cartesian product. It is then self-enriched as a model category, meaning that it is $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched and thus presents an $(\infty, 2)$-category[4].

Definition 1.4.9. A weak factorization system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ is called *accessible* if $\mathcal{M}$ is locally presentable and the functor $E\colon \mathcal{M}^2 \to \mathcal{M}$ sending an arrow $f\colon A \to B$ to the object Ef sitting inside the functorial factorization $A \xrightarrow{\mathcal{L}f} Ef \xrightarrow{\mathcal{R}f} B$ is accessible.

Definition 1.4.10. A model category $\mathcal{M}$ is accessible if the weak factorization systems $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathcal{W})$ are accessible.

Definition 1.4.11. An *accessible $\mathcal{V}$-enriched model category* $\mathcal{M}$ is a $\mathcal{V}$-enriched model category such that the model structure on $\mathcal{M}_0$ is accessible.

Recall that if $\mathcal{M}$ is a model category and I is a category, under some mild conditions on $\mathcal{M}$ (typically to be cofibrantly generated or combinatorial) there exist two model structures on the functor category $[I, \mathcal{M}]$:

- the *projective model structure* in which fibrations and weak equivalences are pointwise,
- the *injective model structure* in which cofibrations and weak equivalences are pointwise.

[4] This is akin to the definition of a 2-category as a **Cat**-enriched category.

For the remainder of this section, $\mathcal{V}$ will be a symmetric monoidal closed model category. The following result, proved in [Mos19], permits to lift such model structures to an enriched setting when certain assumptions hold.

Theorem 1.4.12. *Suppose $\mathcal{M}$ is a locally $\mathcal{V}$-presentable $\mathcal{V}$-category endowed with an accessible $\mathcal{V}$-enriched model structure and let $\mathcal{D}$ be a small $\mathcal{V}$-category.*

(i) If the functors $-\otimes \mathcal{D}(d,d')\colon \mathcal{M}_0 \to \mathcal{M}_0$ preserve cofibrations for all $d,d' \in \mathcal{D}$, the injective model structure on $[\mathcal{D},\mathcal{A}]_0$ exists, and it is again $\mathcal{V}$-enriched.

(ii) If the functors $-\otimes \mathcal{D}(d,d')\colon \mathcal{M}_0 \to \mathcal{M}_0$ preserve trivial cofibrations for all $d,d' \in \mathcal{D}$, the projective model structure on $[\mathcal{D},\mathcal{A}]_0$ exists, and it is again $\mathcal{V}$-enriched.

Moreover, in many cases the enriched injective and projective model structures are Quillen equivalent, as we will see in the proposition below.

Remark 1.4.13. Let $\mathcal{C},\mathcal{D}$ be $\mathcal{V}$-categories and $W\colon \mathcal{D} \to \mathcal{V}$ a weight. Suppose moreover that $\mathcal{C}$ has tensors. We can then characterize a W-weighted limit in $\mathcal{C}$ as the right adjoint to the functor

$$\begin{aligned} W\otimes - \colon\ \mathcal{C} &\to [\mathcal{D},\mathcal{C}] \\ c &\mapsto W\otimes c \end{aligned}$$

where

$$\begin{aligned} W\otimes c\colon\ \mathcal{D} &\to \mathcal{C} \\ d &\mapsto Wd\otimes c. \end{aligned}$$

The right adjoint to $W\otimes -$ sends indeed a functor $F\in[\mathcal{D},\mathcal{C}]$ to an object $x\in\mathcal{C}$ such that $[\mathcal{D},\mathcal{C}](W\otimes c,F)\cong \mathcal{C}(c,x)$. But since $\mathcal{C}$ is tensored (and then so is $[\mathcal{D},\mathcal{C}]$), we have the isomorphism $[\mathcal{D},\mathcal{C}](W\otimes c,F)\cong[\mathcal{D},\mathcal{V}](W,\mathcal{C}(x,F))$ that implies $x\cong \lim^W F$.

The following result is folklore, and we just adapted it to our purposes.

Proposition 1.4.14. *Let $\mathcal{M}$ be a locally $\mathcal{V}$-presentable $\mathcal{V}$-category admitting an accessible $\mathcal{V}$-enriched model structure and take $\mathcal{D}$ to be a small $\mathcal{V}$-category. Suppose that $\mathcal{D}(d,d')$ is a cofibrant object of $\mathcal{V}$ for every couple of objects $d,d' \in \mathcal{D}$. Then the adjunction* $\mathrm{id}\colon [\mathcal{D},\mathcal{M}]^{\mathrm{proj}} \rightleftarrows [\mathcal{D},\mathcal{M}]^{\mathrm{inj}} \colon \mathrm{id}$ *is a Quillen equivalence.*

Proof. We have to show that the identity functor $[\mathcal{D},\mathcal{M}]^{\mathrm{inj}} \to [\mathcal{D},\mathcal{M}]^{\mathrm{proj}}$ is right Quillen, i.e. it preserves fibrations and trivial fibrations. That is, we have to prove the following two statements:

(i) every injective fibration[5] $\alpha\colon F \to G$ in $[\mathcal{D},\mathcal{M}]$ is also a projective fibration (i.e. a pointwise fibration),

(ii) every injective trivial fibration[6] $\beta\colon H \to K$ in $[\mathcal{D},\mathcal{M}]$ is also a projective trivial fibration (i.e. a pointwise trivial fibration).

In order to prove (i), let us take an injective fibration α. Fix $d \in \mathcal{D}$. We claim that $\alpha_d\colon Fd \to Gd$ has the RLP with respect to every trivial cofibration $i\colon A \to B$ in $\mathcal{M}$. The following square of solid arrows admits a dashed lift

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & Fd \arrow[d, "\alpha_d"] \\ B \arrow[r] \arrow[ur, dashed] & Gd \end{tikzcd}$$

if and only if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \lim^{\mathcal{D}(d,-)} F \arrow[d, "\widetilde{\alpha}_d"] \\ B \arrow[r] \arrow[ur, dashed] & \lim^{\mathcal{D}(d,-)} G \end{tikzcd}$$

[5]A morphism that has the RLP with respect to pointwise trivial cofibrations

[6]A morphism that has the RLP with respect to pointwise cofibrations

does, since $\lim^{\mathcal{D}(d,-)} F \cong Fd$ by the Yoneda Lemma. Using the adjunction $- \otimes \mathcal{D}(d,-) \dashv \lim^{\mathcal{D}(d,-)}$ from Remark 1.4.13, we can transpose the latter lifting problem to the following one

$$\begin{tikzcd} A \otimes \mathcal{D}(d,-) \arrow[r] \arrow[d, "i\otimes\mathcal{D}(d,-)"'] & F \arrow[d, "\alpha"] \\ B \otimes \mathcal{D}(d,-) \arrow[r] \arrow[ur, dashed] & G. \end{tikzcd}$$

Since i is a trivial cofibration in $\mathcal{M}$ and $\emptyset \to \mathcal{D}(d,d')$ is a cofibration in $\mathcal{V}$ (because by assumption every hom-object is cofibrant), the pushout-product axiom for enriched model categories guarantees that the morphism

$$A \otimes \mathcal{D}(d,d') \coprod_{A\otimes\emptyset} B \otimes \emptyset \cong A \otimes \mathcal{D}(d,d') \xrightarrow{i\otimes\mathcal{D}(d,d')} B \otimes \mathcal{D}(d,d')$$

is a trivial cofibration in $\mathcal{M}$. This means that $A\otimes\mathcal{D}(d,-) \xrightarrow{i\otimes\mathcal{D}(d,-)} B\otimes\mathcal{D}(d,-)$ is a pointwise trivial cofibration. Thus the last lifting problem can be solved since α has the RLP with respect to pointwise trivial cofibrations, showing (i). The proof of (ii) is exactly the same, with $i\colon A \to B$ a generic cofibration in $\mathcal{M}$. $\square$

Henceforth we will assume that $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched model categories that appear herein satisfy the hypothesis of Theorem 1.4.12. We have the following result.

Corollary 1.4.15. *Let $\mathcal{M}$ be a $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched model category. For every small 2-category $\mathcal{J}$ the projective and the injective enriched model structure on the diagram category $[\mathcal{J},\mathcal{M}]$ are Quillen equivalent.*

Proof. This holds since in $\mathbf{sSet}_{\mathrm{Joyal}}$ every mono is a cofibration; hence $\varnothing \to \mathcal{D}(d,d')$ is a cofibration for every $d, d' \in \mathcal{D}$. We can then apply Proposition 1.4.14 to conclude. $\square$

Chapter 2

Two-dimensional derivator theory

2.1 2-prederivators

Derivators provide a good setting to study the relations between coherent and incoherent diagrams involving homotopy categories of model categories. Here we would like to go one step further and try to formalize the analogous relations we find between homotopy 2-categories of enriched model categories, in particular of model categories enriched over simplicial sets with the Joyal model structure. Size problems aside, in derivator theory we work inside some 2-category of categories, which is a **Cat**-enriched category with respect to the usual cartesian closed structure of **Cat**. Going from an enriched category of categories to an enriched category of 2-categories, it would be natural to replace the base of enrichment from $(\mathbf{Cat}, \times, \mathbb{1})$ to **2-Cat** with the tensor product given by the cartesian product of 2-categories. Nevertheless, we are often interested in situations like the following (in the 2-category of $\mathcal{V}$-categories):

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd}$$

with α a $\mathcal{V}$-natural transformation that is a pointwise equivalence, meaning that α_c is an equivalence in $\mathcal{D}$ for every $c \in \mathcal{C}$ (whenever the notion of equivalence inside $\mathcal{D}$ makes sense). When $\mathcal{V} = \mathbf{Set}$, a $\mathcal{V}$-natural transformation is just an ordinary natural transformation and the equivalences in the category $\mathcal{D}$ (seen as a 2-category with only identity 2-cells) are just the isomorphisms. Therefore our assumption on α is saying that it is a pointwise isomorphism. This implies in turn that α is a natural isomorphism, namely an equivalence[1] in the functor category $[\mathcal{C}, \mathcal{D}]$ of functors and natural transformations. On the other hand, if we set $\mathcal{V} = \mathbf{Cat}$ we are defining α to be a 2-natural transformation which is a pointwise equivalence. It is no longer true that α is an equivalence in the functor 2-category $[\mathcal{C}, \mathcal{D}]$ of 2-functors, 2-natural transformations and modifications. I've learned the following counterexample from Steve Lack.

Proposition 2.1.1. *A* 2*-natural transformation* α *between* 2*-functors* $F, G\colon \mathcal{C} \to \mathcal{D}$ *that is a pointwise equivalence is not in general an equivalence in the functor* 2*-category* $[\mathcal{C}, \mathcal{D}]$.

Proof. To prove the statement we need to find a counterexample. Take for instance $\mathcal{C}$ to be the discrete 2-category $(\rightrightarrows)$ having two objects and only two non-identity parallel arrows between them and $\mathcal{D}$ equal to the 2-category $\mathbf{Cat}$, so that the functor 2-category $\mathcal{D}^{\mathcal{C}}$ is $\mathbf{Cat}^{(\rightrightarrows)}$. Consider the 2-functors $F, G\colon (\rightrightarrows) \to \mathbf{Cat}$ whose images are, respectively, $0, 1\colon \mathbb{1} \rightrightarrows I$ (the inclusions of the endpoints inside the free-living isomorphism) and $\mathrm{id}_I, \mathrm{id}_I\colon I \rightrightarrows I$. Take the 2-natural transformation $\alpha\colon F \Rightarrow G$ whose components are the vertical arrows in

$$\begin{tikzcd}
\mathbb{1} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] \arrow[d, "0"'] & I \arrow[d, "\Delta_0"] \\
I \arrow[r, shift left, "\mathrm{id}_I"] \arrow[r, shift right, "\mathrm{id}_I"'] & I
\end{tikzcd}$$

where Δ_0 is the constant functor with value 0. Both vertical arrows are equivalences of categories since they are fully faithful (the hom-sets are singletons) and essentially surjective ($0 \cong 1$ in I), so α is a pointwise equivalence. Nevertheless, α does not admit a 2-natural quasi-inverse. In fact, assume that such a 2-natural quasi-inverse does exist and has components

[1] Again, equivalences and isomorphisms are the same because we are considering functor *categories*.

that are the vertical arrows of the diagram

$$\begin{tikzcd} I \arrow[r, shift left, "\mathrm{id}_I"] \arrow[r, shift right, "\mathrm{id}_I"'] \arrow[d, "!"'] & I \arrow[d, "m"] \\ \mathbb{1} \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & I \end{tikzcd}$$

such that $m\Delta_0 \cong \mathrm{id}_I \cong \Delta_0 m$. In general, if we call $f\colon 0 \xrightarrow{\cong} 1$ the unique isomorphism inside I, by the commutativity of the last diagram we get the assignments

$$\begin{tikzcd} f \arrow[r, mapsto, "\mathrm{id}_I"] \arrow[d, mapsto, "!"'] & f \arrow[d, mapsto, "m"] \\ \mathrm{id}_* \arrow[r, mapsto, "0"'] & \mathrm{id}_0 \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, mapsto, "\mathrm{id}_I"] \arrow[d, mapsto, "!"'] & f \arrow[d, mapsto, "m"] \\ \mathrm{id}_* \arrow[r, mapsto, "1"'] & \mathrm{id}_1 \end{tikzcd}$$

but $\mathrm{id}_0 \neq \mathrm{id}_1$ since 0 and 1 are different objects (even though they are isomorphic), hence such an m cannot be defined. This implies in particular that we cannot find a 2-natural quasi-inverse to α. □

It turns out that the problem of finding a quasi-inverse to a 2-natural transformation can instead be solved inside the larger 2-category $\mathrm{Ps}(\mathcal{C}, \mathcal{D})$ of 2-functors, *pseudonatural* transformations and modifications.

Lemma 2.1.2. *A pseudonatural transformation is an equivalence in the* 2*-category* $\mathrm{Ps}(\mathcal{C}, \mathcal{D})$ *if and only if it is a pointwise equivalence in* $\mathcal{D}$*.*

Proof. For the implication "⇒" notice that saying that $\alpha\colon F \Rightarrow G\colon \mathcal{C} \to \mathcal{D}$ is an equivalence means there exists $\beta\colon G \Rightarrow F$ and invertible modifications $\Lambda\colon \alpha\beta \cong \mathrm{id}_G$ and $\Theta\colon \beta\alpha \cong \mathrm{id}_F$. Therefore, for every $C \in \mathcal{C}$ the components $\Lambda_C\colon \alpha_C\beta_C \cong \mathrm{id}_{GC}$ and $\Theta_C\colon \beta_C\alpha_C \cong \mathrm{id}_{FC}$ are isomorphisms. Then α is a pointwise equivalence.

For the implication "⇐", for every $C \in \mathcal{C}$, let $\gamma_C\colon GC \to FC$ be a quasi-inverse to $\alpha_C\colon FC \to GC$. By standard results in category theory we can always promote the pointwise equivalences to pointwise adjoint equivalences $\alpha_C \dashv \gamma_C$ with (invertible) units and counits $\eta_C\colon \mathrm{id}_{FC} \Rightarrow \gamma_C\alpha_C$ and $\epsilon_C\colon \alpha_C\gamma_C \Rightarrow \mathrm{id}_{GC}$. For every morphism $f\colon C \to C'$ in $\mathcal{C}$ we define an invertible 2-cell in $\mathcal{D}$ by means of the pasting diagram

$$\begin{tikzcd}
GC \arrow[r, "\gamma_C"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\Swarrow\epsilon_C"] & FC \arrow[r, "Ff"] \arrow[d, "\alpha_C"'] \arrow[dr, phantom, "\cong"] & FC' \arrow[d, "\alpha_{C'}"] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\Swarrow\eta_{C'}"] & \\
& GC \arrow[r, "Gf"'] & GC' \arrow[r, "\gamma_{C'}"'] & FC'
\end{tikzcd}$$

that gives us the naturality-up-to-iso squares of a pseudonatural transformation $\gamma\colon G \Rightarrow F$, with the coherence axioms being satisfied by the triangular identities of the adjoint equivalence. Moreover, this pseudonatural transformation is the quasi-inverse that we were looking for. □

This gives as an immediate corollary the following result.

Corollary 2.1.3. *Every 2-natural transformation that is a pointwise equivalence admits a global pseudonatural quasi-inverse.*

Proof. Every 2-natural transformation is in particular a pseudonatural transformation (with coherences reducing to equalities), hence it admits a global pseudonatural quasi-inverse by Lemma 2.1.2. □

For this reason, it would be a better choice to enrich over a (closed) category of 2-categories whose internal homs are of the form $\mathrm{Ps}(\mathcal{C}, \mathcal{D})$ for some $\mathcal{C}$ and $\mathcal{D}$. This can be done using the so-called *pseudo Gray tensor product*. There exists also a lax version of the Gray tensor product, linked to (op)lax natural transformations in the same way as the pseudo version is related to pseudonatural ones. In the literature it is available an explicit description of the Gray tensor product by means of generators and relations (see [JY20] for instance). The definition provided here is the one given in [BG17].

Definition 2.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. The *pseudo Gray tensor product* $\mathcal{C} \otimes \mathcal{D}$ is a representing object for the functor $\mathbf{2\text{-}Cat}(\mathcal{C}, \mathrm{Ps}(\mathcal{D}, -))\colon \mathbf{2\text{-}Cat} \to \mathbf{Set}$.

In other words, the pseudo Gray tensor product is a 2-category $\mathcal{C} \otimes \mathcal{D}$ realizing the (natural) isomorphism

$$\mathbf{2\text{-}Cat}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \cong \mathbf{2\text{-}Cat}(\mathcal{C}, \mathrm{Ps}(\mathcal{D}, \mathcal{E})).$$

As expected (and proved explicitly e.g. in [JY20] and in [BG17] using its universal property) the pseudo Gray tensor product is actually a bifunctor forming a monoidal structure on the category of 2-categories. Furthermore the isomorphism above witnesses an adjunction $- \otimes \mathcal{D} \dashv \mathrm{Ps}(\mathcal{D}, -)$. That is, we are allowed to give the following definition.

Definition 2.1.5. We define $\mathbf{Gray} \coloneqq (\mathbf{2\text{-}Cat}, \otimes, \mathbb{1})$ to be the (symmetric) monoidal closed category of 2-categories and 2-functors with the pseudo Gray tensor product.

In the following we will use the capitalized word **GRAY** to denote the category of *large* 2-categories with the pseudo Gray tensor product, while **Gray** will indicate that we are considering only *small* 2-categories. Being a symmetric monoidal closed category, **Gray** is self-enriched and provides a good base of enrichment. The categories enriched over it are called **Gray**-categories and from now on we'll consider **Gray** as a **Gray**-category.

Definition 2.1.6. Let **Dia** be a full sub-3-category of **2-Cat**, we call *2-prederivator* a *normal trihomomorphism* $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{GRAY}$. In analogy with ordinary derivators, we'll denote the images of n-cells in **Dia** through $\mathbb{D}$ with an upper star for $n \in \{1, 2, 3\}$.

Notice that we chose a strict category of diagrams since, working inside the image of $\mathbb{D}$, it is enough to give a "pseudo" structure to the target category.

Remark 2.1.7. A 2-prederivator consists of the following data:

- a function $\mathrm{Ob}(\mathbf{Dia}) \to \mathrm{Ob}(\mathbf{GRAY})$,

- a 2-functor $\mathbf{Dia}(\mathcal{D},\mathcal{C}) \to \mathbf{GRAY}(\mathbb{D}(\mathcal{C}),\mathbb{D}(\mathcal{D}))$, meaning that the vertical composition of 2-cells and the identity 2-cells are *strictly* preserved,

- for composable 2-functors f and g, a pseudonatural equivalence $f^*g^* \Rightarrow (gf)^*$ compatible with 2-cells in **Dia**, meaning that the diagram

$$\begin{tikzcd} f^*g^* \arrow[r, Rightarrow, "\simeq"] \arrow[d, Rightarrow, "\alpha^*\beta^*"'] \arrow[dr, phantom, "\cong"] & (gf)^* \arrow[d, Rightarrow, "(\beta\alpha)^*"] \\ l^*m^* \arrow[r, Rightarrow, "\simeq"'] & (ml)^* \end{tikzcd}$$

  commutes up to isomorphism,

- equivalences $\mathrm{id}_{\mathbb{D}(\mathcal{A})} \simeq (\mathrm{id}_{\mathcal{A}})^*$ for every $\mathcal{A} \in \mathbf{Dia}$,

- associativity of 1-cells holds up to equivalence, i.e. the diagram

$$\begin{tikzcd} & f^*(hg)^* \arrow[drr, Rightarrow, "\simeq"] & & \\ f^*g^*h^* \arrow[ur, Rightarrow, "\simeq"] \arrow[dr, Rightarrow, "\simeq"'] & & \cong & ((hg)f)^* \arrow[dl, equal] \\ & (gf)^*h^* \arrow[r, Rightarrow, "\simeq"'] & (h(gf))^* & \end{tikzcd}$$

  commutes up to isomorphism (the 2-cell from $((hg)f)^*$ to $(h(gf))^*$ is an equality since it is the image through $\mathbb{D}$ of the identity 2-cell $(hg)f = h(gf)$ in **Dia**).

- higher coherences between these isomorphisms, as one can find in [Gur06] and [GPS95].

From these points we also get that unitality holds up equivalence, in fact if $f\colon \mathcal{B} \to \mathcal{A}$ is in **Dia** we have

$$f^* \,\mathrm{id}_{\mathbb{D}(\mathcal{A})} \simeq f^*(\mathrm{id}_{\mathcal{A}})^* \simeq (\mathrm{id}_{\mathcal{A}}\, f)^* = f^*$$

and similarly for the other side.

Example 2.1.8. The contravariant hom-functor $[-,\mathcal{D}]\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{GRAY}$ is a 2-prederivator which is called *the 2-prederivator represented by* $\mathcal{D}$. In this case every equivalence involved in the definition of prederivator is an identity.

Example 2.1.9. The category **2-Cat** has products, giving us a bifunctor

$$- \times -\colon \mathbf{2\text{-}Cat} \times \mathbf{2\text{-}Cat} \to \mathbf{2\text{-}Cat}.$$

Looking at **2-Cat** as self-enriched with respect to the cartesian product, for every 2-category $\mathcal{A}$ the product bifunctor gives rise to a 3-functor[2]

$$- \times \mathcal{A}\colon \mathbf{2\text{-}Cat} \to \mathbf{2\text{-}Cat}$$

which in turn, suitably restricted to **Dia** and assuming **Dia** is closed under products, produces a 3-functor

$$- \times \mathcal{A}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{Dia}^{\mathrm{op}}$$

that we can compose with a generic 2-prederivator $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{GRAY}$ to get the *shifted 2-prederivator* $\mathbb{D}^{\mathcal{A}} := \mathbb{D} \circ (- \times \mathcal{A})$, defined on objects by the assignment $\mathcal{J} \mapsto \mathbb{D}(\mathcal{J} \times \mathcal{A})$.

The last part of this section will be devoted to proving that we can associate a 2-prederivator to model categories enriched over $\mathbf{sSet}_{\mathrm{Joyal}}$. We refer back to Remark 1.2.5 and Definition 1.2.6 for the description of the homotopy 2-category.

Lemma 2.1.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be model categories enriched over $\mathbf{sSet}_{\mathrm{Joyal}}$, $F, G\colon \mathcal{A} \rightrightarrows \mathcal{B}$ simplicial functors and $\alpha\colon F \Rightarrow G$ a simplicial natural transformation which is a pointwise homotopy equivalence. Then the 2-natural transformation $h_*\alpha$ is a pointwise equivalence.*

Proof. By assumption we have that for every $X \in \mathcal{A}$ there exists $l_X\colon GX \to FX$ s.t. $l_X\alpha_X \sim \mathrm{id}_{FX}$ and $\alpha_X l_X \sim \mathrm{id}_{GX}$. In the Joyal model structure the cylinder objects are of the form $\mathrm{Cyl}(Y) = Y \times J$ where J is the nerve of the free-living isomorphism $\mathbb{I}$. Hence we can write the first homotopy as

$$\begin{tikzcd}
\Delta^0 \arrow[r, "0"] \arrow[dr, "l_X\alpha_X"'] & J \arrow[d, "H"] & \Delta^0 \arrow[l, "1"'] \arrow[dl, "\mathrm{id}_{FX}"] \\
& \mathcal{B}(FX, FX) &
\end{tikzcd}$$

[2] This can be done more generally for limits, see [Kel05, §3.8].

since $\mathrm{Cyl}(\Delta^0) = \Delta^0 \times J \cong J$. Furthermore, we have denoted by $0, 1$ the two 0-simplices of J. Now we can apply h and we get

$$\begin{tikzcd}
{[0]} \arrow[r, "0"] \arrow[dr, "h(l_X)h(\alpha_X)"'] & \mathbb{I} \arrow[d, "hH"] & {[0]} \arrow[l, "1"'] \arrow[dl, "\mathrm{id}_{FX}"] \\
& h\,\mathcal{B}(FX, FX) &
\end{tikzcd}$$

where we used the isomorphism $hN \cong \mathrm{id}_{\mathbf{Cat}}$ and the functoriality of h (notice that $hFX = FX$). In other words we found an invertible 1-cell inside $h\,\mathcal{B}(FX, FX)$ whose endpoints are $h(l_X)h(\alpha_X)$ and id_{FX}, meaning that they are isomorphic. With the same argument one shows that the other homotopy induces an isomorphism as well. To conclude the proof we see that $(h_*\alpha)_X \cong h(\alpha_X)$ thanks to the action of the change of base 2-functor h_* on the monoidal units of the base monoidal categories: it is precomposed by a map $[0] \to h\Delta^0$ which is necessarily an isomorphism since the endpoints are both trivial categories. □

Corollary 2.1.11. *Suppose we have a pair of 2-functors $G, G' \colon \mathcal{D} \rightrightarrows h_*(\mathcal{M})$ between a 2-category $\mathcal{D}$ and a $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched model category $\mathcal{M}$ such that $GD, G'D$ are fibrant-cofibrant objects in $\mathcal{M}$ for every $D \in \mathcal{D}$. Let $\alpha \colon G \Rightarrow G'$ be a 2-natural transformation s.t. α_D is a weak equivalence for every $D \in \mathcal{D}$. Then α is a pseudonatural equivalence.*

Proof. The proof is an immediate consequence of Lemma 2.1.10 and Corollary 2.1.3, since the components of α are weak equivalences between fibrant-cofibrant objects. □

Definition 2.1.12. Let $\mathcal{M}$ be a $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched model category. We define $F, C \colon \mathcal{M} \to \mathcal{M}$ to be the fibrant and cofibrant replacement simplicial functors. They come with simplicial natural transformations $\alpha \colon \mathrm{id}_{\mathcal{M}} \Rightarrow F$ and $\beta \colon C \Rightarrow \mathrm{id}_{\mathcal{M}}$ whose components are trivial cofibrations and trivial fibrations, respectively. Restricting them to fibrant-cofibrant objects they admit pseudonatural quasi-inverses $\widetilde{\alpha}$ and $\widetilde{\beta}$.

Henceforth we will not distinguish between fibrant/cofibrant replacement simplicial functors and the 2-functors induced on homotopy 2-categories. The context will make it clear.

Corollary 2.1.13. *Suppose we have a pair of 2-functors $H, H' \colon \mathcal{D} \rightrightarrows h_*(\mathcal{M})$ between a 2-category $\mathcal{D}$ and a $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched model category $\mathcal{M}$. Let $\alpha \colon H \Rightarrow H'$ be a 2-natural transformation s.t. α_D is a weak equivalence and $HD, H'D$ are fibrant objects in $\mathcal{M}$ for every $D \in \mathcal{D}$. Then $C\alpha$ is a pseudonatural equivalence. Dually, if HD and $H'D$ are cofibrant for every $D \in \mathcal{D}$ then $F\alpha$ is a pseudonatural equivalence.*

Proof. From the functorial factorizations of the model category $\mathcal{M}$ we know that for every $D \in \mathcal{D}$ the diagram

$$\begin{tikzcd} CHD \arrow[r, two heads, "\sim"] \arrow[d, "(C\alpha)_D"'] & HD \arrow[d, "\alpha_D"] \\ CH'D \arrow[r, two heads, "\sim"'] & H'D \end{tikzcd}$$

commutes and α_D is a weak equivalence. Then $(C\alpha)_D$ is a weak equivalence (for the 2-out-of-3 property of weak equivalences) between fibrant-cofibrant objects (because the cofibrant replacement functor sends fibrant objects to fibrant-cofibrant objects). Therefore Corollary 2.1.11 implies that $C\alpha$ is a pseudonatural equivalence. □

Proposition 2.1.14. *Let $\mathcal{M}$ be a model category enriched in $(\mathbf{sSet}, \times, \Delta^0)$ with the Joyal model structure. For every small 2-category $\mathcal{J}$, seen as a simplicial category whose homs are quasi-categories, the diagram category $[\mathcal{J}, \mathcal{M}]$ is still $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched and admits both the projective and the injective enriched model structure, which are equivalent by Corollary 1.4.15. We can then define the 2-prederivator*

$$\begin{aligned} \mathbb{D}_{\mathcal{M}} \colon \mathbf{Dia}^{\mathrm{op}} &\longrightarrow \mathbf{GRAY} \\ \mathcal{I} &\mapsto h_*[\mathcal{I}, \mathcal{M}]^{proj}_{cf} \\ (\mathcal{I} \xrightarrow{g} \mathcal{J}) &\mapsto h_*([\mathcal{J}, \mathcal{M}]^{proj}_{cf} \xrightarrow{-\circ g} [\mathcal{I}, \mathcal{M}]^{proj}_{f} \xrightarrow{C} [\mathcal{I}, \mathcal{M}]^{proj}_{cf}) \end{aligned}$$

and likewise for higher cells.

Proof. The action of $\mathbb{D}_{\mathcal{M}}$ is 2-functorial on homs, being the composite of the 2-functors $C \circ -$

and $[-,\mathcal{M}]$. For identity 1-cells $\mathrm{id}_{\mathcal{A}}$ in **Dia** we have a diagram

$$\begin{tikzcd}
h_*[\mathcal{A},\mathcal{M}]_{cf} \arrow[r, "\mathrm{id}_{h_*([\mathcal{A},\mathcal{M}]_{cf})}"] \arrow[d, equal] & h_*[\mathcal{A},\mathcal{M}]_{cf} \arrow[d, hook] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\simeq"] & \\
h_*[\mathcal{A},\mathcal{M}]_{cf} \arrow[r, hook] & h_*[\mathcal{A},\mathcal{M}]_f \arrow[r, "C"'] & h_*[\mathcal{A},\mathcal{M}]_{cf}
\end{tikzcd}$$

where the square is strictly commutative and the triangle is filled with an equivalence since $CX \to X$ is an homotopy equivalence for every $X \in [\mathcal{A},\mathcal{M}]_{cf}$. For the composition of 1-cells notice that $G\colon \mathcal{A} \to \mathcal{B}$ and $H\colon \mathcal{B} \to \mathcal{C}$ and their composite are sent to a triangle

$$\begin{tikzcd}
& h_*[\mathcal{B},\mathcal{M}]_{cf} \arrow[dr, "CG^{\#}"] \arrow[d, phantom, "\simeq"] & \\
h_*[\mathcal{C},\mathcal{M}]_{cf} \arrow[ur, "CH^{\#}"] \arrow[rr, "CG^{\#}H^{\#}"'] & {} & h_*[\mathcal{A},\mathcal{M}]_{cf}
\end{tikzcd}$$

where $G^{\#} = h_*(-\circ G)$, and $H^{\#} = h_*(-\circ H)$. The equivalence $CG^{\#}CH^{\#} \overset{\simeq}{\Rightarrow} CG^{\#}H^{\#}$ is a consequence of Corollary 2.1.13 because β is a pointwise weak equivalence, and so is $(-\circ G)\beta$ given that $(-\circ G)$ is right Quillen and so it preserves weak equivalence between fibrant objects by Ken Brown's lemma. Moreover we have that

$$\begin{tikzcd}
CG^{\#}CH^{\#}CK^{\#} \arrow[r, Rightarrow, "CG^{\#}CH^{\#}\beta K^{\#}"] \arrow[d, Rightarrow, "CG^{\#}\beta H^{\#}CK^{\#}"'] & CG^{\#}CH^{\#}K^{\#} \arrow[d, Rightarrow, "CG^{\#}\beta H^{\#}K^{\#}"] \\
CG^{\#}H^{\#}CK^{\#} \arrow[r, Rightarrow, "CG^{\#}H^{\#}\beta K^{\#}"'] & CG^{\#}H^{\#}K^{\#}
\end{tikzcd}$$

commutes because α is 2-natural hence the squares involved in its definition commute. Since h_* sends these identity 2-cells to identities, higher coherences hold trivially. □

Remark 2.1.15. To define the 2-prederivator associated to a $\mathbf{sSet}_{\mathrm{Joyal}}$-enriched model category we considered the projective model structure. Everything works the same if we choose the injective model structure instead. Later on we will verify that this is true for our purposes. Actually, one could prove there exists a *triequivalence* $\mathbb{D}_{\mathcal{M}}^{\mathrm{proj}} \simeq \mathbb{D}_{\mathcal{M}}^{\mathrm{inj}}$ between the 2-prederivator defined using the projective model structure and the one using the injective model structure.

2.2 2-derivators

In analogy with ordinary derivator theory, we will define 2-derivators as 2-prederivators satisfying extra *properties*. We thus provide a series of axioms that a 2-derivator has to satisfy. Henceforth we'll assume that **Dia** is closed under coproducts.

Axiom 1 (HDer 1)**.** $\mathbb{D}(\mathcal{I}\sqcup\mathcal{J}) \cong \mathbb{D}(\mathcal{I}) \times \mathbb{D}(\mathcal{J})$ for every $\mathcal{I},\mathcal{J} \in$ **Dia**. In particular, $\mathbb{D}(\varnothing) \cong \mathbb{1}$.

Remark 2.2.1. Let us see that this axiom holds for familiar examples.

1. The represented 2-prederivator $[-,\mathcal{D}]$ sends weighted colimits to weighted limits and in particular is such that $[\mathcal{I}\sqcup\mathcal{J},\mathcal{D}] \cong [\mathcal{I},\mathcal{D}] \times [\mathcal{J},\mathcal{D}]$ where we are considering the coproduct and the product as enriched limits.

2. $\mathbb{D}_{\mathcal{M}}$ satisfies (HDer 1). In fact, since h commutes with products, it is enough to show that the isomorphism

$$[\mathcal{C}\sqcup\mathcal{D},\mathcal{M}]^{\mathrm{proj}}_{cf} \cong [\mathcal{C},\mathcal{M}]^{\mathrm{proj}}_{cf} \times [\mathcal{D},\mathcal{M}]^{\mathrm{proj}}_{cf}$$

is also a Quillen equivalence between the corresponding enriched projective model structures. In order to prove the Quillen equivalence above, just notice that this isomorphism identifies fibrations and weak equivalences in the LHS with a pair of fibrations and weak equivalence in the RHS serving as components of the formers. On the other hand, cofibrations are completely determined by their left lifting property with respect to trivial fibrations. For instance, in $[\mathcal{C},\mathcal{M}]^{\mathrm{proj}}_{cf} \times [\mathcal{D},\mathcal{M}]^{\mathrm{proj}}_{cf}$ a morphism is a cofibration if and only if the following lifting problem

$$\begin{tikzcd}
(H,K) \arrow[r] \arrow[d, "f\ g"'] & (F,G) \arrow[d, two heads, "\wr\, p\ \wr\, q"] \\
(H',K') \arrow[r] \arrow[ur, dashed] & (F',G')
\end{tikzcd}$$

has a solution. We can then go back to $[\mathcal{C}\sqcup\mathcal{D},\mathcal{M}]^{\mathrm{proj}}_{cf}$ assuming each time $p = \mathrm{id}_F$ or $q = \mathrm{id}_G$. Therefore $\mathbb{D}_{\mathcal{M}}(\mathcal{C}\sqcup\mathcal{D}) \cong \mathbb{D}_{\mathcal{M}}(\mathcal{C}) \times \mathbb{D}_{\mathcal{M}}(\mathcal{D})$.

Remark 2.2.2. If $\mathbb{D}$ satisfies (HDer 1), the shifted prederivator $\mathbb{D}^{\mathcal{A}}$ satisfies (HDer 1) as well. In fact, the category **2-Cat** of 2-categories and 2-functors is cartesian closed, meaning that $- \times \mathcal{A}$ is a left adjoint for every $\mathcal{A} \in$ **2-Cat** hence commuting with colimits and, in particular, with coproducts. It still commutes with coproducts even after the self-enrichment of **2-Cat** since the comparison morphism at the unenriched level remains an isomorphism at the enriched one. It follows that

$$\begin{aligned}\mathbb{D}^{\mathcal{A}}(\mathcal{I} \sqcup \mathcal{J}) &= \mathbb{D}(\mathcal{A} \times (\mathcal{I} \sqcup \mathcal{J})) \\ &\cong \mathbb{D}((\mathcal{A} \times \mathcal{I}) \sqcup (\mathcal{A} \times \mathcal{J})) \\ &\cong \mathbb{D}^{\mathcal{A}}(\mathcal{I}) \times \mathbb{D}^{\mathcal{A}}(\mathcal{J}).\end{aligned}$$

In order to state the next axioms we need a notion of underlying diagram functor analogous to the one we have in the context of one-dimensional derivators.

Remark 2.2.3. Consider the terminal 2-category $\mathbb{1}$ with only one object and the identity 1-cell and 2-cell associated to it. The objects of a 2-category $\mathcal{C}$ correspond to 2-functors $\mathbb{1} \to \mathcal{C}$, so that $\mathbf{Dia}(\mathbb{1}, \mathcal{C}) \cong \mathcal{C}$. The action of a 2-prederivator $\mathbb{D}\colon \mathbf{Dia}^{\mathrm{op}} \to \mathbf{GRAY}$ on homs provides us with a 2-functor

$$\mathcal{C} \cong \mathbf{Dia}(\mathbb{1}, \mathcal{C}) \to \mathbf{GRAY}(\mathbb{D}(\mathcal{C}), \mathbb{D}(\mathbb{1}))$$

that transposes back to a 2-functor

$$\mathcal{C} \otimes \mathbb{D}(\mathcal{C}) \to \mathbb{D}(\mathbb{1})$$

which once again transposes to give a 2-functor

$$\mathbb{D}(\mathcal{C}) \to \mathbb{D}(\mathbb{1})^{\mathcal{C}} = \mathrm{Ps}(\mathcal{C}, \mathbb{D}(\mathbb{1}))$$

that we call *underlying diagram 2-functor*.

The following axioms involving underlying diagram 2-functors are well-known to hold in ∞-cosmology and their verification in our models procedes analogously to the proofs one can find in [RV15] and [RV16].

Axiom 2. (HDer 2) $\mathbb{D}(\mathcal{C}) \to \mathbb{D}(\mathbb{1})^{\mathcal{C}}$ is conservative on 1-cells for every $\mathcal{C} \in \mathbf{Dia}$.

Axiom 3. (HDer 3a) $\mathbb{D}(\mathbf{Adj}) \to \mathbb{D}(\mathbb{1})^{\mathbf{Adj}}$ is a smothering 2-functor.

Axiom 4. (HDer 3b) $\mathbb{D}(\mathbb{2}) \to \mathbb{D}(\mathbb{1})^{\mathbb{2}}$ and $\mathbb{D}(\mathbb{I}) \to \mathbb{D}(\mathbb{1})^{\mathbb{I}}$ are smothering 2-functors.

In particular, axiom (HDer 2) means we can check equivalences levelwise, (HDer 3a) allows us to internalize (fibered) adjunctions, and (HDer 3b) is e.g. useful to construct and compare externally defined notions of slices with internal ones. The remaining axioms will involve a suitable notion of Kan extensions and pointwise Kan extensions. Usually Kan extensions are defined as adjoints to the precomposition functor, but in this setting we need to relax a little this condition and use biadjunctions instead.

Definition 2.2.4. A *biadjunction* in **Gray** is the datum of 2-categories $\mathcal{A}$ and $\mathcal{B}$, 2-functors $f\colon \mathcal{B} \to \mathcal{A}$ and $u\colon \mathcal{A} \to \mathcal{B}$ and pseudonatural transformations $\eta\colon \mathrm{id}_{\mathcal{B}} \Rightarrow uf$ and $\epsilon\colon fu \Rightarrow \mathrm{id}_{\mathcal{A}}$ satisfying the triangular identities up to invertible modifications, i.e. such that there exist invertible modifications filling the triangles

$$\begin{tikzcd} f \arrow[r, Rightarrow, "f\eta"] \arrow[dr, equal] & fuf \arrow[d, Rightarrow, "\epsilon f"] \arrow[dl, phantom, "\cong"] \\ & f \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r, Rightarrow, "\eta u"] \arrow[dr, equal] & ufu \arrow[d, Rightarrow, "u\epsilon"] \arrow[dl, phantom, "\cong"] \\ & u \end{tikzcd}$$

More details on biadjunctions and biequivalences can be found in [Ver11].

Axiom 5. (HDer 4) Every 2-functor $f\colon A \to B$ in **Dia** induces an biadjoint triple

$$\begin{tikzcd} \mathbb{D}(B) \arrow[r, "f^*"] & \mathbb{D}(A) \arrow[l, bend right=60, "\mathrm{Lan}_f"', "\perp_b"] \arrow[l, bend left=60, "\mathrm{Ran}_f", "\perp_b"'] \end{tikzcd}$$

providing *2-categorical homotopy Kan extensions*[3].

[3]An analogous notion can be found in [Nun18] with the name "pseudo-Kan extensions". The main difference is that we are considering strict 2-functors instead of pseudofunctors.

In order to prove that this axiom holds in familiar cases, we will need some preliminary results. The proof of the next result is totally formal and can be adapted representably to other contexts, provided it does hold in **Cat** as we show shortly.

Proposition 2.2.5. *Let $g\colon A \to B$ and $f\colon B \to A$ be functors between the categories A and B. Suppose there exist two natural transformations $\eta\colon \mathrm{id}_B \Rightarrow gf$ and $\epsilon\colon fg \Rightarrow \mathrm{id}_A$ such that $\epsilon f \cdot f\eta$ and $g\epsilon \cdot \eta g$ are natural isomorphisms. Then f is left adjoint to g.*

Proof. We need to show that the triangular identities hold, namely we have to prove

$$
\begin{tikzcd}
f \arrow[r, Rightarrow, "f\eta'"] \arrow[dr, equal] & fgf \arrow[d, Rightarrow, "\epsilon' f"] \\
& f
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, Rightarrow, "\eta' g"] \arrow[dr, equal] & gfg \arrow[d, Rightarrow, "g\epsilon'"] \\
& g
\end{tikzcd}
$$

commute for a suitable choice of η' and ϵ'. If we call $\Phi \coloneqq \epsilon f \cdot f\eta$ and $\Theta \coloneqq g\epsilon \cdot \eta g$, we have $\Phi^{-1}\Phi = \mathrm{id}_f = \Phi\Phi^{-1}$ and $\Theta^{-1}\Theta = \mathrm{id}_g = \Theta\Theta^{-1}$. Taking $\eta' \coloneqq \Theta^{-1}f \cdot \eta$ and $\epsilon' \coloneqq \epsilon$, we observe that

$$
\begin{tikzcd}
\mathrm{id}_B \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta"'] & gf \arrow[r, Rightarrow, "g\Phi^{-1}"] \arrow[d, Rightarrow, "\eta gf"'] & gf \arrow[d, Rightarrow, "\eta gf"] \arrow[ddr, Rightarrow, bend left, "\mathrm{id}_{gf}"] & \\
gf \arrow[r, Rightarrow, "gf\eta"] \arrow[drr, Rightarrow, bend right=50, "\mathrm{id}_{gf}"'] & gfgf \arrow[r, Rightarrow, "gfg\Phi^{-1}"] \arrow[d, Rightarrow, "g\epsilon f"'] & gfgf \arrow[d, Rightarrow, "g\epsilon f"] & \\
& gf \arrow[r, Rightarrow, "g\Phi^{-1}"'] & gf \arrow[r, Rightarrow, "\Theta^{-1}f"'] & gf
\end{tikzcd}
$$

is commutative by the middle four interchange, hence $\Theta^{-1}f \cdot \eta = g\Phi^{-1} \cdot \eta$. Using this, we have that the following diagrams

$$
\begin{tikzcd}
f \arrow[r, Rightarrow, "f\eta"] \arrow[rr, Rightarrow, bend left, "f\eta'"] \arrow[dr, Rightarrow, "\Phi"'] & fgf \arrow[r, Rightarrow, "fg\Phi^{-1}"] \arrow[d, Rightarrow, "\epsilon f"] & fgf \arrow[d, Rightarrow, "\epsilon f"] \\
& f \arrow[r, Rightarrow, "\Phi^{-1}"'] & f
\end{tikzcd}
\qquad
\begin{tikzcd}
g \arrow[r, Rightarrow, "\eta g"] \arrow[rr, Rightarrow, bend left, "\eta' g"] \arrow[dr, Rightarrow, "\Theta"'] & gfg \arrow[r, Rightarrow, "\Theta^{-1}fg"] \arrow[d, Rightarrow, "g\epsilon"] & gfg \arrow[d, Rightarrow, "g\epsilon"] \\
& g \arrow[r, Rightarrow, "\Theta^{-1}"'] & g
\end{tikzcd}
$$

commute. Since $\Theta^{-1}\Theta = \mathrm{id}_g$ and $\Phi^{-1}\Phi = \mathrm{id}_f$ we get the result. $\square$

Lemma 2.2.6. *Suppose we have a pair of 2-functors in* **Gray**

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend right, "u"'] & \mathcal{B} \arrow[l, bend right, "f"'] \end{tikzcd}$$

and a couple of pseudonatural transformations $\eta\colon \mathrm{id}_{\mathcal{B}} \Rightarrow uf$ *and* $\epsilon\colon fu \Rightarrow \mathrm{id}_{\mathcal{A}}$. *If* $\epsilon f \cdot f\eta$ *and* $u\epsilon \cdot \eta u$ *are pseudonatural equivalences, then* f *and* u *can be promoted to a biadjunction.*

Proof. The proof proceeds in the same way as in Proposition 2.2.5, with the only differences being that we have pseudonatural equivalences instead of natural isomorphisms and diagrams commuting up to invertible modifications instead of strictly commutative ones. □

Proposition 2.2.7. *An enriched Quillen adjunction*

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{N} \arrow[l, bend right, "V"'] \end{tikzcd}$$

between two model categories enriched in $\mathbf{sSet}_{\mathrm{Joyal}}$ *induces a biadjunction*

$$\begin{tikzcd} h_*\mathcal{M}_{\mathrm{cf}} \arrow[r, bend right, "\overline{\mathrm{U}}"'] \arrow[r, phantom, "\perp_{\mathrm{b}}"] & h_*\mathcal{N}_{\mathrm{cf}} \arrow[l, bend right, "\overline{\mathrm{V}}"'] \end{tikzcd}$$

between the homotopy 2*-categories of fibrant-cofibrant objects.*

Proof. Since U and C preserve fibrant objects[4], they can be restricted and composed to give a simplicial functor

$$\mathcal{M}_{\mathrm{cf}} \xrightarrow{U} \mathcal{N}_{\mathrm{f}} \xrightarrow{C} \mathcal{N}_{\mathrm{cf}}.$$

Similarly, we get a simplicial functor

$$\mathcal{N}_{\mathrm{cf}} \xrightarrow{V} \mathcal{M}_{\mathrm{c}} \xrightarrow{F} \mathcal{M}_{\mathrm{cf}}.$$

Applying h_* we find 2-functors $\overline{\mathrm{U}} \coloneqq h_*(CU)\colon h_*\mathcal{M}_{\mathrm{cf}} \to h_*\mathcal{N}_{\mathrm{cf}}$ and $\overline{\mathrm{V}} \coloneqq h_*(FV)\colon h_*\mathcal{N}_{\mathrm{cf}} \to h_*\mathcal{M}_{\mathrm{cf}}$. Moreover, the restrictions of the simplicial natural transformations α and β to the

[4]In fact, U is right Quillen and $\beta_X\colon CX \to X$ is a trivial fibration for every $X \in \mathcal{N}$.

full simplicial subcategories $\mathcal{M}_{\mathrm{cf}}$ and $\mathcal{N}_{\mathrm{cf}}$ spanned by fibrant-cofibrant objects are pointwise homotopy equivalences, therefore (by Lemma 2.1.10) h_* sends them to 2-natural transformations that are pointwise equivalences, hence admitting a (global) pseudonatural quasi-inverse, by Corollary 2.1.3. We denote these quasi-inverses by $\widetilde{\alpha}$ and $\widetilde{\beta}$ and we define the following pseudonatural transformations

$$\eta' := \mathrm{id}_{h_*\mathcal{N}_{cf}} \overset{\widetilde{\beta}}{\Rightarrow} C \overset{\overline{\eta}}{\Rightarrow} \overline{\mathrm{U}}\,\overline{\mathrm{V}} \quad \text{and} \quad \epsilon' := \overline{\mathrm{V}}\,\overline{\mathrm{U}} \overset{\overline{\epsilon}}{\Rightarrow} F \overset{\widetilde{\alpha}}{\Rightarrow} \mathrm{id}_{h_*\mathcal{M}_{cf}}$$

where $\overline{\eta}$ is the image through h_* of the restriction of the composition

$$C \overset{C\eta}{\Longrightarrow} CUV \overset{CU\alpha V}{\Longrightarrow} CUFV$$

to $\mathcal{N}_{cf}$ and $\overline{\epsilon}$ is obtained from the composite

$$FVCU \overset{FV\beta U}{\Longrightarrow} FVU \overset{F\epsilon}{\Longrightarrow} F$$

in a similar way (η and ϵ are the unit and the counit of the enriched Quillen adjunction $V \dashv U$). It remains to show that $\epsilon'\,\overline{\mathrm{V}} \cdot \overline{\mathrm{V}}\,\eta'$ and $\overline{\mathrm{U}}\,\epsilon' \cdot \eta'\,\overline{\mathrm{U}}$ are pseudonatural equivalences, so that we can apply Lemma 2.2.6 to get the claim. First notice that the diagram

$$\begin{tikzcd}
CCU \arrow[r, Rightarrow, "C\eta CU"] \arrow[d, Rightarrow, "\beta CU"'] & CUVCU \arrow[r, Rightarrow, "CU\alpha VCU"] \arrow[d, Rightarrow, "CUV\beta U"] & CUFVCU \arrow[d, Rightarrow, "CUFV\beta U"] \\
CU \arrow[r, Rightarrow, "C\eta U"'] \arrow[dr, equal, bend right] & CUVU \arrow[r, Rightarrow, "CU\alpha VU"'] \arrow[d, Rightarrow, "CU\epsilon"] & CUFVU \arrow[d, Rightarrow, "CUF\epsilon"] \\
 & CU \arrow[r, Rightarrow, "CU\alpha"'] & CUF
\end{tikzcd}$$

is made of squares which are commutative thanks to the middle four interchange (they are 2-cells in the 2-category of simplicial categories, simplicial functors and simplicial natural transformations) and a triangle that is commutative by one of the triangular identities of the adjunction $V \dashv U$. Using the middle four interchange we find that the diagram

$$\begin{tikzcd} CCU \arrow[r, Rightarrow, "\beta CU"] \arrow[d, Rightarrow, "\beta CU"'] & CU \arrow[r, Rightarrow, "CU\alpha"] \arrow[d, Rightarrow, "\beta U"] & CUF \arrow[d, Rightarrow, "\beta UF"] \\ CU \arrow[r, Rightarrow, "\beta U"'] & U \arrow[r, Rightarrow, "U\alpha"'] & UF \end{tikzcd}$$

is commutative as well. The vertical 2-cells and the horizontal 2-cells of left square are pointwise weak equivalences because β is so. In addition, U preserves trivial fibrations (it is right Quillen) and hence in particular it takes trivial fibrations between fibrant objects to weak equivalences. Therefore, by Ken Brown's lemma, U takes all weak equivalences between fibrant objects to weak equivalences. If $M \in \mathcal{M}$ is a fibrant object, then $\alpha_M \colon M \to FM$ is a weak equivalence (a trivial cofibration, really) between fibrant objects and so $U(\alpha_M) = (U\alpha)_M$ is a weak equivalence. In other words, the components of $U\alpha$ at fibrant objects are pointwise weak equivalences. The components of $CU\alpha \cdot \beta CU$ at fibrant objects are weak equivalences, by the 2-out-of-3 property. Hence $CU\alpha \cdot \beta CU$ is a pointwise homotopy equivalence when restricted to fibrant-cofibrant objects. Using the commutativity of the first diagram, we get

$$\begin{aligned} CU\alpha \cdot \beta CU &= CUF\epsilon \cdot CUFV\beta U \cdot CU\alpha VCU \cdot C\eta CU \\ &= CU(F\epsilon \cdot FV\beta U) \cdot (CU\alpha V \cdot C\eta)CU, \end{aligned}$$

implying the latter is a pointwise homotopy equivalence at the level of fibrant-cofibrant objects. Taking h_* of the last composition (always restricted to fibrant-cofibrant objects) we get that $\overline{\mathrm{U}}\,\overline{\epsilon} \cdot \overline{\eta}\,\overline{\mathrm{U}}$ is a pointwise equivalence. Finally, $\overline{\mathrm{U}}\,\epsilon' \cdot \eta'\,\overline{\mathrm{U}} = \overline{\mathrm{U}}\,\widetilde{\alpha} \cdot (\overline{\mathrm{U}}\,\overline{\epsilon} \cdot \overline{\eta}\,\overline{\mathrm{U}}) \cdot \widetilde{\beta}\,\overline{\mathrm{U}}$ by definition and each piece of the composition is a pointwise equivalence, so the composite is a pointwise equivalence and hence it is a pseudonatural equivalence by Lemma 2.1.2. The same argument applies to $\epsilon'\,\overline{\mathrm{V}} \cdot \overline{\mathrm{V}}\,\eta'$. Therefore Lemma 2.2.6 gives us the required biadjunction. □

Corollary 2.2.8. *If the Quillen adjunction in Proposition 2.2.7 is an equivalence, then the induced biadjunction is a biequivalence.*

Proof. Since the Quillen adjunction is an equivalence, the components of the unit and the counit at fibrant and cofibrant objects are equivalences and hence are sent to pseudonatural equivalences serving as units and counits of the induced biadjoint biequivalences. □

Proposition 2.2.9. *Let $f\colon \mathcal{D} \to \mathcal{C}$ be in* **Dia**. *The pullback 2-functor $f^*\colon \mathbb{D}_{\mathcal{M}}(\mathcal{C}) \to \mathbb{D}_{\mathcal{M}}(\mathcal{D})$ is uniquely determined up to pseudonatural equivalence.*

Proof. We have to show that the following diagram (the vertical equivalences are a consequence of Corollaries 1.4.15 and 2.2.8)

$$\begin{tikzcd}
h_*[\mathcal{C},\mathcal{M}]^{\mathrm{proj}}_{cf} \arrow[r, "-\circ f"] \arrow[d, bend right, "F^i"'] \arrow[d, phantom, "\wr"] & h_*[\mathcal{D},\mathcal{M}]^{\mathrm{proj}}_{f} \arrow[r, "C^p"] & h_*[\mathcal{D},\mathcal{M}]^{\mathrm{proj}}_{cf} \arrow[dl, hook'] \arrow[d, bend right, "F^i"'] \arrow[d, phantom, "\wr"] \\
h_*[\mathcal{C},\mathcal{M}]^{\mathrm{inj}}_{cf} \arrow[r, "-\circ f"'] \arrow[u, bend right, "C^p"'] & h_*[\mathcal{D},\mathcal{M}]^{\mathrm{inj}}_{c} \arrow[r, "F^i"'] & h_*[\mathcal{D},\mathcal{M}]^{\mathrm{inj}}_{cf} \arrow[u, bend right, "C^p"']
\end{tikzcd}$$

commutes up to a pseudonatural equivalence (we denote with C^p, F^p and C^i, F^i the cofibrant and fibrant replacements with respect to the projective and the injective model structure). The triangle on the right commutes strictly since every cofibration in the projective model structure is a cofibration in the injective model structure and hence every cofibrant object in the projective model structure is a cofibrant object in the injective model structure as well. A 2-functor $g\colon \mathcal{C} \to \mathcal{M}$ in $h_*[\mathcal{C},\mathcal{M}]^{\mathrm{proj}}_{cf}$ is sent to $(F^i g)f$ by $(-\circ f)F^i$, and it is sent to $C^p(gf)$ by the other composite arrow. We have a trivial fibration $C^p(gf) \overset{\sim}{\twoheadrightarrow} gf$ in the projective model structure and a trivial cofibration $gf \overset{\sim}{\hookrightarrow} (F^i g)f$ in the injective model structure (since $g \overset{\sim}{\hookrightarrow} F^i g$ is a trivial cofibration and $-\circ f$ is left Quillen), so in particular they are levelwise weak equivalences. Hence the composite $C^p(gf) \xrightarrow{\sim} (F^i g)f$ is a pointwise weak equivalence between cofibrant objects. Then Corollary 2.1.13 implies that the image through F^i of this morphism is the component in g of a pseudonatural equivalence when we pass at the level of homotopy 2-categories. Hence our diagram commutes up to pseudonatural equivalence meaning f^* is uniquely determined up to pseudonatural equivalence. □

We are now able to prove that the 2-prederivator introduced in Proposition 2.1.14 satisfies the axiom (HDer 4).

Proposition 2.2.10. *The 2-prederivator $\mathbb{D}_{\mathcal{M}}$ satisfies* (HDer 4).

Proof. Suppose we have a $\mathbf{sSet}_{\text{Joyal}}$-enriched model category $\mathcal{M}$ and two 2-categories $\mathcal{C}, \mathcal{D} \in$ **Dia**, that we regard as simplicially enriched categories in the usual way. We want to show that any 2-functor $f\colon \mathcal{D} \to \mathcal{C}$ induces a biadjunction

$$\begin{tikzcd} h_*[\mathcal{C},\mathcal{M}]_{cf} \arrow[r, "f^*" description] & h_*[\mathcal{D},\mathcal{M}]_{cf}. \arrow[l, bend right, "\mathrm{lan}_f"', "\perp_b"] \arrow[l, bend left, "\mathrm{ran}_f", "\perp_b"'] \end{tikzcd}$$

Using Proposition 2.2.7, it is enough to show that the pullback functor f^* sits inside an enriched Quillen adjoint triple

$$\begin{tikzcd} {[\mathcal{C},\mathcal{M}]} \arrow[r, "f^*" description] & {[\mathcal{D},\mathcal{M}].} \arrow[l, bend right, "\mathrm{Lan}_f"', "\perp"] \arrow[l, bend left, "\mathrm{Ran}_f", "\perp"'] \end{tikzcd}$$

Since $\mathcal{M}$ is a complete and cocomplete enriched category, we already know from [Kel05, §4.3] that the adjunctions $\mathrm{Lan}_f \dashv f^*$ and $f^* \dashv \mathrm{Ran}_f$ always exist and are enriched. It remains to show that they are Quillen pairs, meaning that the adjunctions between the underlying categories are Quillen. Recall that Theorem 1.4.12 implies that the underlying categories $[\mathcal{C},\mathcal{M}]_0$ and $[\mathcal{D},\mathcal{M}]_0$ admit both the projective and the injective model structure (which are again enriched) and furthermore Proposition 1.4.14 ensures these model structures are Quillen equivalent (so that we get equivalent homotopy 2-categories), so let us choose them accordingly. We consider $\mathrm{Lan}_f \dashv f^*\colon [\mathcal{C},\mathcal{M}]_0^{\mathrm{proj}} \rightleftarrows [\mathcal{D},\mathcal{M}]_0^{\mathrm{proj}}$ and $f^* \dashv \mathrm{Ran}_f\colon [\mathcal{C},\mathcal{M}]_0^{\mathrm{inj}} \rightleftarrows [\mathcal{D},\mathcal{M}]_0^{\mathrm{inj}}$ and prove that f^* is both right and left Quillen with respect to the chosen model structures. Recall that $[\mathcal{C},\mathcal{M}]_0$ and $[\mathcal{D},\mathcal{M}]_0$ are categories with objects being simplicial functors $\mathcal{C} \to \mathcal{M}$ and $\mathcal{D} \to \mathcal{M}$ and morphisms between any pair of such functors H, K being simplicial natural transformations $H \Rightarrow K$. Let us prove that f^* is right Quillen, the other case being completely analogous. Notice that a simplicial natural transformation $\alpha\colon H \Rightarrow K$ in $[\mathcal{C},\mathcal{M}]_0^{\mathrm{proj}}$ is a fibration (resp. a weak equivalence) if and only if α_c is a fibration (resp. a weak equivalence) in $\mathcal{M}$ (more precisely, in $\mathcal{M}_0$) for every $c \in \mathcal{C}$. Then $f^*\alpha\colon f^*H = Hf \Rightarrow f^*K = Kf$ has components $(f^*\alpha)_d = (\alpha f)_d = \alpha_{fd}$ for a generic $d \in \mathcal{D}$ and so it is a fibration or a weak equivalence whenever α is. In particular if α is a trivial fibration, then $f^*\alpha$ is a trivial fibration as well. □

Proposition 2.2.11. *The shifted 2-prederivator $\mathbb{D}^{\mathcal{A}}$ satisfies* (HDer 4) *if $\mathbb{D}$ satisfies it.*

Proof. We claim that for every $f\colon \mathcal{I} \to \mathcal{J}$ in **Dia** there exists a biadjoint triple

$$\begin{tikzcd} \mathbb{D}^{\mathcal{A}}(\mathcal{J}) \arrow[r, "f^*" description] & \mathbb{D}^{\mathcal{A}}(\mathcal{I}), \arrow[l, bend right, "\mathrm{Lan}_f"', "\perp_b"] \arrow[l, bend left, "\mathrm{Ran}_f", "\perp_b"'] \end{tikzcd}$$

but the pullback 2-functor is given by the action of $\mathbb{D}$ on $\mathcal{A}\times f\colon \mathcal{A}\times\mathcal{I} \to \mathcal{A}\times\mathcal{J}$, which is still in **Dia**, hence $\mathbb{D}^{\mathcal{A}}(f) = \mathbb{D}(\mathcal{A}\times f)$ admits homotopy left and right Kan extensions. $\square$

The following construction is useful for the formulation of the next axiom and for the computations concerning it.

Proposition 2.2.12. *There exists a conservative on* 2*-cells* **Gray***-functor $T\colon$* **Gray** $\to$ **Cat** *sending a* 2*-category $\mathcal{A}$ to the category $T\,\mathcal{A}$ with the same set of objects of $\mathcal{A}$ and morphisms being isomorphism classes of* 1*-cells in $\mathcal{A}$.*

Proof. First of all notice that **Cat** is a 2-category, which we can regard as a discrete **Gray**-category i.e. a **Gray**-category with only identity 3-cells. Therefore T is forced to send modifications in **Gray** to identities. If $F\colon \mathcal{A} \to \mathcal{B}$ is a 2-functor, we define $T(F)$ as F on objects and morphisms. It is well-defined since if f and g belong to the same isomorphism class of 1-cells, there exists an invertible 2-cell $\theta\colon f \Rightarrow g$ that the 2-functor F sends to an invertible 2-cell $F\theta\colon Ff \Rightarrow Fg$ witnessing that Ff and Fg are in the same isomorphism class. Given a pseudonatural transformation $\alpha\colon F \Rightarrow G\colon \mathcal{A} \to \mathcal{B}$, we define $T(\alpha)_A \coloneqq \alpha_A$ for every $A \in \mathcal{A}$. First of all let's check that this assignment defines an actual **Gray**-functor. We have to show that

1. $T_{\mathcal{A},\mathcal{B}}\colon \mathbf{Gray}(\mathcal{A},\mathcal{B}) \to \mathbf{Cat}(T\,\mathcal{A}, T\,\mathcal{B})$ is a 2-functor,

2. the diagrams

$$\begin{tikzcd} \mathbf{Gray}(\mathcal{B},\mathcal{C})\otimes\mathbf{Gray}(\mathcal{A},\mathcal{B}) \arrow[r, "\circ"] \arrow[d, "T_{\mathcal{B},\mathcal{C}}\otimes T_{\mathcal{A},\mathcal{B}}"'] & \mathbf{Gray}(\mathcal{A},\mathcal{C}) \arrow[d, "T_{\mathcal{A},\mathcal{C}}"] \\ \mathbf{Cat}(T\,\mathcal{B},T\,\mathcal{C})\otimes\mathbf{Cat}(T\,\mathcal{A},T\,\mathcal{B}) \arrow[r, "\circ"'] & \mathbf{Cat}(T\,\mathcal{A},T\,\mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "\mathrm{id}_{\mathcal{A}}"] \arrow[dr, "\mathrm{id}_{T\,\mathcal{A}}"'] & \mathbf{Gray}(\mathcal{A},\mathcal{A}) \arrow[d, "T_{\mathcal{A},\mathcal{A}}"] \\ & \mathbf{Cat}(T\,\mathcal{A},T\,\mathcal{A}) \end{tikzcd}$$

commute in **Gray**.

To prove 1. we have to verify that $(T_{\mathcal{A},\mathcal{B}})_{F,G}\colon \mathbf{Gray}(\mathcal{A},\mathcal{B})(F,G) \to \mathbf{Cat}(T\,\mathcal{A}, T\,\mathcal{B})(TF, TG)$ is a functor for every $F, G \in \mathbf{Gray}(\mathcal{A},\mathcal{B})$ and the diagrams for associativity and unitality commute in **Cat**. The functoriality of $(T_{\mathcal{A},\mathcal{B}})_{F,G}$ is clear since it sends every morphism of $\mathbf{Gray}(\mathcal{A},\mathcal{B})(F,G)$ to an identity. For the same reason, it is enough to check associativity and unitality at the level of objects. For a generic $A \in \mathcal{A}$ we have $(T\beta \cdot T\alpha)_A = (T\beta)_A(T\alpha)_A = \beta_A\alpha_A = (\beta\cdot\alpha)_A = T(\beta\cdot\alpha)_A$ and $(T\,\mathrm{id}_F)_A = (\mathrm{id}_F)_A = \mathrm{id}_{FA} = \mathrm{id}_{TFA} = (\mathrm{id}_{TF})_A$. To prove 2. it is enough to check it on 0-cells and 1-cells, since 2-cells are modifications and for these the diagrams trivially commute. Commutativity for 0-cells means that for 2-functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ the equalities $TGTF = T(GF)$ and $T(\mathrm{id}_{\mathcal{A}}) = \mathrm{id}_{T\,\mathcal{A}}$ hold, which is true because T acts as the identity on 1-cells. For the 1-cells we have to check that $T\beta * T\alpha = T(\beta * \alpha)$ and $T(\mathrm{id}_{\mathrm{id}_{\mathcal{A}}}) = \mathrm{id}_{T\,\mathrm{id}_{\mathcal{A}}}$. The former equality is a consequence of the middle four interchange which holds up to isomorphism in **Gray** hence on the nose in the quotient. The latter equality holds since the identities are equalities componentwise. It remains to check that it is conservative on 2-cells, but this is clear since $T(\alpha)$ is invertible in $T\,\mathcal{A}$ if and only if $T(\alpha)_A$ is invertible for every $A \in \mathcal{A}$ if and only if α_A is invertible in $\mathcal{A}$ up to an invertible 2-cell i.e. α_A is an equivalence in $\mathcal{A}$ for every $A \in \mathcal{A}$ if and only if α is a pseudonatural equivalence. □

Definition 2.2.13. The *mate* of the diagram in **Gray**

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f^*"] \arrow[d, "g^*"'] & \mathcal{B} \arrow[d, "k^*"] \arrow[dl, phantom, "\swarrow\alpha"] \\
\mathcal{C} \arrow[r, "h^*"'] & \mathcal{D}
\end{tikzcd}$$

for which there are biadjoint pairs $f_! \dashv_b f^*$ and $h_! \dashv_b h^*$ is the pasting composite

$$\begin{tikzcd}
& \mathcal{B} \arrow[r, "k^*"] \arrow[dr, phantom, "\searrow\alpha"] & \mathcal{D} \arrow[r, "h_!"] \arrow[dr, phantom, "\searrow\epsilon"] & \mathcal{C} \\
\mathcal{B} \arrow[ur, equal] \arrow[r, "f_!"'] \arrow[ur, phantom, "\eta\searrow"] & \mathcal{A} \arrow[u, "f^*"'] \arrow[r, "g^*"'] & \mathcal{C} \arrow[u, "h^*"'] \arrow[ur, equal] &
\end{tikzcd}$$

Remark 2.2.14. Definition 2.2.13 is well posed because pasting diagrams in **Gray** have a well defined composite 2-cell up to canonical isomorphism. Actually, every possible composition is linked to every other one by means of an isomorphism and those isomorphisms compose. In addition, Proposition 2.2.12 gives us the opportunity to perform the calculus of mates as if we were inside of **Cat** rather than in **Gray** which simplifies many of the proofs. In fact, the **Gray**-functor T is conservative on 2-cells and when we compose it with a 2-prederivator $\mathbb{D}$ we obtain the pseudofunctor

$$\mathbf{Dia}^{\mathrm{op}} \subseteq \mathbf{2\text{-}Cat}^{\mathrm{op}} \xrightarrow{\mathbb{D}} \mathbf{GRAY} \xrightarrow{T} \mathbf{CAT},$$

where we are regarding **2-Cat** as a 2-category. Functoriality on hom-categories comes from the (2-)functoriality on homs of T and $\mathbb{D}$. Composition and identity 2-functors are preserved up to equivalence in **GRAY**, and so are preserved up to isomorphism by $T\mathbb{D}$.

Definition 2.2.15. Suppose that **Dia** is closed under collages and consider the following diagrams in **2-Cat**

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "\iota_{f,b}"] \arrow[d, "f"'] & \mathrm{coll}(f,b) \arrow[dl, "\pi_{\mathcal{B}}"'] \\ \mathcal{B} & \mathbb{1} \arrow[l, "b"] \arrow[u, "\bullet"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, hook, "\iota_{\mathcal{B}}"] \arrow[d, "f"'] & \mathrm{coll}(b,f) \arrow[dl, "\pi_{\mathcal{B}}"'] \\ \mathcal{B} & \mathbb{1} \arrow[l, "b"] \arrow[u, "\bullet"'] \end{tikzcd}$$

and suppose they are sent by $\mathbb{D}$ to squares with specified biadjoints

$$\begin{tikzcd} \mathbb{D}(\mathcal{A}) \arrow[r, bend left=50, "\mathrm{Lan}_{\iota_{f,b}}"] \arrow[d, bend right=60, "\mathrm{Lan}_f"'] & \mathbb{D}(\mathrm{coll}(f,b)) \arrow[l, "\iota^*_{f,b}"', "\perp_b"] \arrow[d, "\bullet^*"] \\ \mathbb{D}(\mathcal{B}) \arrow[u, "f^*"', "\dashv_b"] \arrow[ur, "\pi^*_{\mathcal{B}}", "\simeq"'] \arrow[r, "b^*"'] & \mathbb{D}(\mathbb{1}) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(\mathcal{A}) \arrow[r, "\mathrm{Ran}_{\iota_{b,f}}", "\simeq"'] \arrow[d, "\mathrm{Ran}_f"] & \mathbb{D}(\mathrm{coll}(b,f)) \arrow[l, bend right=50, "\iota^*_{b,f}"', "\perp_b"] \arrow[d, "\bullet^*"] \\ \mathbb{D}(\mathcal{B}) \arrow[u, bend left=60, "f^*", "\dashv_b"'] \arrow[ur, "\pi^*_{\mathcal{B}}", "\simeq"'] \arrow[r, "b^*"'] & \mathbb{D}(\mathbb{1}). \end{tikzcd}$$

The biadjoint pairs in the diagrams before induce 2-cells

$$\frac{\dfrac{f^* \simeq \iota^*_{f,b}\pi^*_{\mathcal{B}}}{\mathrm{Lan}_{\iota_{f,b}} f^* \Rightarrow \pi^*_{\mathcal{B}}}}{\mathrm{Lan}_{\iota_{f,b}} \Rightarrow \pi^*_{\mathcal{B}} \mathrm{Lan}_f} \qquad \frac{\dfrac{\iota^*_{b,f}\pi^*_{\mathcal{B}} \simeq f^*}{\pi^*_{\mathcal{B}} \Rightarrow \mathrm{Ran}_{\iota_{b,f}} f^*}}{\pi^*_{\mathcal{B}} \mathrm{Ran}_f \Rightarrow \mathrm{Ran}_{\iota_{b,f}}}$$

We say that the first two diagrams are *exact* if the precomposite 2-cells $\bullet^* \operatorname{Lan}_{\iota_{f,b}} \Rightarrow \bullet^* \pi_{\mathcal{B}}^* \operatorname{Lan}_f \cong b^* \operatorname{Lan}_f$ and $\bullet^* \pi_{\mathcal{B}}^* \operatorname{Ran}_f \cong b^* \operatorname{Ran}_f \Rightarrow \bullet^* \operatorname{Ran}_{\iota_{b,f}}$ are equivalences.

Axiom 6. (HDer 5) For every $f\colon \mathcal{A} \to \mathcal{B}$ and every object $b \in \mathcal{B}$ the diagrams in Definition 2.2.15 are exact.

For instance, we can use the axiom (HDer 5) to prove a version of the classical theorem about Kan extensions along fully faithful functors in this setting. But first we give another result about mates.

Proposition 2.2.16. *Suppose we have an invertible 2-cell α filling a square*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "c"'] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d, "d"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

in **Cat** *with adjunctions $g \dashv v$ and $f \dashv u$. Call $\widehat{\alpha}$ the mate of α, and let η, ϵ be the unit and the counit of $g \dashv v$ and η', ϵ' the unit and the counit of $f \dashv u$. Then we have an equality*

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "d"'] \arrow[dr, phantom, "\nearrow\widehat{\alpha}"] & A \arrow[r, "u"] \arrow[d, "c"] \arrow[dr, phantom, "\nearrow\alpha^{-1}"] & B \arrow[d, "d"] \\ D \arrow[r, "g"'] \arrow[rr, bend right, "\mathrm{id}_D"'] \arrow[rr, phantom, bend right=20, "\nearrow\eta"] & C \arrow[r, "v"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} & A \arrow[dr, "u"] \arrow[d, phantom, "\Uparrow\eta'"] & & \\ B \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_B"'] & {} & B \arrow[r, "d"'] & D. \end{tikzcd}$$

In particular, if $\widehat{\alpha}$ is invertible then ηd is invertible if and only if $d\eta'$ is invertible.

Proof. The proof is an explicit computation. The LHS of the equation is the upper composite in the diagram

$$\begin{tikzcd} d \arrow[r, Rightarrow, "d\eta'"] \arrow[d, Rightarrow, "\eta d"'] & duf \arrow[r, Rightarrow, "\alpha f"] \arrow[d, Rightarrow, "\eta duf"'] \arrow[rrrd, equal, bend left] & vcf \arrow[d, Rightarrow, "\eta vcf"'] \arrow[dr, equal] & & \\ vgd \arrow[r, Rightarrow, "vgd\eta'"'] & vgduf \arrow[r, Rightarrow, "vg\alpha f"'] & vgvcf \arrow[r, Rightarrow, "v\epsilon cf"'] & vcf \arrow[r, Rightarrow, "\alpha^{-1}f"'] & duf \end{tikzcd}$$

while the RHS is the lower one. The squares in the diagram commute for the middle four interchange and the triangle comes from the triangular identities of the adjunction $g \dashv v$. $\square$

Remark 2.2.17. Proposition 2.2.16 can be restated in **Gray** with the equality replaced by an isomorphism. The proof is exactly the same, modulo equivalence.

If $\mathbb{D}$ satisfies (HDer 5), then it satisfies a "generalized elements" version of it.

Proposition 2.2.18. *Suppose* $\mathbb{D}$ *satisfies* (HDer 5) *and consider a cospan* $\mathcal{A} \xrightarrow{f} \mathcal{C} \xleftarrow{g} \mathcal{B}$ *in* **Dia**. *Then the diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "i_0"] \arrow[d, "f"'] & \mathrm{coll}(f,g) \arrow[dl, "\pi_{\mathcal{C}}"'] \\
\mathcal{C} & \mathcal{B} \arrow[l, "g"] \arrow[u, hook, "i_1"']
\end{tikzcd}$$

is exact, i.e. the diagram

$$\begin{tikzcd}
\mathbb{D}(\mathcal{A}) \arrow[r, bend left=50, "\mathrm{Lan}_{i_0}"] \arrow[d, bend right=50, "\mathrm{Lan}_f"'] & \mathbb{D}(\mathrm{coll}(f,g)) \arrow[l, "i_0^*"] \arrow[l, phantom, bend right=25, "\perp_b"'] \arrow[d, "i_1^*"'] \\
\mathbb{D}(\mathcal{C}) \arrow[u, "f^*"'] \arrow[u, phantom, bend left=25, "\dashv_b"] \arrow[ur, "\pi_{\mathcal{C}}^*"] \arrow[r, "g^*"'] \arrow[ur, phantom, shift left=3, "\simeq"] \arrow[ur, phantom, shift right=3, "\simeq"'] & \mathbb{D}(\mathcal{B})
\end{tikzcd}$$

commuting up to equivalence, is such that the 2-cell induced by the biadjunctions composed with i_1^* is invertible.

Proof. There is an equality $\mathrm{coll}(i_0, i_1 b) = \mathrm{coll}(f, gb)$ since they have the same set of objects and $\mathrm{coll}(i_0, i_1 b)(a, \bullet) = \mathrm{coll}(f,g)(i_0 a, i_1 b \bullet) = \mathrm{coll}(f,g)(a,b) = \mathcal{C}(fa, gb) = \mathrm{coll}(f, gb)(a, \bullet)$ and a 2-functor $\mathrm{coll}(f, gb) \to \mathrm{coll}(f,g)$ corresponding to the inclusion of gb in the collage. Considering the diagram

$$\begin{tikzcd}
\mathbb{D}(\mathrm{coll}(i_0,i_1b)) \simeq & \mathbb{D}(\mathrm{coll}(f,gb)) \arrow[dl, "i_0^*"'] \arrow[r, "\bullet^*"] \arrow[d, phantom, "\simeq" very near start] & \mathbb{D}(\mathbb{1}) \\
\mathbb{D}(\mathcal{A}) & \mathbb{D}(\mathrm{coll}(f,g)) \arrow[l, "i_0^*"] \arrow[u] \arrow[r, "i_1^*"] \arrow[ur, phantom, "\simeq"] & \mathbb{D}(\mathcal{B}) \arrow[u, "b^*"'] \\
& \mathbb{D}(\mathcal{C}) \arrow[ul, "f^*"] \arrow[u, "\pi_{\mathcal{C}}^*"] \arrow[ur, "g^*"'] \arrow[ul, phantom, "\simeq" very near end] \arrow[ur, phantom, "\simeq" very near end] &
\end{tikzcd}$$

we notice that the subdiagrams

$$\begin{tikzcd}
\mathbb{D}(\mathcal{A}) \arrow[r, bend left=50, "\mathrm{Lan}_{i_0}"] \arrow[r, phantom, bend left=20, "\perp_b"] \arrow[d, bend right=50, "\mathrm{Lan}_f"'] \arrow[d, phantom, "\perp_b"] & \mathbb{D}(\mathrm{coll}(f,gb)) \arrow[l, "i_0^*"] \arrow[d, "\bullet^*"] \\
\mathbb{D}(\mathcal{C}) \arrow[u, "f^*"'] \arrow[ur, "\pi_{\mathcal{C}}^*"] \arrow[r, "(gb)^*"'] & \mathbb{D}(\mathbb{1})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}(\mathcal{A}) \arrow[r, bend left=50, "\mathrm{Lan}_{i_0}"] \arrow[r, phantom, bend left=20, "\perp_b"] \arrow[d, bend right=50, "\mathrm{Lan}_{i_0}"'] \arrow[d, phantom, "\perp_b"] & \mathbb{D}(\mathrm{coll}(f,gb)) \arrow[l, "i_0^*"] \arrow[d, "\bullet^*"] \\
\mathbb{D}(\mathrm{coll}(f,g)) \arrow[u, "i_0^*"'] \arrow[ur] \arrow[r, "(i_1b)^*"'] & \mathbb{D}(\mathbb{1}).
\end{tikzcd}$$

are both in a form for which we can apply (HDer 5). Therefore, since mates compose, we know that the composition of the mate of the lower square of the pentagonal diagram above with b^* is an equivalence for all $b \in \mathcal{B}$ and so it is an equivalence because pointwise equivalences are equivalences. $\square$

Proposition 2.2.19. *Let $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ be a fully faithful 2-functor, meaning it is an isomorphism at the level of hom-categories, then the unit $\eta\colon \mathrm{id}_{\mathbb{D}(\mathcal{A})} \Rightarrow I^* \circ \mathrm{Lan}_I$ of the biadjunction*

$$\begin{tikzcd}
\mathbb{D}(\mathcal{A}) \arrow[r, bend left, "\mathrm{Lan}_I"] \arrow[r, phantom, "\perp_b"] & \mathbb{D}(\mathcal{B}) \arrow[l, bend left, "I^*"]
\end{tikzcd}$$

is a pseudonatural equivalence.

Proof. Since I is fully faithful, we have an isomorphism $\mathrm{coll}(I,I) \cong \mathrm{coll}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}})$ and so the inclusion $J_0\colon \mathcal{A} \hookrightarrow \mathrm{coll}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}})$ of $\mathcal{A}$ in the collage as the first component has a right adjoint $R\colon \mathrm{coll}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}}) \to \mathcal{A}$, that projects $\mathrm{Ob}(\mathcal{A}) \sqcup \mathrm{Ob}(\mathcal{A})$ back to $\mathrm{Ob}(\mathcal{A})$ and acts on hom-categories as the identity. In particular, there is a 2-natural transformation $J_0R \Rightarrow \mathrm{id}_{\mathrm{coll}(\mathrm{id}_{\mathcal{A}}, \mathrm{id}_{\mathcal{A}})}$ serving as the counit of the adjunction and the identity 2-natural transformation

$\mathrm{id}_{\mathcal{A}} = RJ_0$ which is the unit. $\mathbb{D}$ sends this adjunction to a biadjunction $R^* \dashv_b J_0^*$ (they switch places because $\mathbb{D}$ is contravariant on 1-cells) with a pseudonatural transformation $\alpha\colon R^*J_0^* \Rightarrow \mathrm{id}_{\mathbb{D}(\mathrm{coll}(\mathrm{id}_{\mathcal{A}},\mathrm{id}_{\mathcal{A}}))}$ as counit and a pseudonatural equivalence $\beta\colon \mathrm{id}_{\mathbb{D}(\mathcal{A})} \simeq J_0^*R^*$ as unit. Notice that in

$$\begin{tikzcd}
J_1^*R^* \arrow[r, Rightarrow, "J_1^*R^*\eta"] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "\simeq"] & J_1^*R^*I^*\operatorname{Lan}_I \arrow[r, Rightarrow, "\simeq"] \arrow[d, phantom, "\wr"] \arrow[dr, phantom, "\simeq"] & J_1^*R^*J_0^*\pi_{\mathcal{B}}^*\operatorname{Lan}_I \arrow[r, Rightarrow] \arrow[d, phantom, "\wr"] & J_1^*\pi_{\mathcal{B}}^*\operatorname{Lan}_I \\
\mathrm{id}_{\mathbb{D}(\mathcal{A})} \arrow[r, Rightarrow, "\eta"'] & I^*\operatorname{Lan}_I^* \arrow[r, Rightarrow, "\simeq"'] & J_0^*\pi_{\mathcal{B}}^*\operatorname{Lan}_I \arrow[ur, Rightarrow, "\simeq", "\simeq"']
\end{tikzcd}$$

the second arrows in the top and bottom rows are equivalences because $I^* \simeq J_0^*\pi_{\mathcal{B}}^*$, the vertical arrows are equivalences since $J_1^*R^* \simeq \mathrm{id}_{\mathbb{D}(\mathcal{A})}$ and the oblique arrow at the right is an equivalence since $\pi_{\mathcal{B}}^* \simeq (IR)^* \simeq R^*I^*$ and finally $J_0^*\pi_{\mathcal{B}}^* \simeq J_0^*R^*I^* \simeq J_1^*R^*I^* \simeq J_1^*\pi_{\mathcal{B}}^*$. Since the top row is an equivalence by the extended version of (HDer 5), we have that η must be an equivalence as well. $\square$

Proposition 2.2.20. *If* (HDer 5) *holds for the* 2*-prederivator* $\mathbb{D}$*, then the shifted 2-prederivator* $\mathbb{D}^{\mathcal{C}}$ *satisfies* (HDer 5) *for every* $\mathcal{C} \in$ **Dia**.

Proof. Let us prove the claim for left Kan extensions, the other case being completely analogous. We have to show that for every $f\colon \mathcal{A} \to \mathcal{B}$ in **Dia**, the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "\iota_{f,b}"] \arrow[d, "f"'] & \mathrm{coll}(f,b) \arrow[dl, "\pi_{\mathcal{B}}"'] \\
\mathcal{B} & \mathbb{1} \arrow[l, "b"] \arrow[u, "\bullet"']
\end{tikzcd}$$

is sent to

$$\begin{tikzcd}
\mathbb{D}^{\mathcal{C}}(\mathcal{A}) \arrow[r, bend left=50, "\operatorname{Lan}_{\iota_{f,b}}"] \arrow[d, bend right=50, "\operatorname{Lan}_f"'] \arrow[d, phantom, "\dashv_b"] \arrow[r, phantom, "\perp_b"] & \mathbb{D}^{\mathcal{C}}(\mathrm{coll}(f,b)) \arrow[l, "\iota_{f,b}^*"'] \arrow[d, "\bullet^*"] \\
\mathbb{D}^{\mathcal{C}}(\mathcal{B}) \arrow[u, "f^*"'] \arrow[ur, "\pi_{\mathcal{B}}^*"] \arrow[r, "b^*"'] \arrow[ur, phantom, "\simeq" near end] \arrow[u, phantom, "\simeq" near end] & \mathbb{D}^{\mathcal{C}}(\mathbb{1})
\end{tikzcd}$$

with the precomposite 2-cell $\bullet^* \operatorname{Lan}_{\iota_{f,b}} \Rightarrow \bullet^* \pi^*_{f,b} \operatorname{Lan}_f \cong b^* \operatorname{Lan}_f$ being invertible. If we consider the morphism $\mathcal{C} \times f\colon \mathcal{C} \times \mathcal{A} \to \mathcal{C} \times \mathcal{B}$, we know that

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{A} \arrow[r, hook, "i"] \arrow[d, "C \times f"'] & \operatorname{coll}(\mathcal{C} \times f, \mathcal{C} \times b) \arrow[dl, "\pi_{\mathcal{C} \times \mathcal{B}}"'] \\
\mathcal{C} \times \mathcal{B} & \mathcal{C} \cong \mathcal{C} \times \mathbb{1} \arrow[l, "\mathcal{C} \times b"] \arrow[u, "\mathcal{C} \times \bullet"']
\end{tikzcd}$$

is exact. The exactness of this diagram would imply the exactness of our original diagram, by the generalized version of (HDer 5), as long as we show that $\operatorname{coll}(\mathcal{C} \times f, \mathcal{C} \times b) = \mathcal{C} \times \operatorname{coll}(f, b)$. This is indeed the case, given that they both have $(\mathcal{C} \times \mathcal{A}) \sqcup \mathcal{C}$ as set of objects and the morphisms between $\mathcal{C} \times \mathcal{A}$ and $\mathcal{C}$ are $(\mathcal{C} \times \mathcal{B})((i, fa), (j, b)) = \mathcal{C}(i, j) \times \mathcal{B}(fa, b)$. Therefore the two collages are equal and the claim is proven. □

Proposition 2.2.21. *If $\mathcal{C}$ is a complete and cocomplete 2-category, the represented 2-prederivator $[-, \mathcal{C}]$ satisfies* (HDer 5).

Proof. The square corresponding to left Kan extensions is sent to the diagram

$$\begin{tikzcd}
{[\mathcal{A}, \mathcal{C}]} \arrow[r, bend left=50, "\operatorname{Lan}_{\iota_{f,b}}"] \arrow[r, phantom, bend left=25, "\perp_b"] \arrow[d, bend right=60, "\operatorname{Lan}_f"'] \arrow[d, phantom, "\dashv_b"] \arrow[dr, phantom, "\simeq"] & {[\operatorname{coll}(f, b), \mathcal{C}]} \arrow[l, "\iota^*_{f,b}"'] \arrow[d, "\bullet^*"] \\
{[\mathcal{B}, \mathcal{C}]} \arrow[u, "f^*"'] \arrow[ur, "\pi^*_{\mathcal{B}}", "\simeq"'] \arrow[r, "b^*"'] & {[\mathbb{1}, \mathcal{C}] \cong \mathcal{C}}
\end{tikzcd}$$

with the 2-cell constructed as in Definition 2.2.15 that is invertible by the definition of left Kan extensions in the enriched sense, that exist since $\mathcal{C}$ is a cocomplete 2-category. The same argument shows that the right Kan extensions are pointwise, this time using that $\mathcal{C}$ is complete. Notice that in this case the biadjunctions reduce to genuine adjunctions and the equivalences filling the diagram are actual identities. □

Finally we can provide a sketch of the proof that $\mathbb{D}_{\mathcal{M}}$ satisfies (HDer 5). A full proof would have been too long to fit this thesis.

Proposition 2.2.22. *The axiom* (HDer 5) *holds in* $\mathbb{D}_{\mathcal{M}}$.

Proof. This is an argument involving mates. We know that (HDer 5) holds in enriched category theory, i.e. for the represented 2-prederivator. One can argue that this has to hold also for $\mathbb{D}_{\mathcal{M}}$ by pulling back mates from the represented 2-prederivator (and his image through h_*) to $\mathbb{D}_{\mathcal{M}}$. $\square$

The last axiom is about internalization of comma objects. As a matter of fact, weak comma objects are a cornerstone in ∞-cosmology and the following axiom ensures that commas in 2-derivators have the same weak universal property.

Axiom 7. (HDer 6) Let $\boxslash$ and $\lrcorner$ be the 2-categories realizing respectively the shape of a comma square and of a pullback. Then the underlying diagram 2-functor $\mathbb{D}(\boxslash) \to \mathbb{D}(\mathbb{1})^{\boxslash}$ restricted to the image of $\mathrm{Ran}_i \colon \mathbb{D}(\lrcorner) \to \mathbb{D}(\boxslash)$ (where i is the inclusion of the lower right corner in the square) induces a smothering functor $\mathbb{D}(\mathbb{1})(Y, f \downarrow g) \to \mathbb{D}(\mathbb{1})(Y, f) \downarrow \mathbb{D}(\mathbb{1})(Y, g)$ where $Y \in \mathbb{D}(\mathbb{1})$, f and g are the arrows in the diagram shape of the pullback, $f \downarrow g$ is the internal comma object and the target of the smothering functor is a comma object in **2-Cat**.

Proposition 2.2.23. $\mathbb{D}_{\mathcal{M}}$ *satisfies* (HDer 6).

Proof. The proof is exactly the same one provided in [RV15] for ∞-cosmoi. $\square$

2.3 Future work

In this work we introduced some axioms that 2-derivators have to satisfy as well as some models of these axioms which cover the most common situations we are interested in. There is still work to do in this direction, namely to expand the set of axioms in a non redundant way to include other properties of our models. Once we have that, a natural step forward is to develop basic ∞-category theory in this context. In particular, a proof of concept in favour of the theory of 2-derivators would be to prove Beck's monadicity theorem within this axiomatic framework.

Another interesting application is to develop the theory of fibrations and fibred ∞-category theory in this setting.

A more advanced goal is to use this tool in order to give a synthetic treatment of enriched ∞-category theory and higher algebra.

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