OPTIMAL REINSURANCE AND INVESTMENT STRATEGIES UNDER STRATEGIC INTERACTION AND MODEL UNCERTAINTY

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Abstract

The study of investment and reinsurance optimization problems has developed as a time-honored research area, which has drawn significant interests both in the academia, the insurance and financial sectors. The present study discusses several reinsurance and investment optimization problems in a dynamic environment from two perspectives: (1) application of a robust approach to elucidate how market participants' attitudes towards ambiguity would impact the optimal decision rules; and (2) construction of analytical frameworks to investigate the effects of strategic interactions between two different decision-makers on their optimal strategies.

From the first perspective, considering that the findings yielded by extant experimental studies confirm that the decision-makers are not only risk-averse but also ambiguity-averse, in the current investigation, the focus is on the optimization problems in the presence of model uncertainty. It is further assumed that the reference model available to the decision-makers is an approximating model which may contain a specification error. Specifically, the financial and insurance models are characterized mathematically by the uncertainty about certain parameters in a diffusion model and a jump-type model. Moreover, the decision-makers obtain decision rules that are robust to model misspecification by maximizing their performance functionals over the worst-case scenario across a family of plausible models, which formulates max-min problems. The discrepancy between the alternative models and the reference model is defined through the concept of relative entropy, which serves as a penalty function in the optimization procedure.

Regarding the second perspective, we proceed along two directions. First, we formulate a competition between two insurance companies whose control policies impact those of their competitors due to their relative performance concerns and the correlation between their surplus processes. It is further assumed that they have access to the same financial market and their aggregated claims are governed by a common Poisson process describing the systematic insurance risk. The concept of relative performance concerns is incorporated into robust optimization problems under the expected utility maximization and mean-variance principle criteria. The system comprising the robust optimization problems pertaining to these two companies constitutes a robust non-zero-sum stochastic differential two-player game, for which the explicit expressions for Nash equilibrium strategies can be derived. Second, the bargaining between an insurer and a reinsurer, both of whom are ambiguity-averse, negotiating a reinsurance contract is studied. These two parties in a reinsurance policy form a principal-agent framework, which is essentially a Stackelberg game between the two contracting parties, allowing the insurer's and reinsurer's interests to be combined in a continuous-time setting. Under this framework, the reinsurer is regarded as a principal while the insurer is assigned the role of an agent, allowing the former to adjust the reinsurance premium according to the latter's reinsurance demands. We seek for the equilibrium optimal reinsurance arrangement by a two-step method and discuss the utility loss of the insurer and the reinsurer if they ignore model misspecification.

Finally, based on the study findings, implications for theory and practice are delineated before offering some suggestions for further research in this field.

Keywords: Investment and reinsurance, game, relative performance, principal-agent, model uncertainty, mean-variance, expected utility maximization

Statement of Originality

I, Ning Wang, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy in Actuarial Studies and Business Analytics at Macquarie University (MQU) and Probability and Mathematical Statistics at East China Normal University (ECNU), wholly represents my own work unless otherwise referenced or acknowledged. The document has not been previously included in a thesis, dissertation or report submitted to these two universities or any other institutions for a degree, diploma or other qualifications.

Chapter 2 is based on the paper "Robust non-zero-sum investment and reinsurance game with default risk", co-authored by Nan Zhang, Zhuo Jin and Linyi Qian and published in *Insurance: Mathematics and Economics* **84**, 115-132. Chapter 3 is based on the paper "Reinsurance-investment game between two mean-variance insurers under model uncertainty", co-authored by Nan Zhang, Zhuo Jin and Linyi Qian and published in *Journal of Computational and Applied Mathematics* **382**, 113095. Content of Chapter 4 comprises of an unpublished working paper co-authored by Tak Kuen Siu and Kun Fan. I finished the papers independently with necessary direction from my supervisors: Professor Tak Kuen Siu (MQU), Professor Rongming Wang (ECNU) and Associate Professor Xian Zhou (MQU).

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

This chapter commences with a brief review of pertinent literature, after which the preliminaries and the thesis structure are outlined, before drawing comparisons between the research problems comprising this research study.

1.1 Literature Review

1.1.1 Optimal Reinsurance Contracts

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which is equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying some usual conditions such as right continuity and $\mathbb{P}$-completeness. In such a case, the classical Cramér-Lundberg risk model, which is widely adopted in the literature about insurance risk theory, can be used to describe the surplus process of an insurance company:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,$$

where u is the initial surplus, the claim arrival process $N(t)$ is a Poisson process whose intensity is $\lambda > 0$ and the claim sizes $\{X_i\}_{i=1}^{\infty}$ are independent and identically distributed (i.i.d.) random variables which are supposed to be independent of $N(t)$. The insurance premium rate c can be determined by applying various premium principles such as the expected value principle, the variance principle, the standard deviation principle, etc.

In extant research, reinsurance is considered as a highly useful risk management strategy and a strategic business planning tool for an insurance company. In this context, reinsurance arrangement is defined as a treaty between a reinsurer and an insurer under which any claims that occur during the contract period would be shared between the two parties. Reinsurance contracts can be of proportional and non-proportional type, and excess-of-loss and stop-loss are the most commonly used non-proportional reinsurance policies in the literature about the insurance risk theory. Under a proportional reinsurance treaty, the insurer pays a fixed proportion, denoted as q, of each claim that occurs in the period covered by the reinsurance arrangement. The remaining proportion, $1-q$, of each claim is transferred to the reinsurer. Under an excess-of-loss reinsurance contract, a claim is shared between the two parties only if its value exceeds a predetermined amount denoted as the retention level. Under a stop-loss reinsurance arrangement, the reinsurer becomes liable to pay only if the aggregate claim risks incurred by the insurer exceed a fixed retention level.

In the present investigation, the initial loss incurred by the insurer is denoted as X. To mitigate this risk exposure, the insurer purchases a reinsurance policy f from a reinsurer, where the function f is known as the ceded loss function and it defines a reinsurance contract. When a claim occurs, the insurer cedes $f(X)$ to the reinsurer and pays the reinsurer $P_R^f(X)$ as a reinsurance premium. Consequently, his/her remaining loss decreases to $I_f(X) = X - f(X)$. Hence, under the reinsurance arrangement f, the ceding company's total liability becomes $T_f(X) = I_f(X) + P_R^f(X)$.

The aforementioned example offers insight into the ultimate goal of reinsurance, which is to determine how risk should be ceded and retained under a certain sense of optimality. This issue is discussed in detail in the seminal works of Borch (1960) and Arrow (1963) who considered optimal reinsurance design under the criteria of variance minimization and expected utility maximization, respectively. Since then the study of optimal reinsurance has attracted considerable interest in the actuarial science literature and has been extended from various perspectives in recent decades.

First, the introduction of Solvency II for the insurance and reinsurance companies operating in the European Union region has motivated the incorporation of risk-measure-based optimization criteria and optimization constraints into the reinsurance optimization problems as a means of reducing the probability of insurer insolvency. As a part of this research stream, Cai and Tan (2007) and Cai et al. (2008), for example, studied the optimal reinsurance design problems by minimizing the Value-at-Risk (VaR) and the conditional tail expectation (CTE) of the insurer's total risk exposure. Subsequently, Cheung (2010) re-examined the problems addressed by Cai et al. (2008) and generalized their results by adopting geometric arguments. In a more recent study, Lo (2017) formulated two optimal reinsurance problems under the regulatory constraint described by Tail Value-at-Risk (TVaR) and proposed a unifying approach for solving them.

Second, researchers have started investigating optimal insurance risk sharing problems involving one insurer and multiple reinsurers. For example, Chi and Meng (2014) demonstrated that it was optimal for the insurer to cede two adjacent layers to two reinsurers by minimizing VaR or conditional Value-at-Risk (CVaR) of the insurer's total risk exposure. Meng et al. (2016) showed that, under the ruin probability minimization criterion, a combination of the proportional and excess-of-loss reinsurance was the optimal form when expected value premium principle and variance premium principle were respectively adopted by the two reinsurers. On the other hand, Boonen et al. (2016) discussed a single-period optimal reinsurance design problem involving n reinsurers and one insurer, demonstrating the existence of tranching between the reinsurers when the objective of the insurer was described by a risk measure. Meng et al. (2016) studied a combined reinsurance and dividend optimization problem involving m reinsurers who applied variance premium principles to calculate reinsurance premiums. Their findings revealed that, in such a scenario, the combined proportional reinsurance agreement was optimal.

Third, several authors have considered the two-party nature of reinsurance to design a reinsurance treaty that may be acceptable to both the insurer and the reinsurer. For example, Cai et al. (2013) and Cai et al. (2016) developed objective functions reflecting the interests of both parties. More recently, Chen and Shen (2018, 2019) devised a dynamic Stackelberg stochastic differential game model to quantify the interaction between the insurer and the reinsurer. Subsequently, Bai et al. (2020) extended this research by allowing for asymmetric information and delay feature. On the other hand, Hu et al. (2018a,b) and Hu and Wang

(2019) advocated for the adoption of a principal-agent framework to formulate the strategic interaction between the insurer and the reinsurer. The economic significance of these models stems from the ability to capture the benefits derived by both the insurer and the reinsurer into a unified framework. Contributions to this body of research have also been made by D'Ortona and Marcarelli (2017), Zhang et al. (2018), Wang and Siu (2020), Gu et al. (2020), Zhang et al. (2020), amongst others.

1.1.2 Optimal Reinsurance and Investment Strategies

Once reinsurance protection has been attained, the insurance companies invest their surpluses into financial market products aiming to increase their profits as well as hedge against the insurance risk. Ample body of research has been conducted to date on the insurer's optimal investment and reinsurance strategies in the continuous-time settings. Most commonly used optimization criteria include ruin probability minimization, expected utility maximization and mean-variance criterion, whereby techniques rooted in the stochastic optimal control theory are frequently used to solve these problems. Now we give a brief introduction to these three optimization criteria. The other potential criteria that may be employed in reinsurance practice can be found in, for example, Albrecher et al. (2017). Here, we use $X^{\alpha}(t)$ to denote the surplus generated by the insurer at time t associated with the investment-reinsurance strategy α.

(i) Define $\tau_{\alpha} = \inf\{t : X^{\alpha}(t) < 0\}$, and then the probability of ruin is given by $\psi_{\alpha}(x) = \mathbb{P}(\tau_{\alpha} < \infty | X(0) = x)$. The objective is to find the minimum probability of ruin and an optimal strategy α^* such that $\psi(x) = \inf\limits_{\alpha \in \mathcal{A}} \psi_{\alpha}(x) = \psi_{\alpha^*}(x)$, where $\mathcal{A}$ is the set of admissible strategies. Under this criterion, various optimal reinsurance and/or investment problems have been proposed and further extended in the literature, see, for example, Hipp and Plum (2000), Schmidli (2001), Schmidli (2002), Promislow and Young (2005), Lo (2008), Bai and Guo (2008), Li and Young (2019), Tan et al. (2020), among others.

(ii) Under the expected utility maximization criterion, the value function is defined as:

$$V(t,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}_{t,x}[U(X^{\alpha}(T))] = \mathbb{E}_{t,x}[U(X^{\alpha^*}(T))],$$

where $U(\cdot)$ is the utility function of the insurer and T is a positive parameter for the planning horizon $[0,T]$. More generally, the terminal time T may be random, but in this thesis we assume that it is a given number. The Expected Utility Theory (EUT) was first introduced by Morgenstern and Neumann (1953), and has since given rise to several utility functions, the most common of which include constant absolute risk aversion (CARA), constant relative risk aversion (CRRA) and hyperbolic absolute risk aversion (HARA) utility. Examples of non-concave utility maximization in behavior economics, incentive schemes, aspiration utility, and goal-reaching problems can be found in the works conducted by Carpenter (2000), Chen et al. (2019), Dai et al. (2019) and Dong and Zheng (2020). It is also worth noting that Browne (1995), Yang and Zhang (2005), Wang (2007), Xu et al. (2008), Liang et al. (2017), Brachetta and Ceci (2019), Brachetta and Schmidli (2019), Sun et al. (2019) studied the optimal reinsurance and/or investment strategies based on maximizing the expected utility from the insurer's terminal surplus.

(iii) Under the dynamic mean-variance criterion, the value function is defined as:

$$\begin{aligned} V(t,x) :&= \sup_{\alpha\in\mathcal{A}} \left\{\mathbb{E}_{t,x}[X^{\alpha}(T)] - \frac{m}{2}\mathrm{Var}_{t,x}[X^{\alpha}(T)]\right\} \\ &= \mathbb{E}_{t,x}[X^{\alpha^*}(T)] - \frac{m}{2}\mathrm{Var}_{t,x}[X^{\alpha^*}(T)]. \end{aligned} \tag{1.1.1}$$

It is worth noting that under the criterion given by the expression above, the issue of time-inconsistency arises because the Bellman Optimality Principle is not met, implying that a decision deemed optimal today may not be optimal at some future point in time. This issue can be mitigated by adopting a non-cooperative game framework in order to seek a subgame perfect Nash equilibrium, as described by Björk and Murgoci (2010). In the context of insurance, this approach was used by Zeng and Li (2011), Li and Li (2013), Liang and Song (2015), Shen and Zeng (2015), Zhao and Siu (2019), Zhu et al. (2020) and Chen and Yang (2020), among other authors.

In recent decades, considerable attention has been directed toward more sophisticated models involving insurer's investment activities. First, a number of studies is based on the premise that constant and deterministic volatility of the risky asset is contrary to the empirical results, due to which their authors proposed stochastic volatility models, such as constant elasticity of variance (CEV) model (Cox and Ross (1976)) and Heston model (Heston (1993)). More recently, Li et al. (2012) derived time-consistent optimal proportional reinsurance and investment strategies under Heston model framework. Wang et al. (2018) discussed a reinsurance-investment problem when CEV model was adopted to describe the risky asset price process. Under the mean-variance principle, Wang et al. (2019) studied a reinsurance-investment optimization problem with delay under the consideration of time-varying volatility and defaultable security. Another research stream is guided by the view that treating the interest rate as a fixed constant is unreasonable to some long-term investors, and managing the risk introduced by the interest rate variations is crucial to financial institutions. This has resulted in the emergence of some popular stochastic interest rate models, such as Vasicek model (see Vasicek (1977)), Cox-Ingersoll-Ross (CIR) model (see Cox et al. (1985)), Hull-White model (see Hull and White (1990)), Affine Term Structure model (see Duffie and Kan (1996)), and Arbitrage Free Nelson-Sielgel (AFNS) model (see Christensen et al. (2011)). Some of these models have been incorporated into reinsurance and investment optimization problems under different frameworks, and the contributions by Boyle and Yang (1997), Deelstra et al. (2003), Korn and Kraft (2001), Guan and Liang (2014a), Guan and Liang (2015), Li et al. (2015a), Wang and Li (2018) and Zhang and Zheng (2019) to this research stream are particularly notable. Finally, considering that the economic conditions exhibit significant effects on the asset price dynamics, Markovian regime switching models have been developed by a number of authors to capture the changes in economic trends. The discrete-time Markov-switching autoregressive model was first considered by Hamilton (1989), and was further developed by Kim (1994). Some previous studies on the applications of regime switching models in optimal reinsurance and asset allocation problems include, for example, Elliott et al. (2010), Yiu et al. (2010), Shen and Siu (2012), Chen and Yam (2013), Jang and

Kim (2015), Bi et al. (2019), Wei et al. (2020), amongst others. In such models, the coefficients of the stochastic differential equations are assumed to depend on the states of a continuous-time Markov chain.

1.1.3 Stochastic Differential Games

Although a large body of literature has been devoted to reinsurance and investment optimization problems, in the majority of these studies, researchers have failed to consider strategic interactions and tended to adopt the perspective of a single agent. In the real world, however, financial institutions rely on their competitors' strategies as benchmarks when making investment decisions. This scenario can be captured by the game theory, as it describes strategic interactions among multiple players in a complex dynamic system. The earliest work on the game theory from the mathematical perspective can be traced back to Morgenstern and Neumann (1953). Since then, research on game theory has proliferated, resulting in a wide range of its practical applications. In the context of insurance and finance, several approaches and models have been proposed to date. For example, Elliott and Siu (2011a,b) formulated the optimal investment problem as a zero-sum stochastic differential game, with the two players representing an insurance company and an investment market which was considered as a fictitious player. Zeng (2010) and Taksar and Zeng (2011) were the first to study the proportional and non-proportional reinsurance controls under the zero-sum stochastic differential game framework, respectively. These authors constructed a payoff function based on the difference between two insurers' surplus processes, whereby one insurer's objective was to maximize the payoff function, while the other insurer aimed to minimize it. Liu and Yiu (2013) formulated a two-player zero-sum stochastic differential game, where each insurer was subject to a dynamic VaR constraint to satisfy the solvency capital requirement. Lv (2020) studied a zero-sum stochastic differential game between two players in a regime switching model with an infinite horizon.

Besides the zero-sum game framework, non-zero-sum game approach was also put forward to articulate the competition among different decision-makers. Espinosa and Touzi (2015) established a non-zero-sum stochastic differential game involving N investors by taking into account their performance relative to that of their peers. The objective function of each investor was formulated in the form of a convex combination of his/her wealth and the difference between his/her wealth and the average wealth of his/her competitors. The authors demonstrated the existence and uniqueness of the Nash equilibrium for unconstrained and constrained agents with exponential utility functions in financial markets meeting the Black-Scholes model assumptions. The ideas put forth in this article have been subsequently applied to various investment and reinsurance optimization problems. For instance, Bensoussan et al. (2014) considered a non-zero-sum stochastic differential reinsurance and investment game between two competing insurance companies whose surplus processes were modulated by a continuous-time Markov chain and a market-index process. Meng et al. (2015) similarly investigated a reinsurance game problem involving two insurers, assuming that their surplus processes were quadratic in control terms. Siu et al. (2017) studied a class of non-zero-sum investment and reinsurance games, where the insurers arranged excess-of-loss reinsurance contracts and faced systematic risk and idiosyncratic risk. Yan et al. (2017) discussed a reinsurance and investment game between two insurance companies exhibiting opposing attitudes towards future information. More recently, Chen et al. (2018) considered a game problem and applied a generalized mean-variance principle to determine the reinsurance premium. Deng et al. (2018) studied a non-zero-sum stochastic differential reinsurance and investment game between two competitive insurers presented with several investment opportunities, including a risk-free bond, a risky asset with Heston's SV model, and a defaultable corporate zero-coupon bond.

In all aforementioned models, non-zero-sum stochastic differential games were investigated under expected utility maximization criteria. Although limited work has been conducted to date on adopting dynamic mean-variance criterion to study the interaction between competing insurers, several models have been recently proposed. For example, Hu and Wang (2018) incorporated the relative performance concerns into the mean-variance criterion and derived the time-consistent reinsurance and investment strategies and value functions of the competing insurance companies. The model developed by Hu and Wang (2018) was subsequently extended by Zhu et al. (2019) by adopting Heston model to describe the price process of the risky asset. More recently, Zhu et al. (2020) studied the strategic interaction between two insurers who had mean-variance preference and faced insurance risk, volatility risk and default risk. Yang et al. (2020) constructed a new interaction mechanism involving N competitors by considering common shock dependence in financial and insurance market besides relative performance concerns under the mean-variance criterion.

1.1.4 Model Uncertainty

In recent years, considerable empirical and experimental evidence has emerged, indicating that 'model uncertainty' or 'ambiguity' are distinct from 'risk'. However, more sophisticated quantitative methods and techniques are needed for gaining a better understanding of the decision-making process under model uncertainty. Ellsberg (1961) was among the first authors to discuss model ambiguity or misspecification. Ellsberg demonstrated these concepts using a thought experiment involving two urns, whereby Urn I contained 100 red and black balls in unknown proportions, and Urn II contained 50 red and 50 black balls. A ball would be randomly drawn from one of the urns and the spectators would be allowed to bet on the outcome. In this scenario, 'Red I' ('Black I') would result in a prize if a red (black) ball was drawn from Urn I, and 'Red II' ('Black II') would result in a prize if a red (black) ball was drawn from Urn II. After numerous repetitions of this thought experiment, Ellsberg found that 'Red II' bets outnumbered 'Red I', and 'Black II' was preferred to 'Black I'. This outcome indicates that people favor choices with known probability distributions, as those involving vague probability distributions are perceived as less favorable due to inadequate information. This phenomenon, known as ambiguity aversion, has since been incorporated in numerous models, including those aimed at portfolio selection (e.g., Bergen et al. (2018), Guan and Liang (2019)), derivative analysis (e.g., Escobar et al. (2015), Escobar et al. (2018)) and reinsurance optimization problems (e.g., Wang and Siu (2020), Guan and Wang (2020)). Another motivation for considering model misspecification is that, as pointed out by Merton (1980), Maenhout (2004) and Maenhout (2006), highly accurate models are challenging to develop and some parameters, such as the expected returns, are very difficult to estimate. This has given rise to a number of methodologies aimed at capturing the decision-makers' actions in response to ambiguity, which can be broadly classified under three categories. In the first group are the models proposed by Hansen and Sargent (2001), Anderson et al. (2003) and Hansen and Sargent (2008), based on the premise that the decision-maker aims to find the optimal decisions over the worst-case scenario across the neighbourhood of the baseline model, which formulates a min-max optimization problem where the minimization is performed over the set of the alternative models and the maximization is with respect to the control policies. The second methodology type, as described by Klibanoff et al. (2005), relies on the adoption of smooth ambiguity for investigating the impact of ambiguity on the individual's decision under average case. A key feature of this formulation is a separation between ambiguity and ambiguity aversion attitudes. In the context of insurance, this strategy was applied by Chen et al. (2013), Guan et al. (2018) and Guan and Wang (2020). Finally, the approach proposed by Chen and Epstein (2002) belongs to the third category, as it generalised the stochastic differential utility introduced by Duffie and Epstein (1992) and formulated recursive multiple-priors utility in a continuous-time setting. This approach is widely used to investigate asset pricing, life insurance decisions and consumption-portfolio choice under various market assumptions, and the contributions of Epstein and Miao (2003), Liu (2011), Jensen (2019) and Ruan and Zhang (2020) to this research stream are particularly noteworthy.

1.2 Preliminaries

In this subsection, we will present a brief introduction of the basic techniques and tools to be used in the following chapters. In the literature, there exist several approaches to study stochastic optimal control problems. The first approach is dynamic programming principle and HJB equation, the second approach is martingale method which is based on equivalent martingale measure and martingale representation theorem. The third approach is based on backward stochastic differential equation (BSDE). In Chapter 2, we mainly apply the first approach to discuss the robust optimal reinsurance and investment problems between two competing insurers.

Bellman's dynamic programming principle, one of the milestones in the development of optimal control theory, can be traced back to the pioneering work in Bellman (1952), where the basic idea of dynamic programming in deterministic case was introduced. The first paper incorporating randomness and mentioned "stochastic control" was published in Bellman (1958). A few years later, Kushner (1962) adopted an Itô-type stochastic differential equation (SDE) as the state equation to study the stochastic optimal control. Since then, the continuous-time stochastic version of dynamic programming has been widely applied to solve the stochastic optimal control problems in physics, economics, engineering, management system, etc. Next, we review some conclusions in dynamic programming principle. We also refer the readers to Fleming and Soner (1993), Yong and Zhou (1999) and Pham (2008) for further discussions.

We begin with a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, where T is a fixed positive constant representing a time horizon. Let $\{W(t)\}_{t\in[0,T]}$ be a d-dimensional standard Brownian motion, whose variance-covariance matrix is denoted by Σ. We consider the following controlled diffusion system:

$$\mathrm{d}X(s) = b\left(X(s), \alpha(s)\right)\mathrm{d}s + \sigma\left(X(s), \alpha(s)\right)\mathrm{d}W(s), \tag{1.2.2}$$

where $X(s)$ denotes the state of the system at time s, and $\alpha(\cdot)$ is a progressively measurable process valued in a convex set A. The measurable functions $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ satisfy a uniform Lipschitz condition, i.e., $\forall\ x, y \in \mathbb{R}^n$, $\forall\ a \in A$, we have

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq C|x-y|,$$

for some non-negative constant C. We use $\mathcal{A}$ to denote the set of control processes that satisfy the following condition:

$$\mathbb{E}\left[\int_0^T |b(0,\alpha(t))|^2 + |\sigma(0,\alpha(t))|^2\mathrm{d}t\right] < \infty.$$

$\forall\ (t,x) \in [0,T]\times\mathbb{R}$, $\mathcal{A}(t,x)$ collects the elements in $\mathcal{A}$ such that

$$\mathbb{E}\left[\int_t^T |f\left(s, X_{t,x}(s), \alpha(s)\right)|\mathrm{d}s\right] < \infty,$$

where $X_{t,x}(s)$ is the strong solution of the stochastic differential equation (1.2.2) originating from state x at time t, and f is a function such that $f : [0,T]\times\mathbb{R}^n\times A \to \mathbb{R}$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be another function, and it is supposed that either of the following conditions is satisfied:

(i) g is lower-bounded;

(ii) For some constant K independent of x, g satisfies a quadratic growth condition:

$$|g(x)| \leq K\left(1+|x|^2\right), \quad \forall\, x \in \mathbb{R}^n.$$

For all $(t,x) \in [0,T]\times\mathbb{R}^n$ and $a \in \mathcal{A}(t,x)$, we define the gain functional as follows:

$$J(t,x;\alpha(\cdot)) := \mathbb{E}\left[\int_t^T f\left(s, X_{t,x}(s), \alpha(s)\right) \mathrm{d}s + g(X_{t,x}(T))\right]. \tag{1.2.3}$$

Usually, we state the standard stochastic optimal control problem as follows:
Problem P1. For a given $(t,x) \in [0,T]\times\mathbb{R}^n$, we aim to find the control process in $\mathcal{A}(t,x)$ that maximizes (1.2.3) subject to (1.2.2).
In order to solve **Problem P1**, we first define the associated value function

$$V(t,x) := \sup_{\alpha\in\mathcal{A}(t,x)} J(t,x;\alpha(\cdot)), \tag{1.2.4}$$

and we say that $\alpha^* \in \mathcal{A}(t,x)$ is an optimal control if $V(t,x) = J(t,x;\alpha^*(\cdot))$. On the basis of the previous assumptions, we present the following standard theorem called the dynamic programming principle.

Theorem 1.2.1. *(Dynamic programming principle) For any* $(t,x) \in [0,T]\times\mathbb{R}^n$, *the value function in* (1.2.4) *satisfies:*

$$V(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f\left(s, X_{t,x}(s), \alpha(s)\right) \mathrm{d}s + V(\theta, X_{t,x}(\theta))\right], \quad \forall\, 0 \leq t \leq \theta \leq T. \tag{1.2.5}$$

We may find that the results in Theorem 1.2.1 not that convenient to apply. The following theorem provides an easier way to implement when the value function satisfies some conditions and it states that solving the stochastic optimal control problem given in **Problem P1** can be converted to solving a certain partial differential equation (PDE). To that end, We first define $C^{1,2}([0,T]\times\mathbb{R}^n) :=$

$\{f : [0,T] \times \mathbb{R}^n \to \mathbb{R} | f(t, \cdot)$ is once continuously differentiable on $[0,T]$ and $f(\cdot, x)$ is twice continuously differentiable on $\mathbb{R}^n\}$.

Theorem 1.2.2. *Assume $V(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^n)$, then $V(t,x)$ is a solution to the following terminal value problem of a first-order PDE:*

$$\begin{cases} -\frac{\partial V}{\partial t}(t,x) - H(t,x,D_xV(t,x),D_x^2V(t,x)) = 0, & \forall\ (t,x) \in [0,T) \times \mathbb{R}^n, \\ V(T,x) = g(x), & \forall\ x \in \mathbb{R}^n, \end{cases} \tag{1.2.6}$$

where $H(t,x,p,M)$ is defined as: for all $(t,x,p,M) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times S_n$,

$$H(t,x,p,M) := \sup_{\alpha \in A} \left[b(x,\alpha)p + \frac{1}{2} tr\left(\sigma(x,\alpha)\sigma'(x,\alpha)M\right) + f(t,x,\alpha) \right],$$

with $tr(\cdot)$ denoting the trace of a matrix, S_n the set of symmetric $n \times n$ matrices, D_x the gradient vector and D_x^2 the Hessian matrix of a function.

We call the PDE in (1.2.6) the Hamilton-Jacobi-Bellman (HJB) equation, and $V(t,x)$ solving (1.2.6) is called a classical solution to the HJB equation.

In Chapters 3 and 4, we select mean-variance criteria to study the optimal reinsurance and investment strategies, and we will apply non-cooperative game theoretic approach to deal with the time inconsistency issue. This method can be found in the literature, see, for example, Björk and Murgoci (2010), Björk et al. (2014) and Kronborg and Steffensen (2015). The basic idea of this approach is to view the decision-maker at different time points as different players, and they will choose the optimal strategies from their own standpoint. Following the notation in (1.1.1), we further define that

$$J^\alpha(t,x) := \mathbb{E}_{t,x}[X^\alpha(T)] - \frac{m}{2}\mathrm{Var}_{t,x}[X^\alpha(T)].$$

Accordingly, we provide the following definition of equilibrium strategy and value function with respect to the optimization problem in (1.1.1).

Definition 1.2.1. *For an admissible reinsurance-investment strategy $\alpha^*(t)$ of a decision-maker, with any fixed chosen initial state $(x,t) \in \mathbb{R} \times [0,T]$, we define the following perturbed strategy*

$$\alpha^\epsilon(s) := \begin{cases} \tilde{\alpha}, & t \leq s < t + \varepsilon, \\ \alpha^*(s), & t + \varepsilon \leq s \leq T, \end{cases}$$

where $\varepsilon \in \mathbb{R}^+$. *If* $\forall\ \tilde{\alpha} \in \mathbb{R}^+$, *we have*

$$\liminf_{\varepsilon \to 0} \frac{J^{\alpha^*}(t,x) - J^{\alpha^\varepsilon}(t,x)}{\varepsilon} \geq 0,$$

then $\alpha^*(t)$ *is called an equilibrium strategy and the equilibrium value function of the decision-maker is given by*

$$V(t,x) = J^{\alpha^*}(t,x).$$

In this thesis, considering that it is difficult to estimate the drift coefficients reliably, we assume that the decision-makers face ambiguity about the jump risk and diffusion risk from the insurance business and financial market. Different from the traditional models, the ambiguity-averse decision-makers are skeptical about the reference models due to misspecification errors and wish to consider some alternative models. This will be achieved by formulating robust optimization problems which involve change of probability measure. In what follows, we will introduce the techniques of how to change the original probability measure to an equivalent one. The main instrument in changing measure for stochastic process is Girsanov's theorem. The material presented here is standard in stochastic analysis.

Definition 1.2.2. *We say two probability measures* $\mathbb{P}$ *and* $\widetilde{\mathbb{P}}$ *are equivalent, denoted as* $\mathbb{P} \sim \widetilde{\mathbb{P}}$, *if they have the same null sets, i.e.,* $\mathbb{P}(A) = 0$ *if and only if* $\widetilde{\mathbb{P}}(A) = 0$.

Definition 1.2.3. $\widetilde{\mathbb{P}}$ *is called absolutely continuous with respect to* $\mathbb{P}$, *denoted as* $\widetilde{\mathbb{P}} \ll \mathbb{P}$, *if* $\widetilde{\mathbb{P}}(A) = 0$ *whenever* $\mathbb{P}(A) = 0$. $\widetilde{\mathbb{P}}$ *and* $\mathbb{P}$ *are called equivalent if* $\widetilde{\mathbb{P}} \ll \mathbb{P}$ *and* $\mathbb{P} \ll \widetilde{\mathbb{P}}$.

Theorem 1.2.3. *(Radon-Nikodym derivative) If* $\widetilde{\mathbb{P}} \ll \mathbb{P}$, *then there exists a random variable* $\Lambda \geq 0$, *such that the expectation of* Λ *under probability measure* $\mathbb{P}$, *written as* $\mathbb{E}_{\mathbb{P}}(\Lambda)$, *equals one, and the following formula holds for any measurable set* A :

$$\widetilde{\mathbb{P}}(A) = \int_A \Lambda \mathrm{d}\mathbb{P}. \tag{1.2.7}$$

Conversely, if there exists a random variable Λ *with the above properties and* $\widetilde{\mathbb{P}}$ *is defined by* (1.2.7), *then* $\widetilde{\mathbb{P}}$ *is a probability measure such that* $\widetilde{\mathbb{P}} \ll \mathbb{P}$.

The random variable Λ given in Theorem 1.2.3 is called the Radon-Nikodym derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and usually we denote it as $\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}$. The proof of Theorem 1.2.3 can be found in Klebaner (2005).

Lemma 1.2.1. *We define an exponential process:*

$$Z(t) := \exp\left[\int_0^t \beta'(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^t \beta'(u)\Sigma\beta(u)\mathrm{d}u\right], \qquad (1.2.8)$$

and then we know that it is a martingale under $(\mathcal{F}_t, \mathbb{P})$.

Remark 1.2.1. *A sufficient condition for* $Z(t)$ *to be a martingale is given by:*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \beta'(t)\Sigma\beta(t)\mathrm{d}t\right)\right] < \infty,$$

where $\mathbb{E}_{\mathbb{P}}[\,\cdot\,]$ *denotes the expectation under probability measure* $\mathbb{P}$. *This condition is known as Novikov's condition.*

Theorem 1.2.4. *(Girsanov's theorem) Define a new probability measure on* $\mathcal{F}_T$ *by putting*

$$\widetilde{\mathbb{P}}(A) := \int_A Z(T,w)\mathrm{d}\mathbb{P}, \quad \text{for all } A \in \mathcal{F}_T,$$

then $\widetilde{\mathbb{P}}$ *is a probability measure on* $(\Omega, \mathcal{F}_T)$. *Furthermore,*

$$\widetilde{W}(t) = W(t) - \int_0^t \Sigma\beta(u)\mathrm{d}u, \quad 0 \le t \le T,$$

is a d*-dimensional Brownian motion in the probability space* $(\Omega, \mathcal{F}_T, \widetilde{\mathbb{P}})$ *with respect to the filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$.

From Girsanov's theorem we know that the new measure $\widetilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, and the form of the Radon-Nikodym derivative of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is given by (1.2.8). The proofs of Lemma 1.2.1 and Theorem 1.2.4 can be found in, for example, Shreve (2004) and Elliott and Kopp (2005).

1.3 Structure of the Thesis

This thesis consists of five chapters. The current chapter provides an overview of the stochastic optimal control problems in finance and actuarial science, while the three subsequent self-contained chapters (Chapter 2-4) are designated for three related research problems. The thesis concludes with Chapter 5, where some concluding remarks are made and suggestions for future studies in this field are provided. The content of Chapter 2, 3, and 4 is briefly introduced below.

The work reported in Chapter 2 pertains to a non-zero-sum stochastic differential game between two competitive CARA insurers, both of whom are faced with the potential model uncertainty and seek to identify robust optimal reinsurance and investment strategies. These ambiguity-averse insurers (AAIs) are allowed to purchase reinsurance treaty to mitigate individual claim risks, and can invest in a financial portfolio consisting of one risk-free asset, one risky asset and one defaultable corporate bond. The objective of each insurer is to maximize the expected exponential utility of his/her terminal surplus relative to that of his/her competitor under the worst-case scenario of the alternative measures. Applying the stochastic dynamic programming techniques, in this chapter, robust Nash equilibrium reinsurance and investment policies are explicitly derived and the corresponding verification theorem is presented. Finally, some numerical examples are provided to illustrate the impact of model parameters on the equilibrium reinsurance and investment strategies and we draw some economic interpretations from these results.

In Chapter 3, a family of robust non-zero-sum reinsurance-investment stochastic differential games between two competing insurers are investigated under the time-consistent mean-variance criterion. Each insurer is allowed to purchase a proportional reinsurance treaty and invest his/her surplus into a financial portfolio consisting of one risk-free asset and one risky asset to manage the insurance risk. The surplus processes of both insurers are assumed to be governed by the classical Cramér-Lundberg model and each insurer is concerned about model uncertainty, and is thus designated as AAI. The objective of each insurer is to maximize the expected terminal surplus relative to that of his/her competitor and minimize the variance of this relative terminal surplus under the worst-case scenario of alternative measures. Applying the techniques grounded in the stochastic control theory, extended Hamilton-Jacobi-Bellman (HJB) equations are developed for both insurers. Moreover, robust equilibrium reinsurance-investment strategies are established, along with the corresponding equilibrium value functions of both insurers, by solving the extended HJB equations under both the compound Poisson risk model and its diffusion-approximated model. Finally, several numerical examples are presented to illustrate the influence of different model parameters on the Nash equilibrium strategies.

In Chapter 4, a class of reinsurance contract problems is studied under a continuous-time principal-agent framework with mean-variance criteria, where a reinsurer and an insurer are assigned the roles of the principal and the agent, respectively. Both parties can manage risk by investing in a financial portfolio comprising of a risk-free asset and a risky asset. It is supposed that both the insurer and the reinsurer are concerned about model uncertainty and that they aim to find a robust reinsurance contract and robust investment strategies by maximizing their respective mean-variance cost functionals taking account of sets of probability scenarios. To articulate the time-inconsistency issue attributed to the mean-variance optimization criteria, the optimization procedure of each decision-maker is formulated as a non-cooperative game and is discussed using an extended HJB equation, in line with the extant work on the time-consistent control. Moreover, explicit expressions for the robust reinsurance contract, the robust investment strategies and the value functions of the insurer and the reinsurer are obtained. Then the numerical results and their economic interpretations are discussed.

1.4 Comparison among the Incorporated Papers

The aim of the present study was to explore the reinsurance and investment optimization problems in the presence of strategic interactions and model uncertainty. Advanced knowledge of game theory, stochastic control, and actuarial studies was thus applied to quantify and capture the interactions among different decision-makers who considered model misspecification. The work reported in the three papers comprising this thesis shares some similarities, but each article emphasizes different aspects of the aforementioned problem.

First, the same approach is employed in all cases to define the alternative models and the corresponding penalty functions, assuming that the decision-makers are concerned about specification errors and regard the reference model as an approximation to the true model if it exists. Hence, they seek to attain control rules that are optimal across a set of alternative models. Based on these considerations, in the following three chapters, we will apply the method originally proposed by Anderson et al. (2003) and further developed by Maenhout (2004), which is basically a robust approach based on relative entropy, to describe the decision making under the ambiguity aversion attitudes. Compared with the traditional optimization problems, in this approach, there is an additional penalty term in the objective function allowing the deviation of the alternative model from the reference model to be measured.

Second, two distinct mechanisms are applied to describe the interaction between multiple decision-makers. In the work reported in Chapter 2 and Chapter 3, relative performance concerns were used to model the competition between two insurance companies operating in the same market. In order to examine the implications of considering relative performance, non-zero-sum stochastic differential games are formulated in these two chapters. From the practical perspective, this formulation is interesting, as most financial institutions develop their decision rules with the aim of outperforming their competitors. In Chapter 4, the principal-agent framework is utilised to model the relationship between the insurer (agent) and the reinsurer (principal). This paradigm provides a possibility to design a mutually beneficial reinsurance contract by taking into account the interests of both parties.

Third, different criteria are adopted to formulate the optimization problems discussed in Chapter 2-4. As previously noted when reviewing extant research in this field, three optimization criteria are commonly used to examine the optimal reinsurance and investment strategies, one of which is the expected utility maximization criterion, which is employed in the work reported in Chapter 2. Under this paradigm, both parties would adopt the Nash equilibrium strategies, thereby maximizing the expected utility of their relative terminal surplus in the worst-case scenario of the alternative measures. On the other hand, in the approaches described in Chapter 3 and Chapter 4, it is assumed that the decision-makers' objective is to maximize their respective mean-variance cost functionals in the worst-case models. As dynamic mean-variance problems are affected by the time-inconsistency issue, to overcome this shortcoming, a non-cooperative game played by the future incarnations of the decision-maker is formulated, allowing time-consistent optimal control policies to be obtained. In other words, a non-cooperative game is incorporated into a non-zero-sum stochastic differential game and a principal-agent framework in Chapter 3 and Chapter 4, respectively. Although this approach makes the model more complicated, it also produces richer economic insights.

Finally, although the HJB dynamic programming principle rooted in the stochastic control theory is adopted in this work to derive the closed-form solutions of the optimal strategies for each research problem, the procedures utilized for solving the optimization problems are not the same. In Chapter 2 and Chapter 3, the HJB equation is derived for each insurer's optimization problem, under the assumption that the optimal strategy of his/her competitor is known. Consequently, each player's optimal strategy can be expressed in terms of that of his/her competitor. As a result of this co-dependence, the two players' optimization problems need to be solved simultaneously and the Nash equilibrium is attained by solving a system of equations. In contrast, due to the hierarchical relationship between the insurer and the reinsurer, in Chapter 4, the backward induction method is adopted to solve the reinsurance contracting problem. Consequently, the insurer's optimization problem is solved for a given reinsurance price before proceeding with the reinsurer's optimization problem, assuming that the insurer's optimal reinsurance strategy is already known at this stage. This approach necessitates solutions to two systems of extended HJB equations to identify the equilibrium strategies and the respective value functions.

Chapter 2

Robust Non-zero-sum Investment and Reinsurance Game with Default Risk

2.1 Introduction

Insurance companies invest their surpluses into financial markets to make profits and transfer parts of their risk arising from claim experiences by purchasing reinsurance contracts. Recently, a wide variety of optimal investment-reinsurance problems has been investigated from the perspectives of insurance companies by applying stochastic optimal control theory. Not only can the stochastic optimal control theory provide sound and feasible solutions to optimal investment-reinsurance problems, but also reflect the change of an insurer's strategies over time. Commonly studied objectives of the optimization problems in dynamic settings include maximizing the expected utility of terminal wealth (see, for example, Zhao et al. (2013), Xu et al. (2017)), minimizing the probability of ruin (see, for example, Bai and Guo (2008), Zhang et al. (2016)), and the mean-variance criterion (see, for example, Bi and Guo (2013), Zhang et al. (2017)).

Although the optimal investment-reinsurance problems have been extensively studied in the literature, it appears that the default risk or credit risk has been largely ignored in the modelling framework. The credit risk is regarded as one of the fundamental factors of financial risks, but defaultable bonds are still attractive to many institutional investors due to high rate of return. Many insurance companies actively invest in credit securities. According to the National Association of Insurance Commissioners' Capital Markets Bureau at year-end 2019, corporate bonds have remained the largest bond type for the U.S. insurance industry, amounting to \$ 2.46 trillion and representing 55.2% of total bond exposure and 37.2% of total cash and invested assets[1]. Thus, it may be relevant to investigate the optimal portfolio selection problems with the consideration of defaultable securities and develop appropriate investment strategies for insurance companies. In recent studies, Bielecki and Jang (2006) considered a portfolio optimization problem including a credit-risky asset under the criterion of maximizing the expected constant relative risk aversion (CRRA) utility of the investor's terminal wealth. Barucci and Cosso (2015) studied an optimal portfolio allocation problem subject to a dynamic VaR constraint when a defaultable asset was considered. Shen and Siu (2018) discussed a risk-based approach for an asset allocation optimization problem with default risk. While the efforts to incorporate default risk in the insurer's reinsurance and/or investment optimization problems are still sparse, some models have been proposed in the recent literature. Zhao et al. (2016) introduced a defaultable security into their Markowitz's mean-variance investment-reinsurance optimization problem in a jump-diffusion risk model. Li et al. (2017) also incorporated default risk when deriving equilibrium investment strategies under the mean-variance criterion for a defined contribution (DC) plan under the constant elasticity of variance (CEV) model. Zhu et al. (2015) derived the optimal proportional reinsurance and investment strategies in a defaultable financial market by maximizing the expected constant absolute risk aversion (CARA) utility of an insurer's terminal wealth.

Another notable issue that needs further exploration in the existing literature is ambiguity or model uncertainty. In the traditional settings of optimal reinsurance and/or investment problems, it is assumed that the decision-makers are given knowledge about a real-world probability measure. However, it may be questioned that a real-world probability is given or known when solving optimization problems in practice. Another reason for us to consider model uncertainty is that the parameters, especially the drift parameters, are difficult to estimate with

[1]Please see: https://www.naic.org/capital_markets_archive/special_report_200701.pdf

precision. Thus, it is reasonable to assume that the decision-maker is concerned about model misspecification. There are literature considering model ambiguity. For instance, Maenhout (2004) put forward a new approach to obtain the optimal portfolio decision for an investor with model ambiguity consideration to overcome the difficulty in obtaining accurate estimates of asset return parameters. In the contexts of insurance, Zhou et al. (2016) investigated the optimal reinsurance-investment problem for a general insurance company with ambiguity aversion attitudes under the criterion of maximizing the minimal expected exponential utility of its weighted terminal wealth. Yi et al. (2013) studied the robust optimal reinsurance-investment problem for an ambiguity-averse insurer (AAI), where the Heston's stochastic volatility (SV) model was used to describe the price process of the risky asset. Pun and Wong (2015) discussed the robust optimal reinsurance-investment problem for a general class of utility functions when the risky asset followed a multiscale SV model. Under the variance premium principle, Sun et al. (2017) derived the robust optimal reinsurance and investment strategies by incorporating ambiguity aversion and default risk. Li et al. (2019) studied the robust investment problem for α-maxmin expected utility and solved for the equilibrium strategies of an open-loop type. Further investigation regarding robust optimization theories in insurance applications can be found in Gabrel et al. (2014), Zeng et al. (2016), Li et al. (2018), Hu et al. (2018a,b) and the reference therein.

The aforementioned literature focuses on single-agent optimization problems. However, the real-world economy appears to be a complex and interactive system, where the financial institutions tend to make optimal decisions by taking account of the performance of their competitors. Hence, some scholars start to investigate the competition between two institutions. For example, Browne (2000) proposed a zero-sum stochastic differential portfolio game between two investors. Zeng (2010) developed a zero-sum differential game between two insurance companies and derived the Nash equilibrium for the dynamic proportional reinsurance. Bensoussan et al. (2014) formulated a non-zero-sum stochastic differential game between two insurers by applying the concept of relative performance and obtained explicit solutions for optimal reinsurance and investment strategies under a special case. Deng et al. (2018) studied a non-zero-sum stochastic differential reinsurance-investment game between two competitive CARA insurers, and their investment options included a risk-free bond, a risky asset with Heston's SV model and a defaultable corporate zero-coupon bond. Some attempts have also been made in addressing the robust game. For instance, Zhang and Siu (2009) considered an optimal reinsurance-investment problem in the presence of model uncertainty and formulated the problem into a zero-sum stochastic differential game between the insurer and the market. Elliott and Siu (2011b) extended the model in Zhang and Siu (2009) by assuming that the insurer was exposed to regime-switching risk. Pun and Wong (2016) explored the non-zero-sum stochastic differential reinsurance game between two competitive AAIs who aimed to seek for the robust optimal proportional reinsurance strategies by maximizing the expected utility of the relative surplus at terminal time. However, there appears to be less literature studying robust non-zero-sum reinsurance and investment games between two insurers. We aim to fill in this gap and conduct some investigation on how the AAIs would make decisions regarding reinsurance demands and default risk exposure from a game theoretic perspective.

In this present chapter, we will study a class of non-zero-sum stochastic differential games between two ambiguity-averse insurers who are faced with default risk. To be specific, the surplus process of each insurer is assumed to follow a Brownian motion with drift; the financial market consists of one risk-free asset, one risky asset and one defaultable corporate bond. To take ambiguity aversion into consideration, we assume that both insurers can be ambiguity-averse and seek for optimal strategies among a family of alternative probability measures. Additionally, we incorporate the relative performance concerns into the objective functions of insurers. Under these assumptions, we formulate a robust non-zero-sum game between two insurers who have the option to invest in a more general financial market.

Compared with some existing literature, the work in this chapter has three main points of innovations. First, the impact of model uncertainty on optimal reinsurance and investment strategies is investigated, which was not considered in Deng et al. (2018), where the authors studied the strategic interaction between two CARA insurers who could invest into defaultable bond. However, it is important for a decision-maker to consider parameter ambiguity and stochastic uncertainty because precise estimations of the surplus process and return levels of investment securities are difficult to obtain. Our numerical illustrations show that an AAI would prefer more conservative investment and reinsurance strategies than an ambiguity-neutral insurer (ANI). Second, we extend the robust reinsurance-investment model in Sun et al. (2017), where only a single insurer was considered, to a continuous-time theoretic game framework by taking multiple insurers' relative performance concerns into account. The key reason for formulating non-zero-sum stochastic differential game is that there are always several competing insurers in the market in reality, and they often assess their performance against a relative benchmark of their competitors. Therefore, we derive the Nash equilibrium investment and reinsurance strategies of a non-zero-sum game in this chapter. Numerical examples demonstrate that the relative performance concern makes each insurer more risk-seeking, which is reflected in the increased exposure on risky asset and defaultable bond and the elevated retention level of claims. We also find that the insurer's optimal strategy is affected by his competitor's ambiguity aversion level. More precisely, the competitor's ambiguity-averse attitude makes the insurer more conservative by diminishing the amount invested in the stock market or defaultable bond. When the two players' insurance portfolios are positively correlated, they would also cede larger proportions of claims to the reinsurer. Third, we investigate the default risk in our robust non-zero-sum game. In reality, insurance companies tend to actively participate in various financial activities to make profits from their surpluses. Accordingly, we extend the work in Pun and Wong (2015), where a robust non-zero-sum stochastic differential reinsurance game was studied in a simplified financial market without default risk. To consider the exposed default risk in the financial market, we include an asset of defaultable bond into the classical financial market which consists of risk-free assets and risky assets only. The defaultable securities such as corporate bonds have been increasingly attractive to investors due to their high yields. It seems that this may be related to systemic risk. Hence, our improved model may be relevant to insurers involved in the defaultable market. By adding the additional defaultable bonds into our asset portfolio, the model becomes more versatile and more difficult to solve explicitly.

The remainder of this chapter proceeds as follows. Section 2.2 presents the formulation of the model. After describing the two competing insurers' surplus processes and the dynamics of the financial market, we formulate a robust non-zero-sum stochastic differential game between two CARA insurers. In Section 2.3, we derive the HJBI equations for the pre-default case and the post-default case, respectively. Closed-form expressions for robust equilibrium strategies and corresponding optimal value functions are obtained, and the verification theorem is proved as well. Section 2.4 provides some special cases of our model. Detailed numerical simulations are conducted in Section 2.5 to demonstrate the results. Finally, Section 2.6 concludes this chapter.

2.2 The model formulation

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space indexed by a finite time horizon $[0, T]$, where $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is a right-continuous, $\mathbb{P}$-complete filtration. In the absence of reinsurance and investment, the insurer's surplus process is described by the following classical Cramér-Lundberg risk model:

$$U(t) = u_0 + ct - \sum_{i=1}^{N(t)} Z_i,$$

where $u_0 \geq 0$ is the initial surplus, c is the constant insurance premium rate, the Poisson process $\{N(t)\}_{t\in[0,T]}$ is the claim arrival process with intensity $\tilde{\lambda} > 0$, and the claim sizes $Z_i, i = 1, 2, \ldots$, are independent and identically distributed (i.i.d.) random variables independent of $N(t)$. Suppose that the claim size has finite first and second moments denoted by $\tilde{\mu}$ and $\tilde{\sigma}$, respectively. The insurance premium rate c is assumed to be determined by the expected value premium principle, i.e., $c = (1 + \eta)\tilde{\lambda}\tilde{\mu}$, where $\eta > 0$ is the relative safety loading factor of the insurer.

We further assume that the insurer can purchase proportional reinsurance protection to manage insurance business risks. Denote by $q(t) : [0, T] \to [0, \infty)$ the reinsurance strategy of the insurer at time t. When $q(t) \in [0, 1]$, it means that the reinsurance company will compensate the insurer for $100(1-q(t))\%$ of the claims at time t, in result the net liability for the insurance company will be $100q(t)\%$ of the original claims. When $q(t) \in (1, \infty)$, the insurer performs as a reinsurer for other insurance companies, and we regard this as the acquisition of new business through taking additional insurance risks. Suppose that the reinsurance premium is also determined by the expected value premium principle. Under the proportional reinsurance contract $q(t)$, the reinsurance premium rate is $p(t) = (1+\gamma)(1-q(t))a$ with the reinsurer's safety loading γ satisfying $\gamma \geq \eta$. Therefore, the insurer's surplus process with such a proportional reinsurance treaty becomes:

$$\begin{aligned} R_0(t) &= u_0 + \int_0^t (c - p(s))\mathrm{d}s - \sum_{i=1}^{N(t)} q(T_i)Z_i \\ &= u_0 + \int_0^t a\left[(1+\eta) - (1+\gamma)(1-q(s))\right]\mathrm{d}s - \sum_{i=1}^{N(t)} q(T_i)Z_i, \end{aligned} \tag{2.2.1}$$

where $a := \tilde{\lambda}\tilde{\mu}$, $b := \sqrt{\tilde{\lambda}}\tilde{\sigma}$ and T_i denotes the occurring time of the i-th claim. According to Grandell (1991), the dynamics of $R_0(t)$ in (2.2.1) can be approximated by the following diffusion process:

$$\mathrm{d}R_1(t) = a[\lambda + \gamma q(t)]\mathrm{d}t + bq(t)\mathrm{d}B_0(t),$$

where $\lambda := \eta - \gamma \leq 0$ and $\{B_0(t)\}_{t\in[0,T]}$ is a standard $\mathbb{P}$-Brownian motion.

We consider a financial market consisting of one risk-free asset, one risky asset and one defaultable corporate zero-coupon bond. The price process $S_0(t)$ of the risk-free asset is given by the following ordinary differential equation (ODE):

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t,$$

where $r > 0$ is the risk-free interest rate, and $S_0(0) = s_0 > 0$. The price process $S_1(t)$ of the risky asset follows a geometric Brownian motion:

$$\mathrm{d}S_1(t) = S_1(t)[\mu\mathrm{d}t + \sigma\mathrm{d}B_1(t)],$$

where $\{B_1(t)\}_{t\in[0,T]}$ is a standard Brownian motion under $\mathbb{P}$, $\mu > r$ is the appreciation rate, $\sigma > 0$ denotes the volatility, and $S_1(0) = s_1 > 0$.

Let $T_1 > T$ denote the maturity date of the corporate zero-coupon bond and a nonnegative random variable τ represent the default time of the firm issuing this bond. Define a default indicator process by

$$Z(t) := \mathrm{I}_{\{\tau \leq t\}},$$

where I denotes the indicator function that equals one if $\{\tau \leq t\}$ occurs and zero otherwise. Then, $Z(t) = 0$ and $Z(t) = 1$ correspond to the pre-default case $\tau > t$ and the post-default case $\tau \leq t$, respectively. This implies that the default process makes discrete jumps at the random time τ. We assume that τ is the first arrival of a Poisson process with constant intensity $h^P > 0$ under the probability measure $\mathbb{P}$, and h^P measures the arrival rate of the default. Let $\mathcal{G}_t = \mathscr{F}_t \vee \sigma\{Z(s) : 0 \leq s \leq t\}$ such that $\mathbb{G} := \{\mathcal{G}_t\}_{t\in[0,T]}$ is the smallest filtration, under which τ is a stopping time. Following Bielecki and Jang (2006), we first define a process

$$M^P(t) := Z(t) - \int_0^t (1 - Z(u-))h^P \mathrm{d}u,$$

which is a $\mathbb{G}$-martingale under $\mathbb{P}$. Assume that the investor would recover a fraction of the market value of the defaultable bond prior to default and the value of the defaultable bond after default is zero. Then we use $0 \leq \zeta \leq 1$ to denote the constant loss rate when a default occurs, and $1-\zeta$ is the default recovery rate. By Lemma 2 in Bielecki and Jang (2006), the arrival intensity of the default under a risk-neutral measure $\mathbb{Q}$ is given by $h^Q = h^P/\Delta$, where $1/\Delta$ denotes the default risk premium. As discussed in Duffie and Singleton (2003), the probability of default occurring under a risk-neutral measure $\mathbb{Q}$ is higher than that under the real-world probability $\mathbb{P}$, and hence we have $1/\Delta = h^Q/h^P \geq 1$. According to Bielecki and Jang (2006), the dynamics of the defaultable bond under the measure $\mathbb{P}$ is given by

$$\mathrm{d}p(t, T_1) = p(t-, T_1)\left[r\mathrm{d}t + \delta(1 - Z(t-))(1 - \Delta)\mathrm{d}t - \zeta(1 - Z(t-))\mathrm{d}M^P(t)\right],$$

where $\delta = h^Q\zeta$ denotes the credit spread under the risk-neutral measure $\mathbb{Q}$.

Besides purchasing a proportional reinsurance contract to transfer risk, we assume that the insurer is allowed to invest his/her surplus in the financial market as described above. Then the trading strategy is represented by a three-dimensional stochastic process $\pi(t) := \{(\pi_1(t), \pi_2(t), q(t))\}_{t\in[0,T]}$, where $\pi_1(t)$ and $\pi_2(t)$ denote the dollar amounts invested in the stock and defaultable bond at time t, respectively, $q(t)$ represents the retained proportion of the claims. The remainder of the surplus, $X^\pi(t)-\pi_1(t)-\pi_2(t)$, where $X^\pi(t)$ denotes the surplus process controlled by the strategy $\{\pi(t)\}_{t\in[0,T]}$, is invested in the risk-free asset. Therefore, using the diffusion-approximated surplus process, the insurer's surplus dynamics follow:

$$\begin{aligned}
\mathrm{d}X^\pi(t) =& \frac{X^\pi(t)-\pi_1(t)-\pi_2(t)}{S_0(t)}\mathrm{d}S_0(t)+\frac{\pi_1(t)}{S_1(t)}\mathrm{d}S_1(t)+\frac{\pi_2(t)}{p(t-,T_1)}\mathrm{d}p(t,T_1)+\mathrm{d}R_1(t) \\
=& \left[rX^\pi(t)+(\mu-r)\pi_1(t)+(1-Z(t-))(1-\Delta)\delta\pi_2(t)+(\lambda+\gamma q(t))a\right]\mathrm{d}t \\
&+\sigma\pi_1(t)\mathrm{d}B_1(t)+bq(t)\mathrm{d}B_0(t)-(1-Z(t-))\zeta\pi_2(t)\mathrm{d}M^P(t) \\
=& \left[rX^\pi(t)+(\mu-r)\pi_1(t)+(1-Z(t-))\delta\pi_2(t)+(\lambda+\gamma q(t))a\right]\mathrm{d}t \\
&+\sigma\pi_1(t)\mathrm{d}B_1(t)+bq(t)\mathrm{d}B_0(t)-(1-Z(t-))\zeta\pi_2(t)\mathrm{d}Z(t),
\end{aligned} \tag{2.2.2}$$

where $X^\pi(0)=x_0$ is the initial surplus.

Next, we consider two competing insurers, and insurer i, for $i\in\{1,2\}$, chooses the reinsurance-investment strategy $\pi_i(t):=(\pi_{i,1}(t),\pi_{i,2}(t),q_i(t))$ at time t. Thus, the dynamics of the surplus process $\{X_i^{\pi_i}(t)\}_{t\in[0,T]}$ associated with strategy $\{\pi_i(t)\}_{t\in[0,T]}$ for insurer i is described by

$$\begin{aligned}
\mathrm{d}X_i^{\pi_i}(t) =& \left[rX_i^{\pi_i}(t)+(\mu-r)\pi_{i,1}(t)+(1-Z(t-))\delta\pi_{i,2}(t)+a_i(\lambda_i+\gamma_i q_i(t))\right]\mathrm{d}t \\
&+\sigma\pi_{i,1}(t)\mathrm{d}B_1(t)+b_iq_i(t)\mathrm{d}B_{i,0}(t)-(1-Z(t-))\zeta\pi_{i,2}(t)\mathrm{d}Z(t),
\end{aligned} \tag{2.2.3}$$

where $a_i:=\tilde{\lambda}_i\tilde{\mu},b_i:=\sqrt{\tilde{\lambda}_i}\tilde{\sigma}_i$, $\{B_1(t)\}_{t\in[0,T]}$, $\{B_{1,0}(t)\}_{t\in[0,T]}$, $\{B_{2,0}(t)\}_{t\in[0,T]}$ are standard Brownian motions under probability measure $\mathbb{P}$ such that $B_1(t)$ is independent of $B_{1,0}(t)$ and $B_{2,0}(t)$, and $\mathbb{E}_\mathbb{P}\left[\mathrm{d}B_{1,0}(t)\mathrm{d}B_{2,0}(t)\right]=\rho\mathrm{d}t$.

In this chapter, we adopt the diffusion model to approximate the classical claims model that is described by a jump process. So the insurer actually has ambiguity aversion attitudes towards the approximated claim process under the probability measure $\mathbb{P}$. The insurer only regards $\mathbb{P}$ as a reference measure and would like to consider a set of alternative probability measures. We define a class of probability measures which are equivalent to $\mathbb{P}$ as the alternative measures:

$$\mathcal{Q}:=\{\mathbb{Q}|\mathbb{Q}\sim\mathbb{P}\}.$$

By Girsanov's Theorem, we know that, $\forall\ \mathbb{Q}_i\in\mathcal{Q}$, there exists a process $\phi_i(t):=(\phi_{i,1}(t),\phi_{i,2}(t),\phi_{i,3}(t))$ such that

$$\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}}=\Lambda^{\phi_i}(t),$$

where

$$\Lambda^{\phi_i}(t) = \exp\Bigg\{ -\int_0^t \phi_{i,1}(s)\mathrm{d}B_1(s) - \frac{1}{2}\int_0^t (\phi_{i,1}(s))^2\mathrm{d}s - \int_0^t \phi_{i,2}(s)\mathrm{d}B_{i,0}(s)$$
$$- \frac{1}{2}\int_0^t (\phi_{i,2}(s))^2\mathrm{d}s + \int_0^t \ln\phi_{i,3}(s)\mathrm{d}Z(s)$$
$$+ h^P\int_0^t (1-\phi_{i,3}(s))(1-Z(s))\mathrm{d}s \Bigg\} \tag{2.2.4}$$

is a $\mathbb{P}$-martingale. As with Miao and Rivera (2016), we call $\phi_i(t)$ the density generator of insurer i.

Remark 2.2.1. Process $\phi_i(t) := (\phi_{i,1}(t), \phi_{i,2}(t), \phi_{i,3}(t))$ should satisfy the following conditions:

(i) $\phi_i(t)$ is $\mathcal{G}_t$-measurable for each $t \in [0,T]$,

(ii) $\phi_{i,3}(t) > 0$ almost everywhere in $(t,\omega) \in [0,T] \times \Omega$,

(iii) $\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \|\phi_i(t)\|^2\mathrm{d}t\right)\right] < \infty$ with $\|\phi_i(t)\|^2 = \phi_{i,1}^2(t) + \phi_{i,2}^2(t) + \phi_{i,3}^2(t)$, where $\mathbb{E}_{\mathbb{P}}[\,\cdot\,]$ denotes the expectation under probability measure $\mathbb{P}$. This condition is known as Novikov's condition.

We denote Σ_i as the space of all such processes $\{\phi_i(t)\}_{t\in[0,T]}$.

Under the probability measure $\mathbb{Q}_i$, processes $B_{i,1}^{\mathbb{Q}_i}(t)$ and $B_{i,0}^{\mathbb{Q}_i}(t)$ satisfying

$$\mathrm{d}B_{i,1}^{\mathbb{Q}_i}(t) = \mathrm{d}B_1(t) + \phi_{i,1}(t)\mathrm{d}t$$

and

$$\mathrm{d}B_{i,0}^{\mathbb{Q}_i}(t) = \mathrm{d}B_{i,0}(t) + \phi_{i,2}(t)\mathrm{d}t$$

are standard Brownian motions. Accordingly, the dynamics of the i-th insurer's surplus process under the probability measure $\mathbb{Q}_i$ is given by

$$\mathrm{d}X_i^{\pi_i}(t) = \big[rX_i^{\pi_i}(t) + (\mu - r)\pi_{i,1}(t) + (1 - Z(t-))\delta\pi_{i,2}(t) + a_i(\lambda_i + \gamma_i q_i(t))$$
$$- \sigma\phi_{i,1}(t)\pi_{i,1}(t) - b_i\phi_{i,2}(t)q_i(t)\big]\mathrm{d}t + \sigma\pi_{i,1}(t)\mathrm{d}B_{i,1}^{\mathbb{Q}_i}(t) + b_iq_i(t)\mathrm{d}B_{i,0}^{\mathbb{Q}_i}(t)$$
$$- (1 - Z(t-))\zeta\pi_{i,2}(t)\mathrm{d}Z(t). \tag{2.2.5}$$

It is worth noting that the drift of (2.2.5) alters but its volatility remains the same relative to the model in (2.2.3). Basically, the model ambiguity can be considered as the uncertainty about the drift function, and we can understand this uncertainty as perturbations of the reference model, which is parameterized by $\phi_{i,1}(t)$ and $\phi_{i,2}(t)$.

Next, we provide the definition of admissible strategy.

Definition 2.2.1. *A reinsurance-investment strategy* $\pi_i(t) := (\pi_{i,1}(t), \pi_{i,2}(t), q_i(t))$ *is said to be admissible for insurer* $i, i \in \{1, 2\}$, *if*

1. $\{\pi_i(t)\}_{t\in[0,T]}$ *is a* $\mathbb{G}$*-progressively measurable process and it satisfies that* $\mathbb{E}_{\mathbb{Q}_i^*}\left[\int_0^T \|\pi_i(t)\|^2 \mathrm{d}t\right] < \infty$, *where* $\|\pi_i(t)\|^2 = \pi_{i,1}^2(t) + \pi_{i,2}^2(t) + q_i^2(t)$, *and* $\mathbb{Q}_i^*$ *is the chosen probability measure to describe the worst-case scenario and will be determined later;*

2. $\forall\ x_0 \in \mathbb{R}$, *the stochastic differential equation (SDE)* (2.2.2) *has a pathwise unique solution* $X_i^{\pi_i}(t)$ *satisfying* $\mathbb{E}_{\mathbb{Q}_i^*}\left[\exp\{-m_i X_i^{\pi_i}(t)\}\right] < \infty$.

Let Π_i denote the set of all admissible strategies of insurer i.

Given $X_i^{\pi_i}(t) = x_i$, $X_j^{\pi_j}(t) = x_j$, $Z(t) = z$, for $i \neq j \in \{1, 2\}$. When insurer i is assumed to be ambiguity-neutral, we used to formulate a non-zero-sum stochastic differential game by looking for the strategy $\pi_i^* \in \Pi_i$ that maximizes the following objective function:

$$
\begin{aligned}
&J_i^{\pi_i,\pi_j}(t, x_i, x_j, z) \\
&= \mathbb{E}_{\mathbb{P}}\left[U_i\left((1-n_i)X_i^{\pi_i}(T)+n_i(X_i^{\pi_i}(T)-X_j^{\pi_j}(T))\right)\big|(X_i^{\pi_i}(t), X_j^{\pi_j}(t), Z(t))=(x_i, x_j, z)\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[U_i\left(X_i^{\pi_i}(T) - n_i X_j^{\pi_j}(T)\right)\big|(X_i^{\pi_i}(t), X_j^{\pi_j}(t), Z(t)) = (x_i, x_j, z)\right],
\end{aligned}
\tag{2.2.6}
$$

where U_i is a strictly increasing and strictly concave utility function for insurer i. The constant $n_i \in [0, 1]$ describes the sensitivity of insurer i to the performance of the competing insurer j, $i \neq j \in \{1, 2\}$. A larger n_i implies that insurer i is more concerned about the relative surplus to the competing insurer j at terminal time and the game becomes more competitive. On the other hand, when $n_i = 0$ the objective function would reduce to the traditional expected utility of the terminal wealth in the single-agent problems without competition.

We define the relative performance process of insurer i, for $j \neq i \in \{1,2\}$, as

$$\widehat{X}_i^{\pi_i,\pi_j}(t) := X_i^{\pi_i}(t) - n_i X_j^{\pi_j}(t),$$

and we abbreviate it as $\widehat{X}_i(t)$, whose dynamics under the reference measure $\mathbb{P}$ is governed by

$$\begin{aligned} \mathrm{d}\widehat{X}_i(t) = &\big[r\widehat{X}_i(t) + (\mu - r)(\pi_{i,1}(t) - n_i\pi_{j,1}(t)) + (1 - Z(t-))\delta(\pi_{i,2}(t) - n_i\pi_{j,2}(t)) \\ &+ \lambda_i a_i - n_i\lambda_j a_j + a_i\gamma_i q_i(t) - n_i a_j\gamma_j q_j(t)\big]\mathrm{d}t + \sigma(\pi_{i,1}(t) - n_i\pi_{j,1}(t))\mathrm{d}B_1(t) \\ &+ b_i q_i(t)\mathrm{d}B_{i,0}(t) - n_i b_j q_j(t)\mathrm{d}B_{j,0}(t) - (1 - Z(t-))\zeta(\pi_{i,2}(t) - n_i\pi_{j,2}(t))\mathrm{d}Z(t), \end{aligned} \tag{2.2.7}$$

with the initial condition $\widehat{X}_i(0) = \hat{x}_i = x_i - \kappa_i x_j$. The dynamics of $\widehat{X}_i(t)$ under probability measure $\mathbb{Q}_i$ is described by the following SDE:

$$\begin{aligned} \mathrm{d}\widehat{X}_i(t) = &\Big[r\widehat{X}_i(t) + (\mu - r)(\pi_{i,1}(t) - n_i\pi_{j,1}(t)) + \delta(1 - Z(t-))(\pi_{i,2}(t) - n_i\pi_{j,2}(t)) \\ &+ \lambda_i a_i - n_i\lambda_j a_j + a_i\gamma_i q_i(t) - n_i a_j\gamma_j q_j(t) - \sigma(\phi_{i,1}(t)\pi_{i,1}(t) - n_i\phi_{j,1}(t)\pi_{j,1}(t)) \\ &- (b_i\phi_{i,2}(t)q_i(t) - n_i b_j\phi_{j,2}(t)q_j(t))\Big]\mathrm{d}t + \sigma\pi_{i,1}(t)\mathrm{d}B_{i,1}^{\mathbb{Q}_i}(t) - n_i\sigma\pi_{j,1}(t)\mathrm{d}B_{j,1}^{\mathbb{Q}_j}(t) \\ &+ b_i q_i(t)\mathrm{d}B_{i,0}^{\mathbb{Q}_i}(t) - n_i b_j q_j(t)\mathrm{d}B_{j,0}^{\mathbb{Q}_j}(t) - \zeta(1 - Z(t-))(\pi_{i,2}(t) - n_i\pi_{j,2}(t))\mathrm{d}Z(t). \end{aligned} \tag{2.2.8}$$

Problem 1: The classical non-zero-sum stochastic differential game between two competing insurers is to find a Nash equilibrium $(\pi_1^*, \pi_2^*) \in \Pi_1 \times \Pi_2$ such that for any $(\pi_1, \pi_2) \in \Pi_1 \times \Pi_2$, we have

$$\begin{aligned} J_1^{\pi_1^*,\pi_2^*}(t, x_1, x_2, z) &\geq J_1^{\pi_1,\pi_2^*}(t, x_1, x_2, z), \\ J_2^{\pi_1^*,\pi_2^*}(t, x_1, x_2, z) &\geq J_2^{\pi_1^*,\pi_2}(t, x_1, x_2, z). \end{aligned}$$

Next, we are going to incorporate the concepts of ambiguity aversion into **Problem 1**. Under this situation, each insurer i distrusts the veracity of the reference model $\mathbb{P}$ and chooses $\mathbb{Q}_i$ from $\mathcal{Q}$ as the alternative model. In other words, the insurer aims to maximize the expected utility of his performance at terminal time T relative to that of his competitor under the worst-case scenario of the alternative measure. The robust optimization problem for insurer i becomes

$$\sup_{\pi_i \in \Pi_i} \inf_{\mathbb{Q}_i \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}_i}\left[U_i\left(\widehat{X}_i^{\pi_i,\pi_j}(T)\right) + P_i(\mathbb{P}\|\mathbb{Q}_i)\right], \quad j \neq i \in \{1,2\}, \tag{2.2.9}$$

where $P_i(\mathbb{P}\|\mathbb{Q}_i) \geq 0$ is a penalty function measuring the divergence of $\mathbb{Q}_i$ from $\mathbb{P}$. In the perspective of an ambiguity-averse insurer i, the reference measure $\mathbb{P}$

is an estimation of the underlying real-world probability measure, and he/she is sceptical about the reference model because of the misspecification error and aims to consider the alternative model $\mathbb{Q}_i$. In the case of $P_i(\mathbb{P}\|\mathbb{Q}_i) \to \infty$, insurer i is convinced that the reference model is the true model and any alternative models deviating from the reference model will be heavily penalized. Under this circumstance, the robust optimization problem (2.2.9) reduces to the traditional optimization problem (2.2.6). On the other hand, if $P_i(\mathbb{P}\|\mathbb{Q}_i) \to 0$, i.e., the penalty function term vanishes, insurer i will not penalize any deviation from the reference model, which implies that the decision-maker is extremely ambiguous. Therefore, the penalty function reflects the insurer's degree of confidence in the reference model. In this sense, we can modify **Problem 1** for ambiguity-averse insurers as the following optimization problem.

Problem 2: The robust non-zero-sum stochastic differential game between two competing ambiguity-averse insurers is to find a Nash equilibrium $(\pi_1^*, \pi_2^*) \in \Pi_1 \times \Pi_2$ such that for any $(\pi_1, \pi_2) \in \Pi_1 \times \Pi_2$, we have

$$\inf_{\mathbb{Q}_1 \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}_1}\left[U_1\left(\widehat{X}_1^{\pi_1,\pi_2^*}(T)\right) + P_1(\mathbb{P}\|\mathbb{Q}_1)\right] \le \inf_{\mathbb{Q}_1 \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}_1}\left[U_1\left(\widehat{X}_1^{\pi_1^*,\pi_2^*}(T)\right) + P_1(\mathbb{P}\|\mathbb{Q}_1)\right],$$
$$\inf_{\mathbb{Q}_2 \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}_2}\left[U_2\left(\widehat{X}_2^{\pi_1^*,\pi_2}(T)\right) + P_2(\mathbb{P}\|\mathbb{Q}_2)\right] \le \inf_{\mathbb{Q}_2 \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}_2}\left[U_2\left(\widehat{X}_2^{\pi_1^*,\pi_2^*}(T)\right) + P_2(\mathbb{P}\|\mathbb{Q}_2)\right].$$

Although there may exist uncertainty or errors on the parameters in the reference model, $\mathbb{P}$ is the best description of the model in reality according to the information obtained so far. So the decision-makers wish to consider some alternative models that do not move too far away from the reference model. From (2.2.4), we can see that $\mathbb{Q}_i$ is parameterized by $\phi_i(t)$, and we would add some technical conditions on ϕ_i in the later paragraph to make the formulation analytically tractable. The alternative measures and the penalization term in (2.2.9) demonstrates the trade-off between not completely depending on the reference model and not deviating very far away from it. Considering that relative entropy has a wide application for model detection in statistics and econometrics, inspired by Maenhout (2004), we use relative entropy to measure the deviation of alternative measure $\mathbb{Q}_i$ from reference measure $\mathbb{P}$ for insurer $i, i \in \{1, 2\}$. In Appendix A, we have shown that the increase in relative entropy from t to $t + \mathrm{d}t$ equals

$$\frac{1}{2}\left(\phi_{i,1}(t)\right)^2 \mathrm{d}t + \frac{1}{2}(\phi_{i,2}(t))^2 \mathrm{d}t + h^P(1-z)\left(\phi_{i,3}(t)\ln\phi_{i,3}(t) - \phi_{i,3}(t) + 1\right)\mathrm{d}t,$$

where $z \in \{0,1\}$. It should be noted that the first two terms increase relative entropy due to the diffusion components of the model, while the last term gives the increase due to the jump component arising from the default process.

To solve **Problem 2**, we consider a penalty function of the following form used by Maenhout (2004):

$$P_i(\mathbb{P}\|\mathbb{Q}_i) = \int_t^T \Psi_i\left(s, \phi_i(s), \widehat{X}_i^{\pi_i,\pi_j^*}(s)\right) \mathrm{d}s,$$

and then define the value function of insurer i, for $i \neq j \in \{1,2\}$, as follows:

$$\begin{aligned} &V_i(t,\hat{x}_i,z) \\ &= \sup_{\pi_i\in\Pi_i}\inf_{\mathbb{Q}_i\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}_i}\left[U_i\left(\widehat{X}_i^{\pi_i,\pi_j^*}(T)\right) + \int_t^T \Psi_i\left(s,\phi_i(s),\widehat{X}_i^{\pi_i,\pi_j^*}(s)\right)\mathrm{d}s \middle| \widehat{X}_i^{\pi_i,\pi_j^*}(t)=\hat{x}_i, Z(t)=z\right], \end{aligned}$$

where

$$\begin{aligned} \Psi_i\left(s,\phi_i(s),\widehat{X}_i^{\pi_i,\pi_j^*}(s)\right) = &\frac{(\phi_{i,1}(s))^2}{2\psi_{i,1}\left(s,\widehat{X}_i^{\pi_i,\pi_j^*}(s),Z(s)\right)} + \frac{(\phi_{i,2}(s))^2}{2\psi_{i,2}\left(s,\widehat{X}_i^{\pi_i,\pi_j^*}(s),Z(s)\right)} \\ &+ \frac{(\phi_{i,3}(s)\ln\phi_{i,3}(s) - \phi_{i,3}(s)+1)h^P(1-Z(s))}{\psi_{i,3}\left(s,\widehat{X}_i^{\pi_i,\pi_j^*}(s),Z(s)\right)}, \end{aligned}$$

and $\mathbb{E}_{\mathbb{Q}_i}$ denotes the expectation under alternative probability measure $\mathbb{Q}_i$ which is parameterized by $\phi_i(t) \in \Sigma_i$. For $k \in \{1,2,3\}$, $\psi_{i,k}\left(s,\widehat{X}_i^{\pi_i,\pi_j^*}(s),Z(s)\right)$, the preference parameters for ambiguity aversion, are strictly positive deterministic functions. The larger $\psi_{i,k}\left(s,\widehat{X}_i^{\pi_i,\pi_j^*}(s),Z(s)\right)$ are, the less deviation from the reference model is penalized, then the AAI has less faith in the reference model and has more tendency to consider alternative models. Therefore, the degree of the AAI's ambiguity aversion is increasing with respect to the function $\psi_{i,k}\left(s,\widehat{X}_i^{\pi_i,\pi_j^*}(s),Z(s)\right)$.

In this chapter, we need the following two assumptions to derive explicit solutions.

Assumption 2.2.1. *We assume that both insurers have constant absolute risk aversion (CARA) utility functions defined by*

$$U_i(x) := -\frac{1}{m_i}\exp\{-m_i x\}, \quad m_i > 0,\ i \in \{1,2\},$$

where m_i represents the risk preference of insurer i. The greater the value of m_i, the more risk-averse an insurer is.

Assumption 2.2.2. *We assume that Π_i and Σ_i are compact convex sets.*

We suppress the arguments of the functions, for notational simplicity, in the following paragraphs. According to the dynamic programming principle, the robust Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation can be derived as

$$\sup_{\pi_i\in\Pi_i}\inf_{\phi_i\in\Sigma_i}\left\{\mathcal{L}_z^{\pi_i,\pi_j^*,\phi_i,\phi_j^*}W_i+\frac{\phi_{i,1}^2}{2\psi_{i,1}}+\frac{\phi_{i,2}^2}{2\psi_{i,2}}+\frac{(\phi_{i,3}\ln\phi_{i,3}-\phi_{i,3}+1)h^P(1-z)}{\psi_{i,3}}\right\}=0, \tag{2.2.10}$$

where the operator $\mathcal{L}_z$ is defined as:

$$\begin{aligned}\mathcal{L}_z^{\pi_i,\pi_j,\phi_i,\phi_j}W_i := \frac{\partial W_i}{\partial t} &+ \Big[r\hat{x}_i+(\mu-r)(\pi_{i,1}-n_i\pi_{j,1})+(1-z)\delta(\pi_{i,2}-n_i\pi_{j,2})\\ &+a_i\lambda_i-n_ia_j\lambda_j+a_i\gamma_iq_i-n_ia_j\gamma_jq_j-\sigma(\phi_{i,1}\pi_{i,1}-n_i\phi_{j,1}\pi_{j,1})\\ &-(b_i\phi_{i,2}q_i-n_ib_j\phi_{j,2}q_j)\Big]\frac{\partial W_i}{\partial\hat{x}_i}+\frac{1}{2}\Big(\sigma^2\pi_{i,1}^2+n_i^2\sigma^2\pi_{j,1}^2+b_i^2q_i^2\\ &+n_i^2b_j^2q_j^2-2n_i\sigma^2\pi_{i,1}\pi_{j,1}-2\rho n_ib_ib_jq_iq_j\Big)\frac{\partial^2W_i}{\partial\hat{x}_i^2}\\ &+\Big(W_i(t,\hat{x}_i-\zeta(\pi_{i,2}-n_i\pi_{j,2}),z+1)-W_i(t,\hat{x}_i,z)\Big)h^P(1-z)\phi_{i,3}.\end{aligned} \tag{2.2.11}$$

Applying Theorem 3 in Fan (1952), we then obtain

$$\begin{aligned}&\sup_{\pi_i\in\Pi_i}\inf_{\phi_i\in\Sigma_i}\left\{\mathcal{L}_z^{\pi_i,\pi_j^*,\phi_i,\phi_j^*}W_i+\frac{\phi_{i,1}^2}{2\psi_{i,1}}+\frac{\phi_{i,2}^2}{2\psi_{i,2}}+\frac{(\phi_{i,3}\ln\phi_{i,3}-\phi_{i,3}+1)h^P(1-z)}{\psi_{i,3}}\right\}\\ =&\inf_{\phi_i\in\Sigma_i}\sup_{\pi_i\in\Pi_i}\left\{\mathcal{L}_z^{\pi_i,\pi_j^*,\phi_i,\phi_j^*}W_i+\frac{\phi_{i,1}^2}{2\psi_{i,1}}+\frac{\phi_{i,2}^2}{2\psi_{i,2}}+\frac{(\phi_{i,3}\ln\phi_{i,3}-\phi_{i,3}+1)h^P(1-z)}{\psi_{i,3}}\right\}.\end{aligned}$$

For analytical tractability, we follow the work in Maenhout (2004) and assume that $\psi_{i,k}, i\in\{1,2\}$, $k\in\{1,2,3\}$, is state-dependent by setting:

$$\psi_{i,k}(t,\hat{x}_i,z)=-\frac{\beta_{i,k}}{m_iW_i(t,\hat{x}_i,z)}, \tag{2.2.12}$$

where $\beta_{i,k}\geq 0$ is the ambiguity aversion coefficient of insurer i, which describes the degree of his ambiguity aversion attitudes with respect to the diffusion risk and default risk. When $\beta_{i,k}=0$, insurer i is ambiguity-neutral to that kind of risk.

2.3 Solution to the robust non-zero-sum game

In this section, we will first derive the Nash equilibrium reinsurance-investment strategies under the pre-default case and the post-default case. Then we will prove the verification theorem.

2.3.1 The post-default case

After the default of the corporate zero-coupon bond, which corresponds to the case of $z = 1$, the HJBI equation (2.2.10) turns into a relatively simple form:

$$\inf_{\phi_i\in\Sigma_i}\sup_{\pi_i\in\Pi_i}\left\{\mathcal{L}_1^{\pi_i,\pi_j^*,\phi_i,\phi_j^*}W_i(t,\hat{x}_i,1)-\frac{\phi_{i,1}^2 m_i W_i(t,\hat{x}_i,1)}{2\beta_{i,1}}-\frac{\phi_{i,2}^2 m_i W_i(t,\hat{x}_i,1)}{2\beta_{i,2}}\right\}=0, \tag{2.3.13}$$

with boundary condition $W_i(T, \hat{x}_i, 1) = U_i(\hat{x}_i)$, and $\mathcal{L}_1^{\pi_i,\pi_j^*,\phi_i,\phi_j^*}W_i(t,\hat{x}_i,1)$ is obtained by replacing z with 1, substituting π_j^* and ϕ_j^* for π_j and ϕ_j in (2.2.11). We conjecture that the value function in the post-default case has the following form:

$$W_i(t,\hat{x}_i,1)=-\frac{1}{m_i}\exp\left\{-m_i\hat{x}_i e^{r(T-t)}+g_{i,1}(t)\right\},$$

where $g_{i,1}(t)$ is a deterministic function with $g_{i,1}(T) = 0$. By some simple calculations, we obtain the first-order partial derivatives of W_i shown as follows:

$$\begin{cases}\dfrac{\partial W_i(t,\hat{x}_i,1)}{\partial t}=W_i(t,\hat{x}_i,1)\left(rm_i\hat{x}_ie^{r(T-t)}+g'_{i,1}(t)\right),\\ \dfrac{\partial W_i(t,\hat{x}_i,1)}{\partial \hat{x}_i}=W_i(t,\hat{x}_i,1)\left(-m_ie^{r(T-t)}\right),\\ \dfrac{\partial W_i^2(t,\hat{x}_i,1)}{\partial \hat{x}_i^2}=W_i(t,\hat{x}_i,1)\left(m_i^2e^{2r(T-t)}\right).\end{cases} \tag{2.3.14}$$

Inserting the partial derivatives in (2.3.14) into HJBI equation (2.3.13) yields

$$\begin{aligned}\inf_{\phi_i\in\Sigma_i}\sup_{\pi_i\in\Pi_i}\Bigg\{&g'_{i,1}(t)+\Big[(\mu-r)(\pi_{i,1}-n_i\pi_{j,1}^*)+a_i\lambda_i-n_ia_j\lambda_j+a_i\gamma_iq_i-n_ia_j\gamma_jq_j^*\\&-\sigma(\phi_{i,1}\pi_{i,1}-n_i\phi_{j,1}^*\pi_{j,1}^*)-b_i\phi_{i,2}q_i+n_ib_j\phi_{j,2}^*q_j^*\Big]\left(-m_ie^{r(T-t)}\right)\\&+\frac{1}{2}\big(\sigma^2\pi_{i,1}^2+n_i^2\sigma^2(\pi_{j,1}^*)^2+b_i^2q_i^2+n_i^2b_j^2(q_j^*)^2-2n_i\sigma^2\pi_{i,1}\pi_{j,1}^*\\&-2\rho n_ib_ib_jq_iq_j^*\big)\left(m_i^2e^{2r(T-t)}\right)-\frac{m_i\phi_{i,1}^2}{2\beta_{i,1}}-\frac{m_i\phi_{i,2}^2}{2\beta_{i,2}}\Bigg\}=0.\end{aligned} \tag{2.3.15}$$

Fixing ϕ_i and letting the first-order derivative of the left-hand side of (2.3.15) with respect to π_i equal zero, we can then obtain the supreme point $\pi_i^*(\phi_i) := (\pi_{i,1}^*(\phi_i), \pi_{i,2}^*(\phi_i), q_i^*(\phi_i))$ given by

$$\begin{cases}\pi_{i,1}^*(\phi_i)=\dfrac{\mu-r+m_in_i\sigma^2e^{r(T-t)}\pi_{j,1}^*-\sigma\phi_{i,1}}{m_i\sigma^2e^{r(T-t)}},\\ \pi_{i,2}^*(\phi_i)=0,\\ q_i^*(\phi_i)=\dfrac{a_i\gamma_i-b_i\phi_{i,2}+\rho b_ib_jm_in_ie^{r(T-t)}q_j^*}{b_i^2m_ie^{r(T-t)}}.\end{cases} \tag{2.3.16}$$

Substituting (2.3.16) into (2.3.15), according to the first-order conditions for ϕ_i, the minimum is achieved at $\phi_i^* := (\phi_{i,1}^*, \phi_{i,2}^*)$ (note that there is no ambiguity about default risk after default):

$$\begin{cases} \phi_{i,1}^* = \dfrac{\beta_{i,1}(\mu - r + m_i n_i \sigma^2 e^{r(T-t)} \pi_{j,1}^*)}{\sigma(m_i + \beta_{i,1})}, \\ \phi_{i,2}^* = \dfrac{\beta_{i,2}(a_i\gamma_i + \rho b_i b_j m_i n_i e^{r(T-t)} q_j^*)}{b_i(m_i + \beta_{i,2})}. \end{cases} \tag{2.3.17}$$

Plugging π_i^* and ϕ_i^* into (2.3.15), we obtain the following ODE that $g_{i,1}(t)$ should satisfy:

$$g'_{i,1}(t) - P_{i,1} m_i e^{r(T-t)} + \frac{1}{2} Q_{i,1} m_i^2 e^{2r(T-t)} - \frac{1}{2} O_{i,1} m_i e^{2r(T-t)} = 0,$$

where

$$\begin{cases} P_{i,1} = (\mu - r)(\pi_{i,1}^* - n_i \pi_{j,1}^*) + a_i\lambda_i - n_i a_j \lambda_j + a_i \gamma_i q_i^* - n_i a_j \gamma_j q_j^* \\ \qquad - \sigma^2 \beta_{i,1} (\pi_{i,1}^*)^2 e^{r(T-t)} + \sigma n_i \phi_{j,1}^* \pi_{j,1}^* - \beta_{i,2} b_i^2 (q_i^*)^2 e^{r(T-t)} + n_i b_j \phi_{j,2}^* q_j^*, \\ Q_{i,1} = \sigma^2 (\pi_{i,1}^*)^2 + n_i^2 \sigma^2 (\pi_{j,1}^*)^2 + b_i^2 (q_i^*)^2 + n_i^2 b_j^2 (q_j^*)^2 - 2n_i\sigma^2 \pi_{i,1}^* \pi_{j,1}^* - 2\rho n_i b_i b_j q_i^* q_j^*, \\ O_{i,1} = \sigma^2 \beta_{i,1} (\pi_{i,1}^*)^2 + b_i^2 \beta_{i,2} (q_i^*)^2, \end{cases}$$

$\pi_{i,1}^*$ and q_i^* are given by (2.3.16), and ϕ_i^* is defined by (2.3.17). Noting the terminal condition $g_{i,1}(T) = 0$, we obtain

$$g_{i,1}(t) = \int_t^T \left[-P_{i,1} m_i e^{r(T-s)} + \frac{1}{2} Q_{i,1} m_i^2 e^{2r(T-s)} - \frac{1}{2} O_{i,1} m_i e^{2r(T-s)} \right] \mathrm{d}s. \tag{2.3.18}$$

2.3.2 The pre-default case

Before the default of the corporate zero-coupon bond occurs, i.e., $z = 0$, the HJBI equation (2.2.10) becomes

$$\inf_{\phi_i \in \Sigma_i} \sup_{\pi_i \in \Pi_i} \left\{ \mathcal{L}_0^{\pi_i, \pi_j^*, \phi_i, \phi_j^*} W_i(t, \hat{x}_i, 0) - \frac{\phi_{i,1}^2 m_i W_i(t, \hat{x}_i, 0)}{2\beta_{i,1}} \right. \\ \left. - \frac{\phi_{i,2}^2 m_i W_i(t, \hat{x}_i, 0)}{2\beta_{i,2}} - \frac{(\phi_{i,3} \ln \phi_{i,3} - \phi_{i,3} + 1) h^P m_i W_i(t, \hat{x}_i, 0)}{\beta_{i,3}} \right\} = 0, \tag{2.3.19}$$

with boundary condition $W_i(T, \hat{x}_i, 0) = U_i(\hat{x}_i)$, and $\mathcal{L}_0^{\pi_i, \pi_j^*, \phi_i, \phi_j^*} W_i(t, \hat{x}_i, 0)$ is obtained by replacing z with 0, substituting π_j^* and ϕ_j^* for π_j and ϕ_j in (2.2.11).

We attempt to find a solution to the value function of the following form in the pre-default case:

$$W_i(t, \hat{x}_i, 0) = -\frac{1}{m_i} \exp\left\{-m_i \hat{x}_i e^{r(T-t)} + g_{i,0}(t)\right\}, \tag{2.3.20}$$

where $g_{i,0}(t)$ is a deterministic function satisfying $g_{i,0}(T) = 0$. A direct calculation yields

$$\begin{cases} \dfrac{\partial W_i(t, \hat{x}_i, 0)}{\partial t} = W_i(t, \hat{x}_i, 0)\left(r m_i \hat{x}_i e^{r(T-t)} + g'_{i,0}(t)\right), \\ \dfrac{\partial W_i(t, \hat{x}_i, 0)}{\partial \hat{x}_i} = W_i(t, \hat{x}_i, 0)\left(-m_i e^{r(T-t)}\right), \\ \dfrac{\partial W_i^2(t, \hat{x}_i, 0)}{\partial \hat{x}_i^2} = W_i(t, \hat{x}_i, 0)\left(m_i^2 e^{2r(T-t)}\right), \\ W_i(t, \hat{x}_i - \zeta(\pi_{i,2} - n_i \pi^*_{j,2}), 1) - W_i(t, \hat{x}_i, 0) \\ \qquad = W_i(t, \hat{x}_i, 0)\left[\exp\left\{m_i \zeta e^{r(T-t)}(\pi_{i,2} - n_i \pi^*_{j,2}) + g_{i,1}(t) - g_{i,0}(t)\right\} - 1\right]. \end{cases} \tag{2.3.21}$$

Inserting (2.3.21) back into HJBI equation (2.3.19), we obtain

$$\begin{aligned} \inf_{\phi_i \in \Sigma_i} \sup_{\pi_i \in \Pi_i} \Bigg\{ & g'_{i,0}(t) + \Big[(\mu - r)(\pi_{i,1} - n_i \pi^*_{j,1}) + \delta(\pi_{i,2} - n_i \pi^*_{j,2}) + a_i \lambda_i - n_i a_j \lambda_j \\ & + a_i \gamma_i q_i - n_i a_j \gamma_j q_j^* - \sigma(\phi_{i,1}\pi_{i,1} - n_i \phi^*_{j,1}\pi^*_{j,1}) - b_i \phi_{i,2} q_i \\ & + n_i b_j \phi^*_{j,2} q_j^*\Big]\left(-m_i e^{r(T-t)}\right) + \frac{1}{2}\Big[\sigma^2 \pi_{i,1}^2 + n_i^2 \sigma^2 (\pi^*_{j,1})^2 + b_i^2 q_i^2 + n_i^2 b_j^2 (q_j^*)^2 \\ & - 2 n_i \sigma^2 \pi_{i,1} \pi^*_{j,1} - 2\rho n_i b_i b_j q_i q_j^*\Big]\left(m_i^2 e^{2r(T-t)}\right) - \frac{m_i \phi_{i,1}^2}{2\beta_{i,1}} - \frac{m_i \phi_{i,2}^2}{2\beta_{i,2}} \\ & + \Big[\exp\left\{m_i \zeta e^{r(T-t)}(\pi_{i,2} - n_i \pi^*_{j,2}) + g_{i,1}(t) - g_{i,0}(t)\right\} - 1\Big]\phi_{i,3} h^P \\ & - \frac{(\phi_{i,3} \ln \phi_{i,3} - \phi_{i,3} + 1) h^P m_i}{\beta_{i,3}} \Bigg\} = 0. \end{aligned} \tag{2.3.22}$$

We first fix ϕ_i and the first-order conditions give the following optimal reinsurance and investment strategies

$$\begin{cases} \pi^*_{i,1}(\phi_i) = \dfrac{\mu - r + m_i n_i \sigma^2 e^{r(T-t)} \pi^*_{j,1} - \sigma \phi_{i,1}}{m_i \sigma^2 e^{r(T-t)}}, \\ \pi^*_{i,2}(\phi_i) = \dfrac{\ln\left[\delta / (\zeta h^P \phi_{i,3})\right] - g_{i,1}(t) + g_{i,0}(t)}{m_i \zeta e^{r(T-t)}} + n_i \pi^*_{j,2}, \\ q_i^*(\phi_i) = \dfrac{a_i \gamma_i - b_i \phi_{i,2} + \rho b_i b_j m_i n_i e^{r(T-t)} q_j^*}{b_i^2 m_i e^{r(T-t)}}. \end{cases} \tag{2.3.23}$$

Substituting (2.3.23) into (2.3.22) and minimizing over ϕ_i, we obtain

$$\begin{cases} \phi_{i,1}^* = \dfrac{\beta_{i,1}(\mu - r + m_i n_i \sigma^2 e^{r(T-t)} \pi_{j,1}^*)}{\sigma(m_i + \beta_{i,1})}, \\ \phi_{i,2}^* = \dfrac{\beta_{i,2}(a_i\gamma_i + \rho b_i b_j m_i n_i e^{r(T-t)} q_j^*)}{b_i(m_i + \beta_{i,2})}, \\ \dfrac{\delta}{\zeta} - h^P \phi_{i,3}^* - \dfrac{h^P m_i}{\beta_{i,3}} \phi_{i,3}^* \ln \phi_{i,3}^* = 0. \end{cases} \tag{2.3.24}$$

Proposition 2.3.1. *Let* $H(\phi_{i,3}) = \frac{\delta}{\zeta} - h^P \phi_{i,3} - \frac{h^P m_i}{\beta_{i,3}} \phi_{i,3} \ln \phi_{i,3}$, *then* $H(\phi_{i,3})$ *has a unique positive root* $\phi_{i,3}^*$.

Proof. The proof is the same as that of Proposition 4.2 in Sun et al. (2017), and hence we omit it here. □

Inserting π_i^* and ϕ_i^* into (2.3.22) yields the following ODE satisfied by $g_{i,0}(t)$:

$$g_{i,0}'(t) - \frac{\delta}{\zeta} g_{i,0}(t) - P_{i,0} m_i e^{r(T-t)} + \frac{1}{2} Q_{i,0} m_i^2 e^{2r(T-t)} - O_{i,0} = 0,$$

where

$$\begin{cases} P_{i,0} = (\mu - r)(\pi_{i,1}^* - n_i \pi_{j,1}^*) + \dfrac{\delta}{m_i \zeta e^{r(T-t)}} \left(\ln \dfrac{\delta}{\zeta h^P \phi_{i,3}^*} - g_{i,1}(t) \right) + a_i \lambda_i - n_i a_j \lambda_j \\ \qquad + a_i \gamma_i q_i^* - n_i a_j \gamma_j q_j^* - \sigma(\phi_{i,1}^* \pi_{i,1}^* - n_i \phi_{j,1}^* \pi_{j,1}^*) - b_i \phi_{i,2}^* q_i^* + n_i b_j \phi_{j,2}^* q_j^*, \\ Q_{i,0} = \sigma^2 (\pi_{i,1}^*)^2 + n_i^2 \sigma^2 (\pi_{j,1}^*)^2 + b_i^2 (q_i^*)^2 + n_i^2 b_j^2 (q_j^*)^2 - 2 n_i \sigma^2 \pi_{i,1}^* \pi_{j,1}^* - 2\rho n_i b_i b_j q_i^* q_j^*, \\ O_{i,0} = \dfrac{m_i (\phi_{i,1}^*)^2}{2\beta_{i,1}} + \dfrac{m_i (\phi_{i,2}^*)^2}{2\beta_{i,2}} - \dfrac{\delta}{\zeta} + h^P \phi_{i,3}^* + \dfrac{(\phi_{i,3}^* \ln \phi_{i,3}^* - \phi_{i,3}^* + 1) h^P m_i}{\beta_{i,3}}, \end{cases}$$

$\pi_{i,1}^*$ and q_i^* are given in (2.3.23), ϕ_i^* is defined by (2.3.24) and $g_{i,1}(s)$ is defined by (2.3.18). Combining with the boundary condition $g_{i,0}(T) = 0$, we obtain the following expression for $g_{i,0}(t)$:

$$g_{i,0}(t) = e^{-\frac{\delta}{\zeta}(T-t)} \int_t^T e^{\frac{\delta}{\zeta}(T-s)} \left[-P_{i,0} m_i e^{r(T-s)} + \frac{1}{2} Q_{i,0} m_i^2 e^{2r(T-s)} - O_{i,0} \right] \mathrm{d}s.$$

Combining the results in the pre-default case and the post-default case, we have the following solution to the HJBI equation (2.2.10) for insurer i ($i \in \{1, 2\}$):

$$\widetilde{W}_i(t, \hat{x}_i, z) = (1 - z) W_i(t, \hat{x}_i, 0) + z W_i(t, \hat{x}_i, 1), \quad \text{where } z = 0 \text{ or } 1. \tag{2.3.25}$$

Additionally, we define the candidate optimal strategies $\pi_i^* := (\pi_{i,1}^*, \pi_{i,2}^*, q_i^*)$ as follows:

$$\begin{cases} \pi_{i,1}^* = \dfrac{\mu - r - \sigma\phi_{i,1}^*}{m_i\sigma^2 e^{r(T-t)}} + n_i\pi_{j,1}^*, & t \in [0,T], \\ \pi_{i,2}^* = \begin{cases} \frac{C_i(t)}{m_i\zeta e^{r(T-t)}} + n_i\pi_{j,2}^*, \\ 0, \end{cases} & \begin{array}{l} t \in [0,\tau\wedge T], \\ t \in [\tau\wedge T, T], \end{array} \\ q_i^* = \dfrac{a_i\gamma_i - b_i\phi_{i,2}^* + \rho b_i b_j m_i n_i e^{r(T-t)} q_j^*}{b_i^2 m_i e^{r(T-t)}}, & t \in [0,T], \end{cases} \tag{2.3.26}$$

where

$$C_i(t) = \ln\left(\frac{\delta}{\zeta h^P \phi_{i,3}^*}\right) - g_{i,1}(t) + g_{i,0}(t), \quad i \in \{1,2\}.$$

The worst-case density generator $\phi_i^* := (\phi_{i,1}^*, \phi_{i,2}^*, \phi_{i,3}^*)$ of insurer i is given by

$$\begin{cases} \phi_{i,1}^* = \dfrac{\beta_{i,1}(\mu - r + m_i n_i \sigma^2 e^{r(T-t)}\pi_{j,1}^*)}{\sigma(m_i + \beta_{i,1})}, & t \in [0,T], \\ \phi_{i,2}^* = \dfrac{\beta_{i,2}(a_i\gamma_i + \rho b_i b_j m_i n_i e^{r(T-t)} q_j^*)}{b_i(m_i + \beta_{i,2})}, & t \in [0,T], \end{cases} \tag{2.3.27}$$

and $\phi_{i,3}^*$ is given by Proposition 2.3.1 for $t \in [0,\tau\wedge T]$.

Proposition 2.3.2. *The candidate optimal strategies π_i^* given in* (2.3.26) *derived by the first-order conditions solves the maximization problem in HJBI equation* (2.2.10).

Proof. For a fixed ϕ_i, we gather the terms of $\pi_{i,1}$ in the left-hand side of HJBI equation (2.2.10) and define

$$f_{i,1}(\pi_{i,1}) := [(\mu - r)\pi_{i,1} - \sigma\phi_{i,1}\pi_{i,1}]\frac{\partial W_i}{\partial \hat{x}_i} + \frac{1}{2}\left(\sigma^2\pi_{i,1}^2 - 2n_i\sigma^2\pi_{i,1}\pi_{j,1}\right)\frac{\partial^2 W_i}{\partial \hat{x}_i^2}.$$

Noting that $\frac{\partial^2 W_i}{\partial \hat{x}_i^2} < 0$, we have

$$f_{i,1}''(\pi_{i,1}) = \sigma^2\frac{\partial^2 W_i}{\partial \hat{x}_i^2} < 0.$$

Furthermore, for the reinsurance strategy we define

$$f_{i,2}(q_i) := (a_i\gamma_i q_i - b_i\phi_{i,2}q_i)\frac{\partial W_i}{\partial \hat{x}_i} + \frac{1}{2}b_i^2 q_i^2\frac{\partial^2 W_i}{\partial \hat{x}_i^2},$$

and then we have

$$f_{i,2}''(q_i) = b_i^2\frac{\partial^2 W_i}{\partial \hat{x}_i^2} < 0.$$

Similarly, for the defaultable bond investment strategy we let

$$f_{i,3}(\pi_{i,2}) = (1-z)\delta\pi_{i,2}\frac{\partial W_i}{\partial \hat{x}_i} + h^P(1-z)\phi_{i,3}W_i(t,\hat{x}_i - \zeta(\pi_{i,2} - n_i\pi_{j,2}), z+1),$$

and hence

$$f''_{i,3}(\pi_{i,2}) = \zeta^2 h^P(1-z)\phi_{i,3}\frac{\partial^2 W_i}{\partial \hat{x}_i^2}(t,\hat{x}_i - \zeta(\pi_{i,2} - n_i\pi_{j,2}), z+1),$$

which is less than zero when $z=0$ since $\phi_{i,3} > 0$. When $z=1$, $f_{i,3}(\pi_{i,2}) = 0$ and in this case $\pi^*_{i,2} = 0$. Therefore, the first-order optimality conditions provide the optimal reinsurance-investment strategy. □

2.3.3 Verification theorem

In order to give the verification theorem, we need the following lemma.

Lemma 2.3.1. *We have the following properties of the candidate optimal strategy* π_i^**, the worst-case density generator* ϕ_i^* *and the corresponding function* $\widetilde{W}_i(t,\hat{x}_i,z)$ *(which are given by* (2.3.26), (2.3.27) *and* (2.3.25)*, respectively):*

(i) Candidate optimal strategy π_i^* *is an admissible strategy and the alternative probability measure* $\mathbb{Q}_i^*$ *determined by* $\Lambda^{\phi_i^*}(t)$ *is well defined;*

(ii) $\mathbb{E}_{\mathbb{Q}_i^*}\left(\sup_{t\in[0,T]}\left|\widetilde{W}_i\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t),Z(t)\right)\right|^4\right) < \infty;$

(iii) $\mathbb{E}_{\mathbb{Q}_i^*}\left(\sup_{t\in[0,T]}\left|\frac{\left(\phi^*_{i,1}(t)\right)^2}{2\psi_{i,1}\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t)\right)} + \frac{\left(\phi^*_{i,2}(t)\right)^2}{2\psi_{i,2}\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t)\right)} + \frac{\left(\phi^*_{i,3}\ln\phi^*_{i,3} - \phi^*_{i,3}+1\right)h^P(1-z)}{\psi_{i,3}\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t)\right)}\right|^2\right)$
$< \infty$*, where* $\mathbb{Q}_i^*$ *and* $\Lambda^{\phi_i^*}(t)$ *satisfy* $\frac{\mathrm{d}\mathbb{Q}_i^*}{\mathrm{d}\mathbb{P}} = \Lambda^{\phi_i^*}(t)$.

Proof. See Appendix B. □

Theorem 2.3.1. *If there exist a function* $\widetilde{W}_i(t,\hat{x}_i,z) \in C^{1,2}([0,T]\times\mathbb{R}\times\{0,1\})$ *and a Markovian control policy* $(\pi_i^*,\phi_i^*) \in \Pi_i \times \Sigma_i$ *such that*

(a) for any $\phi_i \in \Sigma_i$, $\mathcal{L}_z^{\pi_i^*,\pi_j^*,\phi_i,\phi_j^*}\widetilde{W}_i(t,\hat{x}_i,z) + \Psi_i(t,\phi_i,\hat{x}_i) \geq 0;$

(b) for any $\pi_i \in \Pi_i$, $\mathcal{L}_z^{\pi_i,\pi_j^*,\phi_i^*,\phi_j^*}\widetilde{W}_i(t,\hat{x}_i,z) + \Psi_i(t,\phi_i^*,\hat{x}_i) \leq 0;$

(c) $\mathcal{L}_z^{\pi_i^*,\pi_j^*,\phi_i^*,\phi_j^*}\widetilde{W}_i(t,\hat{x}_i,z) + \Psi_i(t,\phi_i^*,\hat{x}_i) = 0;$

(d) for all $(\pi_i, \phi_i) \in \Pi_i \times \Sigma_i$, $\lim_{t \to T-} \widetilde{W}_i\left(t, \widehat{X}_i^{\pi_i,\pi_j^*}(t), Z(t)\right) = U_i\left(\widehat{X}_i^{\pi_i,\pi_j^*}(T)\right)$;

(e) $\left\{\widetilde{W}_i\left(\tau, \widehat{X}_i^{\pi_i,\pi_j^*}(\tau), Z(\tau)\right)\right\}_{\tau\in\mathcal{T}}$ *and* $\Psi_i\left(\tau, \phi_i(\tau), \widehat{X}_i^{\pi_i,\pi_j^*}(\tau)\right)_{\tau\in\mathcal{T}}$ *are uniformly integrable, where* $\mathcal{T}$ *denotes the set of all stopping times satisfying* $\tau \leq T$.

Then π_i^* *is the optimal strategies and* $\widetilde{W}_i(t, \hat{x}_i, z) = V_i(t, \hat{x}_i, z)$ *is the associated value function.*

Proof. We obtain this verification theorem by using Lemma 2.3.1 stated above and Corollary 1.2 in Kraft (2004). The proof can also be referred to Mataramvura and Øksendal (2008) and so it is omitted here. □

We are now in a position to state the following results, which present the Nash equilibrium reinsurance and investment strategies and the optimal value functions of both insurers when the two competing insurers are AAIs.

Theorem 2.3.2. *When both insurers are AAIs, the Nash equilibrium investment strategies are given by*

$$\begin{cases} \pi_{1,1}^*(t) = \dfrac{(\mu - r)(m_2 + \beta_{2,1} + m_1 n_1)}{\left[(m_1 + \beta_{1,1})(m_2 + \beta_{2,1}) - m_1 n_1 m_2 n_2\right]\sigma^2 e^{r(T-t)}}, & t \in [0, T], \\ \pi_{2,1}^*(t) = \dfrac{(\mu - r)(m_1 + \beta_{1,1} + m_2 n_2)}{\left[(m_1 + \beta_{1,1})(m_2 + \beta_{2,1}) - m_1 n_1 m_2 n_2\right]\sigma^2 e^{r(T-t)}}, & t \in [0, T], \end{cases}$$

and

$$\begin{cases} \pi_{1,2}^*(t) = \left(\dfrac{m_1 n_1 C_2(t) + m_2 C_1(t)}{(1 - n_1 n_2) m_1 m_2 \zeta e^{r(T-t)}}\right)^+, & t \in [0, \tau \wedge T], \\ \pi_{2,2}^*(t) = \left(\dfrac{m_2 n_2 C_1(t) + m_1 C_2(t)}{(1 - n_1 n_2) m_1 m_2 \zeta e^{r(T-t)}}\right)^+, & t \in [0, \tau \wedge T], \\ \pi_{1,2}^*(t) = \pi_{2,2}^*(t) = 0, & t \in [\tau \wedge T, T], \end{cases}$$

where $x^+ = \max\{0, x\}$*, and* $\phi_{i,3}^*$ *in* $C_i(t)$ *is given by Proposition* 2.3.1.

The Nash equilibrium reinsurance strategy is given by

$$\begin{cases} q_1^*(t) = \dfrac{a_1\gamma_1 b_2(m_2 + \beta_{2,2}) + \rho b_1 m_1 n_1 a_2 \gamma_2}{\left[(m_1 + \beta_{1,2})(m_2 + \beta_{2,2}) - \rho^2 m_1 n_1 m_2 n_2\right] b_1^2 b_2 e^{r(T-t)}}, & t \in [0, T], \\ q_2^*(t) = \dfrac{a_2\gamma_2 b_1(m_1 + \beta_{1,2}) + \rho b_2 m_2 n_2 a_1 \gamma_1}{\left[(m_1 + \beta_{1,2})(m_2 + \beta_{2,2}) - \rho^2 m_1 n_1 m_2 n_2\right] b_2^2 b_1 e^{r(T-t)}}, & t \in [0, T]. \end{cases} \tag{2.3.28}$$

The worst-case density generators $\phi_{i,j}^*(t)$, for $i,j \in \{1,2\}$, are given by

$$\begin{cases} \phi_{1,1}^*(t) = \dfrac{\beta_{1,1}(\mu - r)(m_2 + \beta_{2,1} + m_1 n_1)}{\sigma[(m_1 + \beta_{1,1})(m_2 + \beta_{2,1}) - m_1 n_1 m_2 n_2]}, & t \in [0,T], \\ \phi_{2,1}^*(t) = \dfrac{\beta_{2,1}(\mu - r)(m_1 + \beta_{1,1} + m_2 n_2)}{\sigma[(m_1 + \beta_{1,1})(m_2 + \beta_{2,1}) - m_1 n_1 m_2 n_2]}, & t \in [0,T], \end{cases} \tag{2.3.29}$$

$$\begin{cases} \phi_{1,2}^*(t) = \dfrac{\beta_{1,2}[a_1\gamma_1 b_2(m_2 + \beta_{2,2}) + \rho m_1 n_1 a_2 \gamma_2 b_1]}{b_1 b_2[(m_1 + \beta_{1,2})(m_2 + \beta_{2,2}) - \rho^2 m_1 n_1 m_2 n_2]}, & t \in [0,T], \\ \phi_{2,2}^*(t) = \dfrac{\beta_{2,2}[a_2\gamma_2 b_1(m_1 + \beta_{1,2}) + \rho m_2 n_2 a_1 \gamma_1 b_2]}{b_1 b_2[(m_1 + \beta_{1,2})(m_2 + \beta_{2,2}) - \rho^2 m_1 n_1 m_2 n_2]}, & t \in [0,T]. \end{cases} \tag{2.3.30}$$

The optimal value functions of both insurers are given by (2.3.25), *for* $i \in \{1,2\}$.

Proof. When both insurers are AAIs, the value function of insurer i is given by (2.3.25), and the optimal reinsurance and investment strategies of insurer i are given by (2.3.26), for $i \in \{1,2\}$. If we insert the expression of $\phi_{i,1}^*$ in (2.3.27) into $\pi_{i,1}^*$ in (2.3.26), we obtain the following representation of $\pi_{i,1}^*$:

$$\pi_{i,1}^* = \frac{\mu - r + m_i n_i \sigma^2 e^{r(T-t)} \pi_{j,1}^*}{(m_i + \beta_{i,1})\sigma^2 e^{r(T-t)}}, \quad t \in [0,T].$$

So we can derive the Nash equilibrium strategy of stock investment by solving the following system of equations:

$$\begin{cases} \pi_{i,1}^* = \dfrac{\mu - r + m_i n_i \sigma^2 e^{r(T-t)} \pi_{j,1}^*}{(m_i + \beta_{i,1})\sigma^2 e^{r(T-t)}}, & t \in [0,T], \\ \pi_{j,1}^* = \dfrac{\mu - r + m_j n_j \sigma^2 e^{r(T-t)} \pi_{i,1}^*}{(m_j + \beta_{j,1})\sigma^2 e^{r(T-t)}}, & t \in [0,T]. \end{cases}$$

The Nash equilibrium strategy of the amount invested in the defaultable bond for the pre-default case can be obtained by solving

$$\begin{cases} \pi_{i,2}^* = \dfrac{C_i(t)}{m_i \zeta e^{r(T-t)}} + n_i \pi_{j,2}^*, & t \in [0, \tau \wedge T], \\ \pi_{j,2}^* = \dfrac{C_j(t)}{m_j \zeta e^{r(T-t)}} + n_j \pi_{i,2}^*, & t \in [0, \tau \wedge T]. \end{cases}$$

The Nash equilibrium reinsurance strategy is the solution of the following system of equations:

$$\begin{cases} q_i^* = \dfrac{a_i \gamma_i + \rho b_i b_j m_i n_i e^{r(T-t)} q_j^*}{(m_i + \beta_{i,2}) b_i^2 e^{r(T-t)}}, & t \in [0,T], \\ q_j^* = \dfrac{a_j \gamma_j + \rho b_i b_j m_j n_j e^{r(T-t)} q_i^*}{(m_j + \beta_{j,2}) b_j^2 e^{r(T-t)}}, & t \in [0,T]. \end{cases}$$

Combining the expressions of $\pi_{i,1}^*$, q_i^* and (2.3.27) yields (2.3.29) and (2.3.30). Since $\mu > r$, $n_i \in [0,1]$ and $\rho^2 \in [0,1]$, we have $m_1 m_2 \geq m_1 m_2 n_1 n_2$ and $m_1 m_2 \geq \rho^2 m_1 m_2 n_1 n_2$. Hence $\pi_{i,1}^*(t) > 0$ and $q_i^*(t) > 0$. This completes the proof. $\square$

2.4 Special case: ANI case

Our model would reduce to the analysis of insurer i who is an ANI in the stochastic differential reinsurance and investment game if we set the ambiguity aversion coefficients to be 0, i.e., $\beta_{i,k} = 0$, for $k \in \{1,2,3\}$. Under this circumstance, the relative performance process of insurer i under probability measure $\mathbb{P}$ is described by (2.2.7), and the value function is defined as

$$\widehat{V}_i(t, \hat{x}_i, z) := \sup_{\hat{\pi}_i \in \widehat{\Pi}_i} \mathbb{E}_{\mathbb{P}} \left[U_i \left(\widehat{X}_i^{\hat{\pi}_i, \hat{\pi}_j^*}(T) \right) \middle| \widehat{X}_i^{\hat{\pi}_i, \hat{\pi}_j^*}(t) = \hat{x}_i, Z(t) = z \right],$$

where $\hat{\pi}_i := (\hat{\pi}_{i,1}, \hat{\pi}_{i,2}, \hat{q}_i)$, and $\widehat{\Pi}_i$ is the set of admissible strategies of an ANI. For notational convenience, we define an operator $\hat{\mathcal{L}}_z$ by

$$\begin{aligned}
\hat{\mathcal{L}}_z^{\hat{\pi}_i, \hat{\pi}_j} \widehat{W}_i(t, \hat{x}_i, z) :=& \frac{\partial \widehat{W}_i(t, \hat{x}_i, z)}{\partial t} + \Big[r\hat{x}_i + (\mu - r)(\hat{\pi}_{i,1} - n_i \hat{\pi}_{j,1}) + (1-z)\delta(\hat{\pi}_{i,2} - n_i \hat{\pi}_{j,2}) \\
& + a_i \lambda_i - n_i a_j \lambda_j + a_i \gamma_i \hat{q}_i - n_i a_j \gamma_j \hat{q}_j \Big] \frac{\partial \widehat{W}_i(t, \hat{x}_i, z)}{\partial \hat{x}_i} + \frac{1}{2} \Big(\sigma^2 \hat{\pi}_{i,1}^2 \\
& + n_i^2 \sigma^2 \hat{\pi}_{j,1}^2 + b_i^2 \hat{q}_i^2 + n_i^2 b_j^2 \hat{q}_j^2 - 2 n_i \sigma^2 \hat{\pi}_{i,1} \hat{\pi}_{j,1} - 2\rho n_i b_i b_j \hat{q}_i \hat{q}_j \Big) \frac{\partial^2 \widehat{W}_i(t, \hat{x}_i, z)}{\partial \hat{x}_i^2} \\
& + \Big(\widehat{W}_i(t, \hat{x}_i - \zeta(\hat{\pi}_{i,2} - n_i \hat{\pi}_{j,2}), z+1) - \widehat{W}_i(t, \hat{x}_i, z) \Big) h^P (1-z),
\end{aligned}$$

and then the corresponding Hamilton-Jacobi-Bellman (HJB) equation becomes

$$\sup_{\hat{\pi}_i \in \widehat{\Pi}_i} \hat{\mathcal{L}}_z^{\hat{\pi}_i, \hat{\pi}_j^*} \widehat{W}_i(t, \hat{x}_i, z) = 0.$$

Similar to the derivation of the robust case, we have the optimal reinsurance and investment strategies and the corresponding optimal value functions for the ANI case as follows.

Proposition 2.4.1. *The optimal reinsurance and investment strategies for an*

ANI are given by

$$
\begin{cases}
\hat{\pi}_{i,1}^{*} = \dfrac{\mu - r}{m_i \sigma^2 e^{r(T-t)}} + n_i \hat{\pi}_{j,1}^{*}, & t \in [0,T], \\
\hat{\pi}_{i,2}^{*} = \begin{cases} \frac{\widehat{C}_i(t)}{m_i \zeta e^{r(T-t)}} + n_i \hat{\pi}_{j,2}^{*}, & t \in [0, \tau \wedge T], \\ 0, & t \in [\tau \wedge T, T], \end{cases} \\
\hat{q}_i^{*} = \dfrac{a_i \gamma_i + \rho b_i b_j m_i n_i e^{r(T-t)} \hat{q}_j^{*}}{b_i^2 m_i e^{r(T-t)}}, & t \in [0,T],
\end{cases}
$$

where

$$
\begin{aligned}
\widehat{C}_i(t) &= \ln\left(\frac{\delta}{\zeta h^P}\right) - \hat{g}_{i,1}(t) + \hat{g}_{i,0}(t), \\
\hat{g}_{i,1}(t) &= \int_t^T \Bigg\{ \Big[(\mu - r)(\hat{\pi}_{i,1}^{*} - n_i \hat{\pi}_{j,1}^{*}) + a_i \lambda_i - n_i a_j \lambda_j + a_i \gamma_i \hat{q}_i^{*} - n_i a_j \gamma_j \hat{q}_j^{*} \Big] \left(-m_i e^{r(T-s)}\right) \\
&\quad + \frac{1}{2} \Big[\sigma^2 (\hat{\pi}_{i,1}^{*})^2 + n_i^2 \sigma^2 (\hat{\pi}_{j,1}^{*})^2 + b_i^2 (\hat{q}_i^{*})^2 + n_i^2 b_j^2 (\hat{q}_j^{*})^2 \\
&\quad - 2 n_i \sigma^2 \hat{\pi}_{i,1}^{*} \hat{\pi}_{j,1}^{*} - 2 \rho n_i b_i b_j \hat{q}_i^{*} \hat{q}_j^{*} \Big] \left(m_i^2 e^{2r(T-s)} \right) \Bigg\} \mathrm{d}s, \\
\hat{g}_{i,0}(t) &= e^{-\frac{\delta}{\zeta}(T-t)} \int_t^T e^{\frac{\delta}{\zeta}(T-s)} \Bigg\{ \Big[(\mu - r)(\hat{\pi}_{i,1}^{*} - n_i \hat{\pi}_{j,1}^{*}) + \frac{\delta}{m_i \zeta e^{r(T-s)}} \left(\ln \frac{\delta}{\zeta h^P} - \hat{g}_{i,1}(s) \right) \\
&\quad + a_i \lambda_i - n_i a_j \lambda_j + a_i \gamma_i \hat{q}_i^{*} - n_i a_j \gamma_j \hat{q}_j^{*} \Big] \left(-m_i e^{r(T-s)}\right) \\
&\quad + \frac{1}{2} \Big[\sigma^2 (\hat{\pi}_{i,1}^{*})^2 + n_i^2 \sigma^2 (\hat{\pi}_{j,1}^{*})^2 + b_i^2 (\hat{q}_i^{*})^2 + n_i^2 b_j^2 (\hat{q}_j^{*})^2 \\
&\quad - 2 n_i \sigma^2 \hat{\pi}_{i,1}^{*} \hat{\pi}_{j,1}^{*} - 2 \rho n_i b_i b_j \hat{q}_i^{*} \hat{q}_j^{*} \Big] \left(m_i^2 e^{2r(T-s)} \right) + \frac{\delta}{\zeta} - h^P \Bigg\} \mathrm{d}s.
\end{aligned}
$$

The optimal value function is given by

$$
\widehat{V}_i(t, \hat{x}_i, z) = (1 - z) \widehat{W}_i(t, \hat{x}_i, 0) + z \widehat{W}_i(t, \hat{x}_i, 1), \qquad \textit{where } z = 0 \textit{ or } 1, \tag{2.4.31}
$$

with

$$
\widehat{W}_i(t, \hat{x}_i, 0) = -\frac{1}{m_i} \exp\left\{ -m_i \hat{x}_i e^{r(T-t)} + \hat{g}_{i,0}(t) \right\},
$$

and

$$
\widehat{W}_i(t, \hat{x}_i, 1) = -\frac{1}{m_i} \exp\left\{ -m_i \hat{x}_i e^{r(T-t)} + \hat{g}_{i,1}(t) \right\}.
$$

On the basis of Theorem 2.3.2 and Proposition 2.4.1, we obtain analytical solutions to the Nash equilibrium reinsurance and investment strategies for the following two situations in Corollary 2.4.1 and Corollary 2.4.2. In Corollary 2.4.1, insurer i is assumed to mistrust the approximation model and have robustness preferences for the diffusion risk and default risk, while his competitor insurer j is ambiguity-neutral and uses the reference probability measure to evaluate his expected terminal utility.

Corollary 2.4.1. *When insurer i is AAI while insurer j is ANI, for $i,j \in \{1,2\}, i \neq j$, the Nash equilibrium investment strategies are given by*

$$\begin{cases} \pi^*_{i,1}(t) = \dfrac{(\mu - r)(m_i n_i + m_j)}{(m_i + \beta_{i,1} - m_i n_i n_j)\sigma^2 e^{r(T-t)} m_j}, & t \in [0,T], \\ \pi^*_{j,1}(t) = \dfrac{(\mu - r)(m_j n_j + m_i + \beta_{i,1})}{(m_i + \beta_{i,1} - m_i n_i n_j)\sigma^2 e^{r(T-t)} m_j}, & t \in [0,T], \end{cases} \tag{2.4.32}$$

and

$$\begin{cases} \pi^*_{i,2}(t) = \left(\dfrac{m_i n_i \widehat{C}_j(t) + m_j C_i(t)}{(1 - n_i n_j) m_i m_j \zeta e^{r(T-t)}} \right)^+, & t \in [0, \tau \wedge T], \\ \pi^*_{j,2}(t) = \left(\dfrac{m_j n_j C_i(t) + m_i \widehat{C}_j(t)}{(1 - n_i n_j) m_i m_j \zeta e^{r(T-t)}} \right)^+, & t \in [0, \tau \wedge T], \\ \pi^*_{i,2}(t) = \pi^*_{j,2}(t) = 0, & t \in [\tau \wedge T, T], \end{cases}$$

with $\phi^*_{i,3}$ in $C_i(t)$ given by Proposition 2.3.1.

The Nash equilibrium reinsurance strategies are given by

$$\begin{cases} q^*_i(t) = \dfrac{a_i \gamma_i b_j m_j + \rho b_i m_i n_i a_j \gamma_j}{b_i^2 b_j m_j e^{r(T-t)} (m_i + \beta_{i,2} - \rho^2 m_i n_i n_j)}, & t \in [0,T], \\ q^*_j(t) = \dfrac{a_j \gamma_j b_i (m_i + \beta_{i,2}) + \rho a_i \gamma_i b_j m_j n_j}{b_i b_j^2 m_j e^{r(T-t)} (m_i + \beta_{i,2} - \rho^2 m_i n_i n_j)}, & t \in [0,T]. \end{cases} \tag{2.4.33}$$

The functions $\phi^*_{i,1}(t)$ and $\phi^*_{i,2}(t)$ are given by

$$\begin{cases} \phi^*_{i,1}(t) = \dfrac{\beta_{i,1}(\mu - r)(m_j + m_i n_i)}{\sigma m_j (m_i + \beta_{i,1} - m_i n_i n_j)}, & t \in [0,T], \\ \phi^*_{i,2}(t) = \dfrac{\beta_{i,2}(a_i \gamma_i b_j m_j + \rho b_i m_i n_i a_j \gamma_j)}{b_i b_j m_j (m_i + \beta_{i,2} - \rho^2 m_i n_i n_j)}, & t \in [0,T]. \end{cases} \tag{2.4.34}$$

The optimal value function of insurer i is given by (2.3.25), *and the optimal value function of insurer j is given by* (2.4.31).

Proof. For the proof, please see Appendix C. □

The following corollary derives the closed-form expressions for Nash equilibrium reinsurance and investment strategies when both insurers are completely convinced of the reference model.

Corollary 2.4.2. *When both insurers are ANIs, the Nash equilibrium investment strategies are given by*

$$\begin{cases} \pi_{1,1}^*(t) = \dfrac{(\mu - r)(m_1 n_1 + m_2)}{(1 - n_1 n_2)\sigma^2 e^{r(T-t)} m_1 m_2}, & t \in [0, T], \\ \pi_{2,1}^*(t) = \dfrac{(\mu - r)(m_2 n_2 + m_1)}{(1 - n_1 n_2)\sigma^2 e^{r(T-t)} m_1 m_2}, & t \in [0, T], \end{cases} \tag{2.4.35}$$

and

$$\begin{cases} \pi_{1,2}^*(t) = \left(\dfrac{m_1 n_1 \widehat{C}_2(t) + m_2 \widehat{C}_1(t)}{(1 - n_1 n_2) m_1 m_2 \zeta e^{r(T-t)}} \right)^+, & t \in [0, \tau \wedge T], \\ \pi_{2,2}^*(t) = \left(\dfrac{m_2 n_2 \widehat{C}_1(t) + m_1 \widehat{C}_2(t)}{(1 - n_1 n_2) m_1 m_2 \zeta e^{r(T-t)}} \right)^+, & t \in [0, \tau \wedge T], \\ \pi_{1,2}^*(t) = \pi_{2,2}^*(t) = 0, & t \in [\tau \wedge T, T]. \end{cases}$$

The Nash equilibrium reinsurance strategy is presented as

$$\begin{cases} q_1^*(t) = \dfrac{a_1 \gamma_1 b_2 m_2 + \rho b_1 m_1 n_1 a_2 \gamma_2}{b_1^2 m_1 e^{r(T-t)} b_2 m_2 (1 - \rho^2 n_1 n_2)}, & t \in [0, T], \\ q_2^*(t) = \dfrac{a_2 \gamma_2 b_1 m_1 + \rho b_2 m_2 n_2 a_1 \gamma_1}{b_2^2 m_2 e^{r(T-t)} b_1 m_1 (1 - \rho^2 n_1 n_2)}, & t \in [0, T]. \end{cases} \tag{2.4.36}$$

Furthermore, we require that $n_i = 1$ and $n_j = 1$ cannot hold at the same time to guarantee that $\pi_{i,1}^*(t)$ and $q_i^*(t)$ are positive. The optimal value function of insurer i is given by (2.4.31)*, for $i \in \{1, 2\}$.*

Proof. When both insurers are ANIs, the value functions and optimal reinsurance and investment strategies are given in Proposition 2.4.1. From (2.4.34), we find that the worst-case density generators $\phi_{i,j}^* = 0$ when $\beta_{i,j} = 0$, $j \in \{1, 2\}$. Then we can obtain (2.4.35) and (2.4.36) by setting $\beta_{i,j} = 0$ in (2.4.32) and (2.4.33), respectively. The Nash equilibrium bond investment strategy for the pre-default case can be easily obtained by solving the following system of equations:

$$\begin{cases} \pi_{i,2}^* = \dfrac{\widehat{C}_i(t)}{m_i \zeta e^{r(T-t)}} + n_i \pi_{j,2}^*, & t \in [0, \tau \wedge T], \\ \pi_{j,2}^* = \dfrac{\widehat{C}_j(t)}{m_j \zeta e^{r(T-t)}} + n_j \pi_{i,2}^*, & t \in [0, \tau \wedge T]. \end{cases}$$

Furthermore, we require that $n_i = 1$ and $n_j = 1$ cannot hold at the same time. Under this case, $n_1 n_2 \in [0, 1)$, and hence $1 - n_1 n_2 \in (0, 1]$, which implies that $\pi_{i,1}^*(t)$ in (2.4.35) is positive. Similarly, $\rho^2 \in [0, 1]$, and so $1 - \rho^2 n_1 n_2 \in (0, 1]$, and this leads to that $q_i^*(t)$ in (2.4.36) is positive. □

Remark 2.4.1. According to the expressions of Nash equilibrium reinsurance strategy in Theorem 2.3.2 and Corollary 2.4.1, we know that the robust optimal reinsurance strategy of insurer i decreases with respect to the coefficient $\beta_{i,2}$ which reveals the ambiguity aversion level for diffusion risk caused by claims. This property coincides with the intuition that an AAI who is more ambiguity-averse tends to purchase more reinsurance. This phenomenon has also been shown in Sun et al. (2017) and Li et al. (2018). However, our conclusions here are more elaborate than theirs because we find that the ambiguity aversion level of insurer i affects the optimal reinsurance strategy of his competitor insurer j, which is shown by (2.4.33) and (2.3.28). Similarly, we can analyze the effects of the ambiguity aversion coefficients on the Nash equilibrium investment strategies. Detailed study will be conducted in the next section by using numerical examples.

2.5 Numerical examples

In this section, we conduct some numerical experiments to provide sensitivity analyses for the Nash equilibrium reinsurance-investment strategy $\pi_i^*(t)$, $i \in \{1, 2\}$, in three different scenarios. In Case I we assume that both insurers are ANIs; in Case II one insurer is AAI, and we might as well assume that insurer 1 is AAI while insurer 2 is ANI; in Case III both insurers are AAIs. Without loss of generality, we assume the model parameters under current time, that is, $t = 0$. Unless otherwise specified, the basic model parameters are shown in Table 2.1.

Table 2.1: Model parameters

Common parameters								
t	T	r	μ	σ	ζ	h^P	δ	ρ
0	5	0.05	0.1	0.6	0.4	0.125	0.2	0.5
Insurer 1								
a_1	b_1	η_1	n_1	$\beta_{1,1}$	$\beta_{1,2}$	$\beta_{1,3}$	m_1	γ_1
50	16	0.15	0.2	0.2	0.4	0.6	0.3	0.5
Insurer 2								
a_2	b_2	η_2	n_2	$\beta_{2,1}$	$\beta_{2,2}$	$\beta_{2,3}$	m_2	γ_2
40	10	0.2	0.4	0.5	0.6	0.7	0.1	0.5

Figure 2.1 depicts the effects of relative performance parameters n_i, $i \in \{1, 2\}$, on the optimal investment strategies $\pi_{i,1}^*(0)$ in three cases, respectively. We find

that $\pi^*_{i,1}(0)$ is an increasing function of n_i. That is to say, the more concerned about outperforming his competitor, the more wealth that insurer i would like to invest in the stock market. As a result, the probability of accumulating greater wealth than his competitor at terminal time T would be enhanced. In particular, $n_i = 0$ represents the absence of relative performance concerns, then the game is simplified as a single-agent optimization problem. We find that the competition makes each insurer more risk-seeking, which is reflected by the increased exposure on risky asset. For the same level of n_i, a larger n_j implies that insurer i would face a higher competition intensity, and this also leads insurer i to hold more stock shares. Additionally, the ranges of $\pi^*_{i,1}(0)$ in three cases decrease gradually in Figure 2.1. We conclude that the insurer's ambiguity aversion attitudes make both players in the game more conservative to risks and reduced amount of surplus would be invested in the stock comparing with the results in Case I, even when only one insurer is AAI, as shown in Figures 2.1(c) and 2.1(d). This also implies that the consideration of model uncertainty can offset the effects of relative performance parameters on equilibrium stock investment strategies $\pi^*_{i,1}(0)$.

Figure 2.2 displays the effects of n_i on the optimal reinsurance strategies $q^*_i(0)$ at equilibrium in three different cases. From the subfigures, we observe that $q^*_i(0)$ increases as n_i increases and $q^*_i(0)$ is a linear increasing function of n_i when $n_j = 0$. Additionally, we can see that the insurer with relative performance concerns becomes less risk-averse than that without competition (i.e., $n_i = 0$), and the insurer tends to increase his respective retention level. We also find $q^*_2(0) > 1$ in Figures 2.2(b) and 2.2(d), which implies that the second insurer would act as a reinsurer for some insurance companies in Cases I and II. However, $q^*_2(0) < 1$ in Case III when both insurers have an ambiguity aversion attitude. In all subfigures, the ranges of $q^*_i(0)$ for three different cases decrease gradually, which shows that the player's ambiguity aversion attitudes in the game make both insurers more conservative to risks and hence they tend to transfer more risks to the reinsurer. For insurer 1, the retained proportion of the claims increases with the rising value of n_1. A reasonable explanation for this is that a larger n_1 indicates the insurer cares more about its relative terminal wealth, and hence he tends to retain more risks, so less reinsurance premium would be paid out and more capital could be obtained at terminal time. For a fixed n_1, a higher competition intensity will cause insurer 1 to bear more risk exposure in the insurance market and purchase less reinsurance contracts. Finally, the acceleration speed of $q_i^*(0)$ for each insurer increasingly declines in these six subfigures, which illustrates that the insurer's robustness preference plays a more important role than the relative performance parameter in his equilibrium reinsurance strategy.

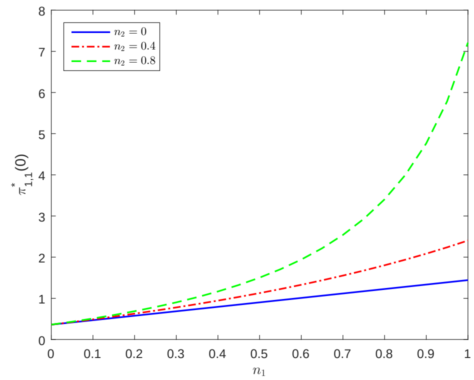
Figure 2.3 demonstrates how the equilibrium pre-default bond investment strategies $\pi_{i,2}^*(0)$ vary with respect to the competition parameter n_i. From these subfigures, we find that $\pi_{i,2}^*(0)$ is an increasing function of n_i in all three cases. In other words, if an insurer is more concerned about his competitor's performance at terminal time, he would choose riskier investment strategy, i.e., investing more in the corporate bond. In addition, for a fixed n_i, a greater n_j produces a larger $\pi_{i,2}^*(0)$, which implies that insurer i would invest more surplus in the corporate bond when his competitor is aggressive. Similar to the analysis for $\pi_{i,1}^*(0)$ and $q_i^*(0)$, we find that the variation rate of the curves in Case III is the smallest, and that of Case II is less than Case I. This is because the consideration of model ambiguity makes the insurer have less confidence in the reference model and he/she would like to select a more conservative and cautious investment strategy.

From Figure 2.4, we can see the changes of the equilibrium bond investment strategies $\pi_{i,2}^*(0)$ in the pre-default case under different loss rate ζ. It is not surprising to find a negative relationship between the optimal investment strategies of the defaultable bond $\pi_{i,2}^*(0)$ and the loss rate ζ since a higher loss rate induces a lower recovery amount, which implies a higher potential loss of the insurer. Moreover, for a fixed level of ζ, a higher default risk premium $1/\Delta$ leads to a higher investment amount in the defaultable bond. This result is intuitive because a greater default risk premium produces a higher potential yield. The ranges of $\pi_{i,2}^*(0)$ reduce gradually in these three cases because insurer's model ambiguity consideration makes both players in the game more conservative to financial risks. The curves in Figures 2.4(c), 2.4(e) and 2.4(f) are truncated by $\pi_{i,2}^*(0) = 0$, which indicates that the insurer would short sell the defaultable bond. In Case III, when both insurers are AAIs, $\pi_{i,2}^*(0)$ shows relatively less sensitivity with respect to the default risk premium $1/\Delta$. That is to say, the deviation in $1/\Delta$ would not largely change the insurers' defaultable bond investment strategies.

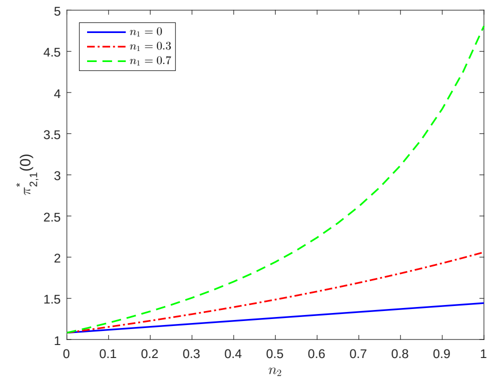
Figures 2.5-2.7 reveal the equilibrium reinsurance and investment strategies as decreasing functions of the ambiguity aversion coefficients. $\beta_{i,1}$ represents the ambiguity aversion coefficient of insurer i with respect to the diffusion risk of stocks. From Figure 2.5, we can see that the insurer with a higher level of $\beta_{i,1}$ would invest less amount in the stock market. Intuitively, this conclusion is reasonable because the insurer would not invest a great deal of money in such assets that he has less information or confidence (higher $\beta_{i,1}$) so as to mitigate financial risks. $\beta_{i,2}$ indicates the ambiguity aversion level of insurer i for the default risk. The results in Figure 2.6 also coincide with our intuition in the sense that the insurer is prone to investing less amount of his surplus in the defaultable bond when he faces more model uncertainty (larger $\beta_{i,2}$) of the bond price. $\beta_{i,3}$ denotes ambiguity aversion coefficient of insurer i with respect to the diffusion risk for claims. As shown in Figure 2.7, the insurer with a higher ambiguity aversion level tends to purchase more reinsurance to spread the underlying insurance business risks. Furthermore, for a fixed ambiguity aversion level, the equilibrium reinsurance and investment strategies decrease with the growth of m_i. Note that the greater m_i is, the more risk-averse the insurer i is, so he/she tends to reduce the amount invested in the stock market and defaultable corporate bond market, and purchase more reinsurance treaties to avoid risks.

2.6 Conclusion

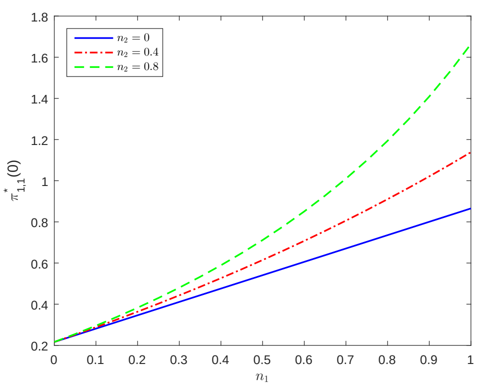
In this chapter, we study a class of non-zero-sum investment and reinsurance games subject to default risk between two insurers who are faced with model misspecification or model uncertainty by developing robust optimal strategies. Specifically, we allow each insurer to dynamically purchase proportional reinsurance protection and allocate his surplus to a financial market consisting of a risk-free asset, a risky asset and a defaultable corporate bond. Applying the stochastic dynamic programming method, we derive the HJBI equations for pre-default and post-default cases. Explicit expressions for the robust equilibrium investment and reinsurance strategies that maximize the expected exponential



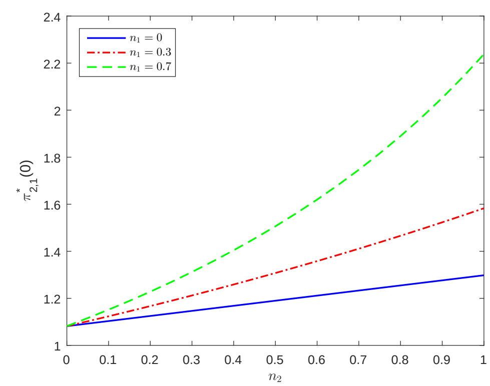
(a) n_1 on $\pi^*_{1,1}(0)$ in Case I



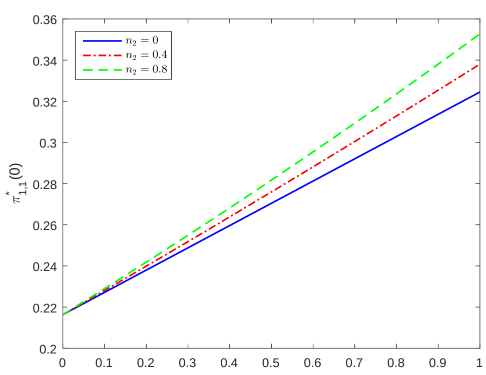
(b) n_2 on $\pi^*_{2,1}(0)$ in Case I



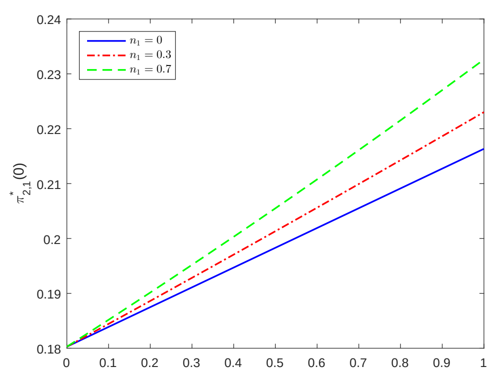
(c) n_1 on $\pi^*_{1,1}(0)$ in Case II



(d) n_2 on $\pi^*_{2,1}(0)$ in Case II

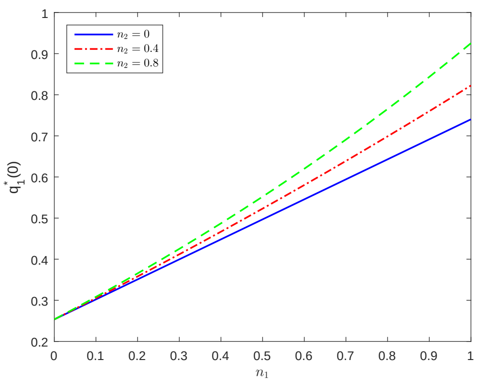


(e) n_1 on $\pi^*_{1,1}(0)$ in Case III

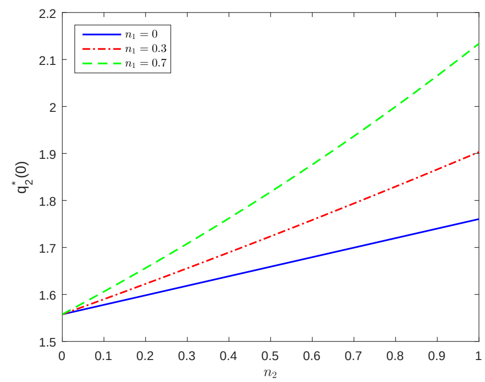


(f) n_2 on $\pi^*_{2,1}(0)$ in Case III

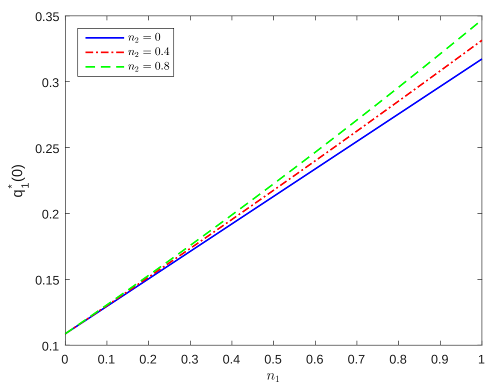
Figure 2.1: Effects of n_i on $\pi^*_{i,1}(0)$ in three cases, for $i \in \{1, 2\}$.



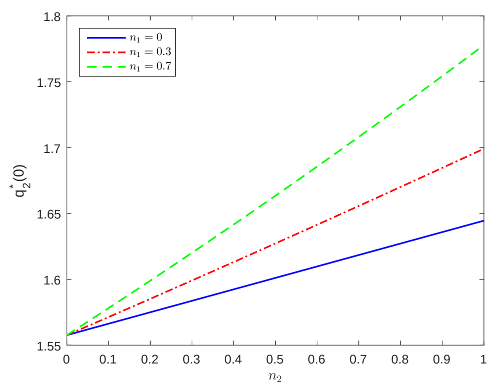
(a) n_1 on $q_1^*(0)$ in Case I



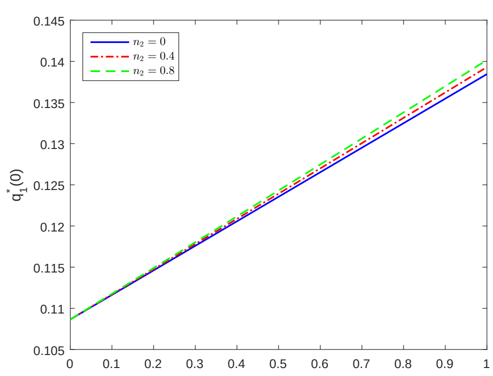
(b) n_2 on $q_2^*(0)$ in Case I



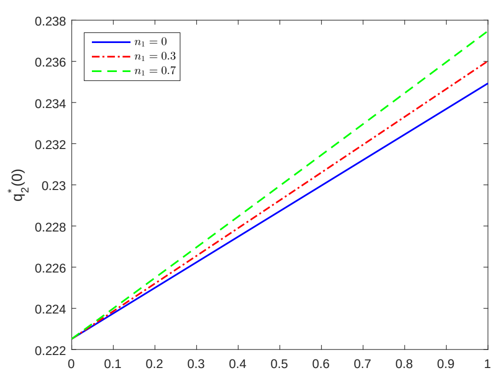
(c) n_1 on $q_1^*(0)$ in Case II



(d) n_2 on $q_2^*(0)$ in Case II

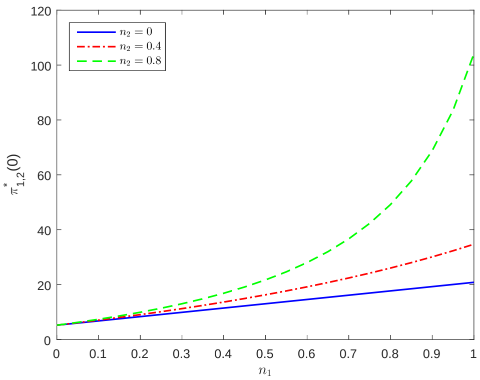


(e) n_1 on $q_1^*(0)$ in Case III

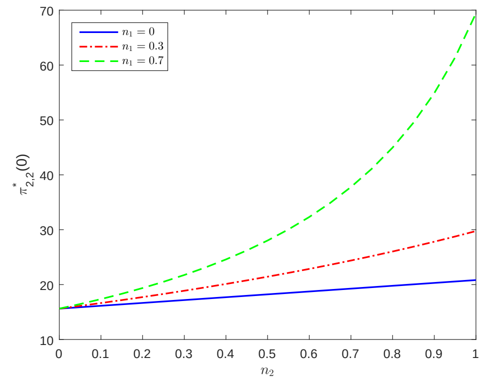


(f) n_2 on $q_2^*(0)$ in Case III

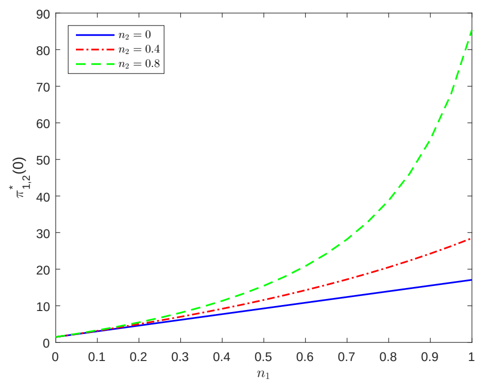
Figure 2.2: Effects of n_i on $q_i^*(0)$ in three cases, for $i \in \{1, 2\}$.



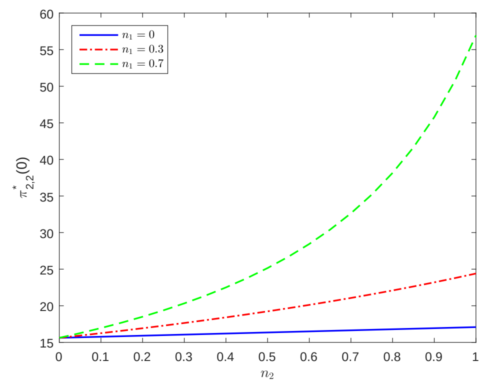
(a) n_1 on $\pi^*_{1,2}(0)$ in Case I



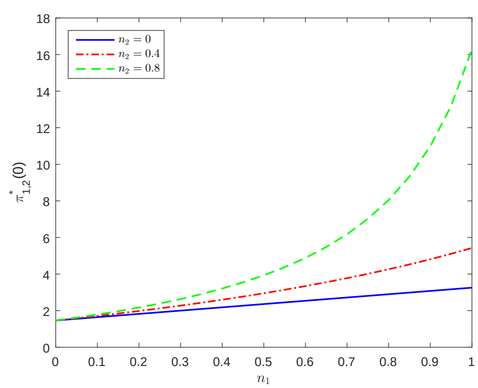
(b) n_2 on $\pi^*_{2,2}(0)$ in Case I



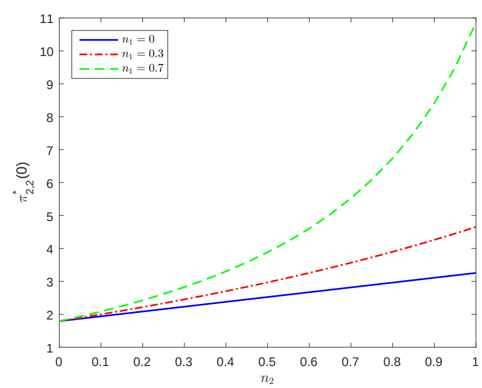
(c) n_1 on $\pi^*_{1,2}(0)$ in Case II



(d) n_2 on $\pi^*_{2,2}(0)$ in Case II

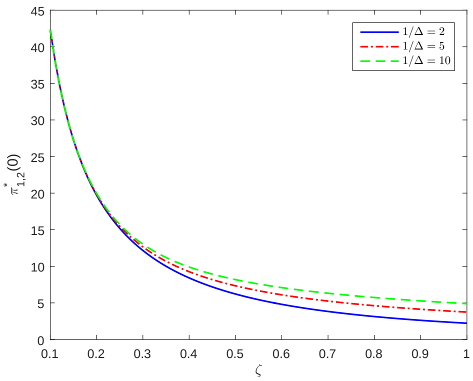


(e) n_1 on $\pi^*_{1,2}(0)$ in Case III

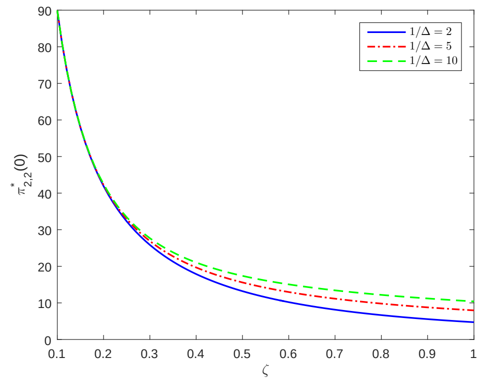


(f) n_2 on $\pi^*_{2,2}(0)$ in Case III

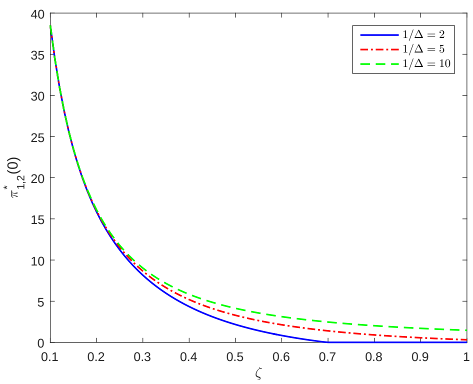
Figure 2.3: Effects of n_i on $\pi^*_{i,2}(0)$ in three cases, for $i \in \{1, 2\}$.



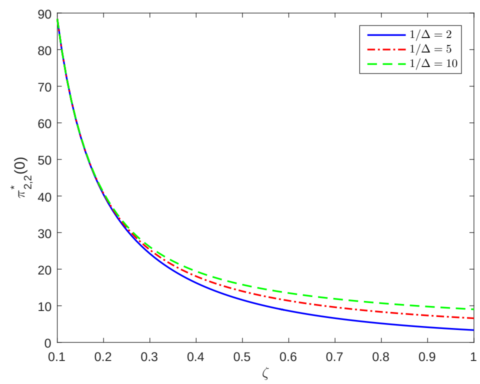
(a) ζ on $\pi^*_{1,2}(0)$ in Case I



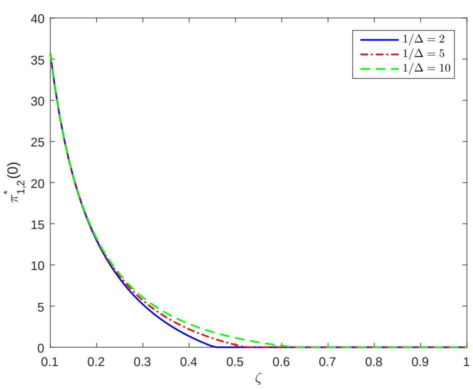
(b) ζ on $\pi^*_{2,2}(0)$ in Case I



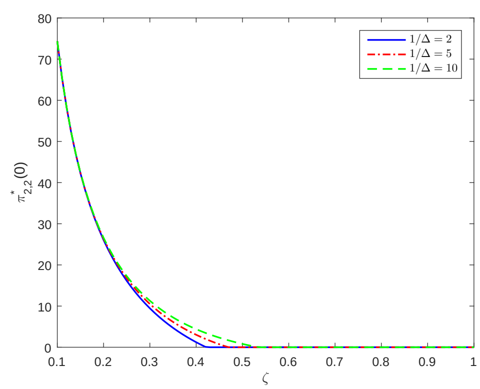
(c) ζ on $\pi^*_{1,2}(0)$ in Case II



(d) ζ on $\pi^*_{2,2}(0)$ in Case II



(e) ζ on $\pi^*_{1,2}(0)$ in Case III



(f) ζ on $\pi^*_{2,2}(0)$ in Case III

Figure 2.4: Effects of ζ on $\pi^*_{i,2}(0)$ in three cases, for $i \in \{1, 2\}$.

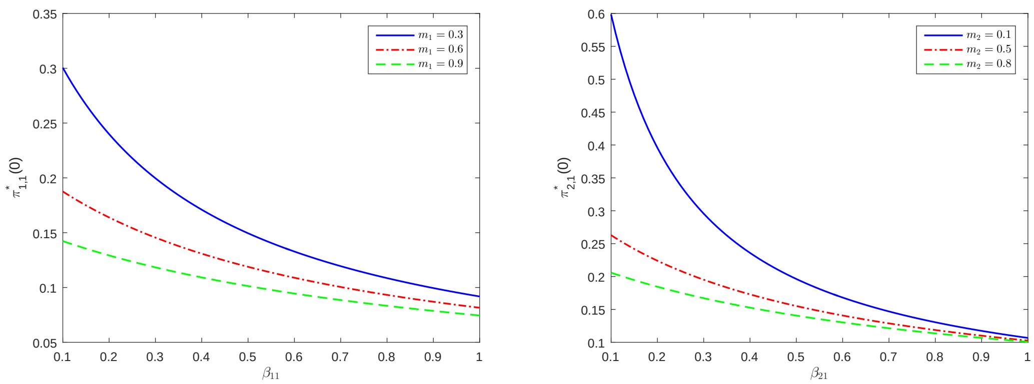


Figure 2.5: Effects of $\beta_{i,1}$ on $\pi^*_{i,1}(0)$ in Case III, for $i \in \{1, 2\}$.

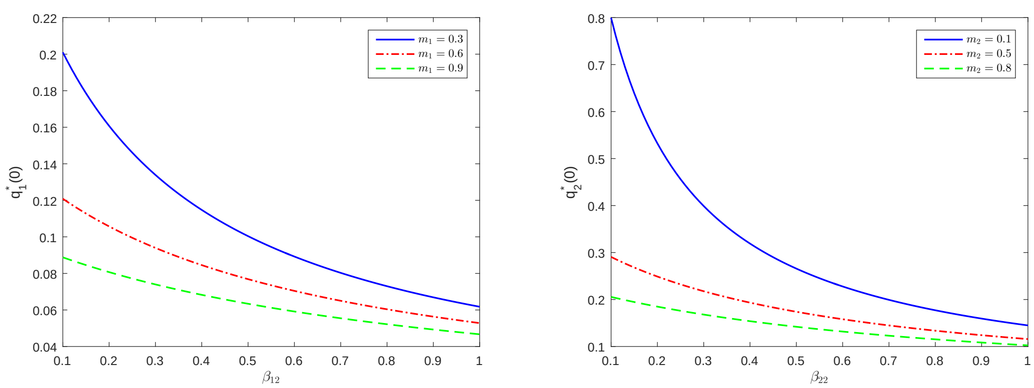


Figure 2.6: Effects of $\beta_{i,2}$ on $q^*_i(0)$ in Case III, for $i \in \{1, 2\}$.

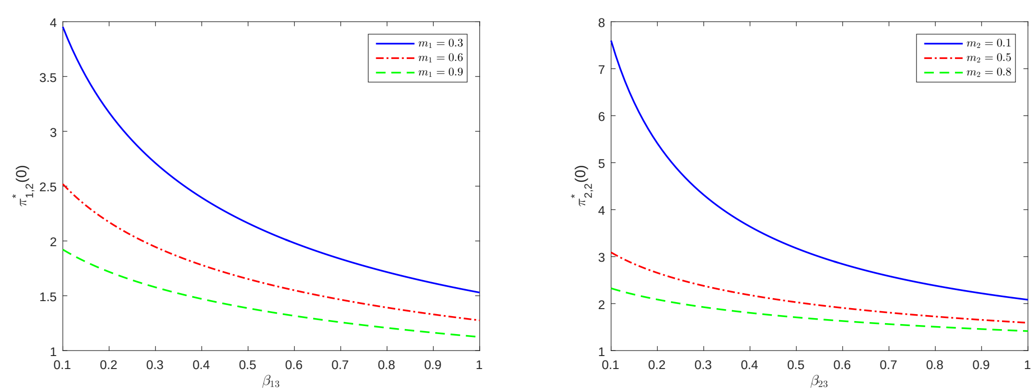


Figure 2.7: Effects of $\beta_{i,3}$ on $\pi^*_{i,2}(0)$ in Case III, for $i \in \{1, 2\}$.

utility of the terminal wealth relative to that of his competitor and corresponding optimal value functions are obtained. We also provide some special cases of our model and explore the economic implications from numerical examples. Results indicate that the relative performance concerns of the insurer increase the retained proportional of claims and the amount invested in the risky asset and defaultable bond, which implies that the competition would lead the insurers to be much more risk-seeking. Besides, the equilibrium strategies of an AAI are significantly affected by his attitudes towards model uncertainty. The AAI would choose more conservative strategies than the ANI, which is reflected in transferring more risks through reinsurance contracts and reducing the risky-asset and defaultable bond investment. We also find that ambiguity aversion attitudes can offset certain effects of the competition. Last but not least, the insurer's optimal strategies are influenced by his competitor's attitudes towards model uncertainty. That is to say, the strategies of two AAIs in a game would be more conservative than these of a game consisting one AAI and one ANI. And not surprisingly, two ANIs in a game would have the least conservative strategies.

Chapter 3

Time-consistent Reinsurance-investment Games under Model Uncertainty

3.1 Introduction

In practice, insurance companies can purchase reinsurance contracts to transfer parts of the underwriting risk to a reinsurer and invest in the financial markets to increase their profits. The investigation of the insurer's optimal investment and reinsurance strategies has attracted considerable attention in the area of actuarial science. For example, Bai and Guo (2008) considered the optimal investment-reinsurance problem by maximizing the expected exponential utility of the insurer's terminal wealth under the no-shorting constraint. Shen and Zeng (2015) applied the mean-variance criterion to derive the insurer's optimal reinsurance and investment strategies. Zhang et al. (2016) aimed to seek the optimal reinsurance and investment strategies by minimizing the probability of ruin, where the reinsurance premium was determined by the generalized mean-variance principle. For more literature, the readers may refer to Gu et al. (2012), Guan and Liang (2014b), Zhao et al. (2016) and Sun and Guo (2018), just to name a few.

As was shown in Ellsberg (1961), decision-makers were not only risk-averse but also ambiguity-averse. In many situations, the parameters, especially the drift parameters, are difficult to estimate with precision, and thus it is reasonable to assume that the decision-maker concerns about model misspecification. In the literature, one popular approach to describe model ambiguity was proposed in Anderson et al. (2003) which studied asset pricing problems in stochastic continuous-time settings by incorporating the investor's consideration of model misspecification. Under their assumption, the investor regarded the specific probability measure as his reference measure and could then find robust strategies that worked over the nearby measures known as alternative measures. Since then, due to its analytical tractability, the formulation of the robust optimization procedures conducted in Anderson et al. (2003) has been adopted in portfolio selection, asset pricing and optimal reinsurance-investment problems. For example, Maenhout (2004) obtained the robust optimal portfolio decision for an investor with ambiguity aversion attitudes. Pun and Wong (2015) discussed a robust optimal reinsurance-investment problem for an ambiguity-averse insurer (AAI) under a general class of utility functions when the risky asset followed a multiscale stochastic volatility (SV) model. Zeng et al. (2016) derived optimal time-consistent investment and proportional reinsurance strategies for an AAI under mean-variance criterion. They assumed that the surplus of the AAI followed Cramér-Lundberg model and the price of the risky asset could be characterized by a jump-diffusion process. Li et al. (2018) articulated the optimal investment and excess-of-loss reinsurance problem for an AAI who was concerned about ambiguity with respect to the diffusion and jump components arising from the financial and insurance markets. Gu et al. (2018) investigated a robust optimal investment and proportional reinsurance problem for an AAI who could invest his surplus into one risk-free asset, one market index and a pair of mispriced stocks. Liu and Zhou (2020) determined the robust optimal investment strategy for an individual who concerned about drift misspecification by minimizing the probability of lifetime ruin.

The aforementioned literature is devoted to developing optimal reinsurance and investment strategy for a single representative insurer. However, in reality, different insurance companies are usually closely related and they tend to compare themselves with their peers. There also exists literature showing that relative concerns play an important role in human behaviors (see, for example, Gómez (2007) and DeMarzo et al. (2008)). Therefore, it is meaningful to investigate the reinsurance-investment strategy selection problems considering the interactions between different insurers. Some researchers formulated the problem into a zero-sum stochastic differential game, for example, Browne (2000) proposed a zero-sum stochastic differential portfolio game between two investors. Mataramvura and Øksendal (2008) studied a zero-sum portfolio game between an agent and a market by minimizing convex risk measures in a Lévy market. Liu and Yiu (2013) studied a zero-sum stochastic differential reinsurance and investment game for two competing insurance companies, and they imposed constant VaR constraints for the purpose of risk management. Xu et al. (2014) investigated a zero-sum investment game under regime switching framework. On the other hand, some works studied the relative performance concerns and formulated non-zero-sum game problems. Along this direction, Bensoussan et al. (2014) applied dynamic programming techniques to consider a non-zero-sum reinsurance and investment game under the regime-switching framework. Siu et al. (2017) extended the results obtained in Bensoussan et al. (2014) by allowing the insurers to purchase an excess-of-loss reinsurance contract and invest in the risky asset whose price was described by Heston SV model. Moreover, Siu et al. (2017) studied the effects of systematic risks on the equilibrium reinsurance strategy. Subsequently, Deng et al. (2018) studied a non-zero-sum stochastic differential reinsurance-investment game between two competitive constant absolute risk aversion (CARA) insurers, and their investment options included a risk-free bond, a risky asset with Heston's SV model and a defaultable corporate zero-coupon bond. Hu and Wang (2018) investigated a class of non-zero-sum reinsurance and investment games under time-consistent mean-variance criterion. Zhu et al. (2020) discussed a time-consistent non-zero-sum stochastic differential reinsurance and investment game between two insurers who were faced with insurance risk, volatility risk and default risk. Some attempts have also been made in addressing the robust game problems. For instance, Zhang and Siu (2009) considered an optimal reinsurance-investment problem in the presence of model uncertainty and formulated the problem into a zero-sum stochastic differential game between the insurer and the market. Pun and Wong (2016) explored the non-zero-sum stochastic differential game between two competitive AAIs who aimed to seek the robust optimal proportional reinsurance strategies by maximizing the expected utility of the terminal surplus relative to that of his competitor. Pun et al. (2016) studied the economic implications of ambiguous correlation in a non-zero-sum game between two insurers. Wang et al. (2019a) formulated a class of non-zero-sum reinsurance and investment games between two AAIs who faced default risk and explicit expressions for Nash equilibrium investment and reinsurance strategies were established.

It appears that there is less literature studying the robust non-zero-sum stochastic differential game between two AAIs under the mean-variance criterion. In this chapter, we aim to fill this gap and investigate how ambiguity aversion and relative performance concerns impact the equilibrium reinsurance and investment strategies. More specifically, we suppose that each insurer has the choice to purchase a proportional reinsurance contract and invest in one risk-free asset and one risky asset. The surplus process of each insurer is assumed to follow the classical Cramér-Lundberg model. Inspired by Siu et al. (2017), we assume that the insurers face both idiosyncratic and systematic jump risks. In regard to the objective function, we incorporate relative performance and model ambiguity concerns into mean-variance criterion. In the traditional mean-variance optimization problems, most of the literature obtained the pre-commitment strategies which were time-inconsistent and only optimal at the initial time. However, time-consistency of the optimal strategies is a basic requirement for a rational decision-maker. On account of this opinion, in this chapter we follow the approach proposed in Björk et al. (2014) and Kronborg and Steffensen (2015) in order to obtain time-consistent reinsurance-investment strategy. In fact, this approach tackles the problem within a non-cooperative game theoretic framework, where the players are the future incarnations of the decision-maker. In the contexts of insurance, this approach was applied by many researchers, see, for example, Li et al. (2015a), Lin and Qian (2016) and Li et al. (2017). More recently, Pun (2018) established a general and tractable framework for stochastic control problems when model uncertainty was incorporated with time-inconsistent preference. Therefore, the formulation in this chapter induces each insurer not only to compete with himself but also with his competitor under the worst-case scenario of the alternative measures. The non-cooperative game with himself stems from the approach applied to obtain time-consistent reinsurance and investment strategies under the mean-variance criterion and the game with the competing insurer derives from relative performance consideration. Employing the Hamilton-Jacobi-Bellman (HJB) dynamic programming principle in stochastic optimal control theory, we obtain the robust equilibrium reinsurance and investment strategies as well as the corresponding equilibrium value functions.

Compared with the existing literature, the main contributions of this chapter are summarized as follows. First, we extend the robust optimal reinsurance-investment problem under the mean-variance criterion in Zeng et al. (2016), where only a single insurer was considered, to a continuous-time game framework by utilization of relative performance concerns. The key reason for considering strategic interaction between two insurers is that there always exist several competitors in the insurance market, and they often assess their performance against that of their competitors. Therefore, we derive Nash equilibrium investment and reinsurance strategies of a non-zero-sum game in this chapter. Numerical examples show that the competition makes each insurer more risk-seeking compared with the case without competition because they would increase their exposure on the risky asset and their respective retention levels of the claims. Second, the effects of ambiguity aversion on the optimal reinsurance and investment strategies under a non-zero-sum stochastic differential game framework are investigated, which was not considered in Siu et al. (2017) or Hu and Wang (2018), although these two papers formulated non-zero-sum games under expected utility maximization criterion and mean-variance criterion, respectively. Our numerical experiments show that an AAI would prefer more conservative investment and reinsurance strategies than an ambiguity-neutral insurer (ANI), which is reflected in the reduction of retention level of insurance risks and the amount invested in the risky asset.

The remainder of this chapter is organized as follows. Section 3.2 formulates a non-zero-sum stochastic differential reinsurance and investment game between two AAIs under the mean-variance criterion. In Section 3.3, we derive the extended HJB equation and present the time-consistent equilibrium reinsurance and investment strategies under the classic Cramér-Lundberg model. Section 3.4 provides the results under the diffusion-approximated model. In Section 3.5, we carry out some numerical examples to illustrate the effects of some important parameters on the time-consistent equilibrium reinsurance and investment strategies. Finally, we provide some concluding remarks in Section 3.6.

3.2 Model formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a complete filtered probability space, where the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is right continuous and $\mathbb{P}$-complete; $[0,T]$ is a fixed time horizon for investment and reinsurance. In what follows, all stochastic processes are assumed to be adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$.

We consider an insurance market consisting of two competing insurance companies, for simplicity, referred to as insurer 1 and insurer 2, whose surplus processes $S_k(t), k \in \{1,2\}$, are described by the classical Cramér-Lundberg model:

$$S_k(t) = x_k + p_k t - \sum_{i=1}^{N_k(t)+N(t)} Z_{ki}, \qquad k \in \{1,2\}, \tag{3.2.1}$$

where $x_k \geq 0$ is the initial surplus, $p_k > 0$ is the constant insurance premium rate, $\sum_{i=1}^{N_k(t)+N(t)} Z_{ki}$ is a compound Poisson process, $N_k(t)$ and $N(t)$ are mutually independent and homogeneous Poisson processes with constant intensities λ_k and λ, respectively. Hence, Poisson process $N_k(t) + N(t)$ has an intensity $\lambda_k + \lambda$, and this process denotes the number of claims received by insurer k up to time t. Note that each insurer faces both a common systematic insurance risk denoted by $N(t)$ and his idiosyncratic insurance risk represented as $N_k(t)$, and this assumption captures the dependence between these two insurers through a common Poisson process $N(t)$. The claim sizes $\{Z_{ki}\}_{i\in\mathbb{N}}$ are independent and identically distributed (i.i.d.) random variables that are independent of $N_k(t)$ and $N(t)$. Furthermore, we assume that $\{Z_{ki}\}_{i\in\mathbb{N}}$ have finite mean $\mu_k := \mathbb{E}^{\mathbb{P}}[Z_{ki}]$ and second moment $\sigma_k^2 := \mathbb{E}^{\mathbb{P}}[Z_{ki}^2]$, where $\mathbb{E}^{\mathbb{P}}[\,\cdot\,]$ denotes the expectation under probability measure $\mathbb{P}$. The premium rate of each insurer is assumed to be calculated according to the expected value premium principle, that is, $p_k = \mu_k(\lambda_k + \lambda)(1+\theta_k)$, where $\theta_k > 0$ is the relative safety loading factor of insurer k.

We further suppose that both insurers can purchase proportional reinsurance contracts or acquire new business to manage insurance business risks. Denote by $q_k(t) : [0,T] \to [0,\infty)$ the value of risk exposure for insurer k at time t. When $q_k(t) \in [0,1]$, it means that the reinsurer would indemnify the insurer

$100(1-q_k(t))\%$ of each claim occurring at time t, and hence the insurer's risk exposure reduces to $100q_k(t)\%$. In this case, insurer k must allocate parts of his premium rate $p_k^q(t)$ to the reinsurer. Here, $p_k^q(t)$ is also assumed, for simplicity, to be determined by the expected value premium principle, i.e.,

$$p_k^q(t) = \mu_k(\lambda_k + \lambda)(1+\eta_k)(1-q_k(t)),$$

where $\eta_k \geq \theta_k$ is the safety loading of the reinsurer. For simplicity, we call $\{q_k(t)\}_{t\in[0,T]}$, for $k \in \{1,2\}$, a reinsurance strategy of insurer k henceforth. Under this reinsurance treaty, the surplus process of insurer k becomes

$$\mathrm{d}U_k(t) = \mu_k(\lambda_k+\lambda)\left[\gamma_k + (1+\eta_k)q_k(t)\right]\mathrm{d}t - \mathrm{d}\sum_{i=1}^{N_k(t)+N(t)} q_k(t)Z_{ki}, \qquad k \in \{1,2\}, \tag{3.2.2}$$

where $\gamma_k = \theta_k - \eta_k \leq 0$.

In addition, we assume that both insurers have access to a financial market consisting of one risk-free asset and one risky asset. The price process $S_0(t)$ of the risk-free asset is given by the following ordinary differential equation (ODE):

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t,$$

where $r > 0$ is the constant risk-free interest rate. The price process $S_1(t)$ of the risky asset follows a geometric Brownian motion (GBM):

$$\mathrm{d}S_1(t) = S_1(t)\left[\mu\mathrm{d}t + \sigma\mathrm{d}B(t)\right],$$

where $\mu > r$ denotes the appreciation rate and $\sigma > 0$ is the volatility of the risky asset, $\{B(t)\}_{t\in[0,T]}$ is a standard one-dimensional Brownian motion under probability measure $\mathbb{P}$, and we assume that $\{B(t)\}_{t\in[0,T]}$ and $\sum_{i=1}^{N_k(t)+N(t)} Z_{ki}$ are independent.

Next, $\forall\ t \in [0,T]$, $k \in \{1,2\}$, we use Poisson random measures $N_k(\cdot,\cdot)$ and $N(\cdot,\cdot)$ on $\Omega \times [0,T] \times [0,\infty)$ to represent the compound Poisson process $\sum_{i=1}^{N_k(t)+N(t)} Z_{ki}$ as

$$\sum_{i=1}^{N_k(t)+N(t)} Z_{ki} = \int_0^t \int_0^\infty z_k N_k(\mathrm{d}s, \mathrm{d}z_k) + \int_0^t \int_0^\infty z_k N(\mathrm{d}s, \mathrm{d}z_k).$$

If we denote by $\nu_k(\mathrm{d}t, \mathrm{d}z_k) = \lambda_k \mathrm{d}t \mathrm{d}F(z_k)$ and $\nu(\mathrm{d}t, \mathrm{d}z_k) = \lambda \mathrm{d}t \mathrm{d}F(z_k)$, then we have, $\forall\ t \in [0, T]$, $k \in \{1, 2\}$,

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{N_k(t)+N(t)} Z_{ki}\right] = \int_0^t \int_0^\infty z_k \nu_k(\mathrm{d}s, \mathrm{d}z_k) + \int_0^t \int_0^\infty z_k \nu(\mathrm{d}s, \mathrm{d}z_k),$$

where $\nu_k(\cdot, \cdot)$ and $\nu(\cdot, \cdot)$ are compensators of the Poisson random measures $N_k(\cdot, \cdot)$ and $N(\cdot, \cdot)$, respectively. Therefore, the relationship between the compensated measures $\widehat{N}_k(\cdot, \cdot) = N_k(\cdot, \cdot) - \nu_k(\cdot, \cdot)$ and $\widehat{N}(\cdot, \cdot) = N(\cdot, \cdot) - \nu(\cdot, \cdot)$ and the compound Poisson process $\sum_{i=1}^{N_k(t)+N(t)} Z_{ki}$ is given as follows:

$$\begin{aligned}
&\int_0^t \int_0^\infty z_k \widehat{N}_k(\mathrm{d}s, \mathrm{d}z_k) + \int_0^t \int_0^\infty z_k \widehat{N}(\mathrm{d}s, \mathrm{d}z_k) \\
&= \sum_{i=1}^{N_k(t)+N(t)} Z_{ki} - \mathbb{E}^{\mathbb{P}}\left[\sum_{i=1}^{N_k(t)+N(t)} Z_{ki}\right], \qquad \forall\ t \in [0, T], \qquad k \in \{1, 2\}.
\end{aligned}$$

Denote $\pi_k(t)$ as the dollar amount invested by insurer k in the risky asset at time t. The remainder of his surplus, $X_k^{u_k}(t) - \pi_k(t)$, is invested in the risk-free asset, where $X_k^{u_k}(t)$ is the surplus process of insurer k controlled by the reinsurance-investment strategy $u_k(t) := (q_k(t), \pi_k(t))$, $\forall\ t \in [0, T]$. Hence, the surplus process $\{X_k^{u_k}(t)\}_{t \in [0,T]}$ of insurer k follows

$$\begin{aligned}
\mathrm{d}X_k^{u_k}(t) = &\left[rX_k^{u_k}(t) + (\mu - r)\pi_k(t) + \mu_k(\lambda_k + \lambda)(\gamma_k + (1 + \eta_k)q_k(t))\right] \mathrm{d}t \\
&+ \sigma\pi_k(t)\mathrm{d}B(t) - \int_0^\infty q_k(t) z_k N_k(\mathrm{d}t, \mathrm{d}z_k) - \int_0^\infty q_k(t) z_k N(\mathrm{d}t, \mathrm{d}z_k),
\end{aligned} \tag{3.2.3}$$

where $X_k^{u_k}(0) = x_k^0$ is the initial surplus of insurer k.

In most of the literature on the reinsurance-investment optimization problems, the insurers are assumed to completely believe in the models describing the real-world probability. However, it should be noted that there exist many uncertainties in the financial markets and insurance industries. It may be questioned if a real-world probability measure is given in the optimization models. Even if the model has been selected appropriately, the parameters on it are very difficult to estimate accurately. So it is of interest to investigate how the insurers having ambiguity aversion attitudes make their decisions in investment and reinsurance opportunities. In this current chapter, we take model uncertainty or ambiguity into account, which implies that we consider ambiguity-averse insurers (AAIs) instead of ambiguity-neutral insurers (ANIs). From the perspectives of an AAI, probability measure $\mathbb{P}$ is only a reference measure and he would like to consider plausible alternative probability measures. We define a class of probability measures which are equivalent to $\mathbb{P}$. That is,

$$\mathcal{Q} := \{\mathbb{Q} | \mathbb{Q} \sim \mathbb{P}\}.$$

Define, for each $k \in \{1, 2\}$, an exponential process $\{\Lambda^{\phi_k}(t)\}_{t\in[0,T]}$ by

$$\begin{aligned}\Lambda^{\phi_k}(t) = \exp\Bigg\{ & \int_0^t \phi_{k1}(s)\mathrm{d}B(s) - \frac{1}{2}\int_0^t \phi_{k1}^2(s)\mathrm{d}s \\ & + \int_0^t\int_0^\infty \ln\phi_{k2}(s) N_k(\mathrm{d}s, \mathrm{d}z_k) + \int_0^t\int_0^\infty (1-\phi_{k2}(s))\nu_k(\mathrm{d}s, \mathrm{d}z_k) \\ & + \int_0^t\int_0^\infty \ln\phi_{k3}(s) N(\mathrm{d}s, \mathrm{d}z_k) + \int_0^t\int_0^\infty (1-\phi_{k3}(s))\nu(\mathrm{d}s, \mathrm{d}z_k)\Bigg\},\end{aligned} \tag{3.2.4}$$

where $\phi_k(t) := (\phi_{k1}(t), \phi_{k2}(t), \phi_{k3}(t))$ is a measurable real-valued process.

Assumption 3.2.1. *Suppose that, for each $k \in \{1, 2\}$, the density generator process $\phi_k(t) := (\phi_{k1}(t), \phi_{k2}(t), \phi_{k3}(t))$ satisfies the following three conditions:*

(i) $\{\phi_k(t)\}_{t\in[0,T]}$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted;

(ii) $\phi_{k2}(t), \phi_{k3}(t) > 0$ for a.s. $(t, \omega) \in [0, T] \times \Omega$;

(iii) $\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \|\phi_k(t)\|^2\mathrm{d}t\right)\right] < \infty$ with $\|\phi_k(t)\|^2 = \phi_{k1}^2(t) + \phi_{k2}^2(t) + \phi_{k3}^2(t)$. This condition is called Novikov's condition.

We denote Σ_k as the collection of all processes $\phi_k(t)$ satisfying Assumption 3.2.1. Under Assumption 3.2.1, we know that the exponential process $\{\Lambda^{\phi_k}(t)\}_{t\in[0,T]}$ is a $\big(\{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\big)$-martingale. This then indicates that $\mathbb{E}^{\mathbb{P}}[\Lambda^{\phi_k}(T)] = 1$, for each $k \in \{1, 2\}$. As a result, by Girsanov's Theorem, a new probability measure $\mathbb{Q}_k$ equivalent to $\mathbb{P}$ can be constructed by

$$\frac{\mathrm{d}\mathbb{Q}_k}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_T} = \Lambda^{\phi_k}(T).$$

We can see that $\mathbb{Q}_k$ is parameterized by $\phi_k(t)$, and hence for the given probability measure $\mathbb{P}$, selecting an alternative probability measure $\mathbb{Q}_k$ is equivalent to determining the density generator ϕ_k.

Under an alternative probability measure $\mathbb{Q}_k$, the stochastic process $\{B^{\mathbb{Q}_k}(t)\}_{t\in[0,T]}$ satisfying

$$\mathrm{d}B^{\mathbb{Q}_k}(t) = \mathrm{d}B(t) - \phi_{k1}(t)\mathrm{d}t$$

is a standard Brownian motion, and the Poisson processes $N_k(t)$ and $N(t)$ change to $N_k^{\mathbb{Q}_k}(t)$ and $N^{\mathbb{Q}_k}(t)$ with intensities $\lambda_k\phi_{k2}(t)$ and $\lambda\phi_{k3}(t)$, respectively. For analytical tractability, as in Branger and Larsen (2013), we assume that the distributions of Z_{ki} are known and identical under probability measures $\mathbb{P}$ and $\mathbb{Q}_k$. Accordingly, the dynamics of the k-th insurer's surplus process under probability measure $\mathbb{Q}_k$ are given by

$$\begin{aligned}\mathrm{d}X_k^{u_k}(t) = &\left[rX_k^{u_k}(t) + (\mu - r)\pi_k(t) + \mu_k(\lambda_k + \lambda)(\gamma_k + (1+\eta_k)q_k(t)) + \sigma\pi_k(t)\phi_{k1}(t)\right]\mathrm{d}t\\ &+ \sigma\pi_k(t)\mathrm{d}B^{\mathbb{Q}_k}(t) - \int_0^\infty q_k(t)z_k N_k^{\mathbb{Q}_k}(\mathrm{d}t, \mathrm{d}z_k) - \int_0^\infty q_k(t)z_k N^{\mathbb{Q}_k}(\mathrm{d}t, \mathrm{d}z_k),\end{aligned}$$

where $N_k^{\mathbb{Q}_k}(\mathrm{d}t, \mathrm{d}z_k)$ and $N^{\mathbb{Q}_k}(\mathrm{d}t, \mathrm{d}z_k)$ are Poisson random measures under the alternative probability measure $\mathbb{Q}_k$.

The following definition of admissible strategy is similar to that in Zeng et al. (2016).

Definition 3.2.1. *A reinsurance-investment strategy* $u_k(t) := (q_k(t), \pi_k(t))$ *is said to be admissible for insurer* $k, k \in \{1, 2\}$*, if*

(i) $q_k(t), \pi_k(t) \in [0, \infty)$, $\forall\ t \in [0, T]$;

(ii) $u_k(t)$ *is a progressively measurable process w.r.t* $\{\mathcal{F}_t\}_{t\in[0,T]}$ *and* $\mathbb{E}^{\mathbb{Q}_k^*}\left[\int_0^T \|u_k(t)\|^2\mathrm{d}t\right] < \infty$*, where* $\|u_k(t)\|^2 = q_k^2(t) + \pi_k^2(t)$*, and* $\mathbb{Q}_k^*$ *is the chosen probability measure to describe the worst-case scenario;*

(iii) $\forall\ (x_k, t) \in \mathbb{R} \times [0, T]$*, the stochastic differential equation (SDE)* (3.2.3) *has a pathwise unique solution* $\{X_k^{u_k}(t)\}_{t\in[0,T]}$.

Let $\mathcal{U}_k$ denote the set of all admissible strategies of insurer k.

The conventional analysis of optimal reinsurance and investment focuses on the single representative agent's decision. However, as pointed out in DeMarzo et al. (2008), the financial institutions concerned about their relative performance across industry peers, and the achieved equilibrium provided meaningful insights into explaining financial bubbles. DeMarzo and Kaniel (2018) demonstrated that "Keeping up with the Joneses" preference significantly altered contract incentives. A tractable framework was proposed in Espinosa and Touzi (2015) to model the mutual interaction mechanism among heterogeneous agents. In this present chapter, we formulate the competition between two insurers by incorporating relative performance concerns used in Espinosa and Touzi (2015). Specifically, not only does the insurer want to control risks by purchasing reinsurance protection and make profits by investing in the financial market, also hopes to outperform the competing insurer in terms of terminal surplus. We define the relative performance process of insurer k, for $k, l \in \{1, 2\}$, $k \neq l$, as follows:

$$\begin{aligned}\widehat{X}_k^{u_k,u_l}(t) &:= (1 - n_k)X_k^{u_k}(t) + n_k\left(X_k^{u_k}(t) - X_l^{u_l}(t)\right)\\ &= X_k^{u_k}(t) - n_k X_l^{u_l}(t),\end{aligned}$$

where the constant $n_k \in [0, 1]$ measures the sensitivity of insurer k to the performance of the competing insurer l. A larger n_k implies that insurer k is more concerned with increasing the relative surplus to the competing insurer l and the game becomes more competitive. The dynamics of $\widehat{X}_k^{u_k,u_l}(t)$ under probability measure $\mathbb{P}$ are governed by

$$\begin{aligned}\mathrm{d}\widehat{X}_k^{u_k,u_l}(t) = \Big[&r\widehat{X}_k^{u_k,u_l}(t) + (\mu - r)(\pi_k(t) - n_k\pi_l(t)) + \mu_k(\lambda_k + \lambda)(\gamma_k + (1 + \eta_k)q_k(t))\\ &- n_k\mu_l(\lambda_l + \lambda)(\gamma_l + (1 + \eta_l)q_l(t))\Big]\mathrm{d}t + \sigma(\pi_k(t) - n_k\pi_l(t))\mathrm{d}B(t)\\ &+ n_k\int_0^\infty q_l(t)z_l N_l(\mathrm{d}t, \mathrm{d}z_l) - \int_0^\infty q_k(t)z_k N_k(\mathrm{d}t, \mathrm{d}z_k)\\ &+ n_k\int_0^\infty q_l(t)z_l N(\mathrm{d}t, \mathrm{d}z_l) - \int_0^\infty q_k(t)z_k N(\mathrm{d}t, \mathrm{d}z_k),\end{aligned}$$

with the initial condition $\widehat{X}_k^{u_k,u_l}(0) = \hat{x}_k^0 = x_k^0 - n_k x_l^0$. The dynamics of $\widehat{X}_k^{u_k,u_l}(t)$

under $\mathbb{Q}_k$ are described by the following SDE:

$$\begin{aligned}
\mathrm{d}\widehat{X}_k^{u_k,u_l}(t) = \Big[& r\widehat{X}_k^{u_k,u_l}(t) + (\mu - r)(\pi_k(t) - n_k\pi_l(t)) + \mu_k(\lambda_k + \lambda)(\gamma_k + (1+\eta_k)q_k(t)) \\
& - n_k\mu_l(\lambda_l + \lambda)(\gamma_l + (1+\eta_l)q_l(t)) + \sigma\pi_k(t)\phi_{k1}(t) - n_k\sigma\pi_l(t)\phi_{l1}(t)\Big]\mathrm{d}t \\
& + \sigma\pi_k(t)\mathrm{d}B^{\mathbb{Q}_k}(t) - n_k\sigma\pi_l(t)\mathrm{d}B^{\mathbb{Q}_l}(t) - \int_0^\infty q_k(t)z_k N_k^{\mathbb{Q}_k}(\mathrm{d}t, \mathrm{d}z_k) \\
& + n_k\int_0^\infty q_l(t)z_l N_l^{\mathbb{Q}_l}(\mathrm{d}t, \mathrm{d}z_l) - \int_0^\infty q_k(t)z_k N^{\mathbb{Q}_k}(\mathrm{d}t, \mathrm{d}z_k) \\
& + n_k\int_0^\infty q_l(t)z_l N^{\mathbb{Q}_l}(\mathrm{d}t, \mathrm{d}z_l).
\end{aligned}$$

Under this competitive circumstance, we assume that each insurer has a mean-variance preference and aims to maximize the mean-variance criterion of the relative surplus at terminal time. When insurer k is ambiguity-neutral, some of the existing papers derive the insurer's optimal reinsurance-investment strategy by considering the optimality of the solution at the initial time, where the value function is defined by

$$\check{J}_k^{u_k,u_l}(0, \hat{x}_k^0) := \sup_{u_k\in\mathcal{U}_k}\left\{\mathbb{E}^{\mathbb{P}}_{0,\hat{x}_k^0}\left[\widehat{X}_k^{u_k,u_l}(T)\right] - \frac{m_k}{2}\mathrm{Var}^{\mathbb{P}}_{0,\hat{x}_k^0}\left[\widehat{X}_k^{u_k,u_l}(T)\right]\right\}, \qquad (3.2.5)$$

where $\mathbb{E}^{\mathbb{P}}_{t,x}[\,\cdot\,] = \mathbb{E}^{\mathbb{P}}\left[\,\cdot\,\middle|\widehat{X}_k^{u_k,u_l}(t) = x\right]$, $\mathrm{Var}^{\mathbb{P}}_{t,x}[\,\cdot\,] = \mathrm{Var}^{\mathbb{P}}\left[\,\cdot\,\middle|\widehat{X}_k^{u_k,u_l}(t) = x\right]$, and $m_k > 0$ is the risk-averse coefficient of insurer k. Some literature on this topic can be found in, for instance, Zhou and Li (2000), Cong and Oosterlee (2016), Sun et al. (2016) and Bian et al. (2018). It is obvious that we can only obtain the strategies that are optimal at time zero by solving the static optimization problem (3.2.5). In Björk et al. (2014) and Kronborg and Steffensen (2015), problem (3.2.5) was called a mean-variance optimization problem with pre-commitment. The corresponding optimal strategy was called a pre-commitment strategy because it was not updated when new information emerged. A shortcoming of the pre-commitment strategy is that it is time-inconsistent. However, from both the theoretical and practical perspectives, it could be a basic requirement for a rational decision-maker to adopt time-consistent strategies. Following Björk et al. (2014) and Kronborg and Steffensen (2015), we develop time-consistent reinsurance and investment strategies by defining a time-varying value function $\widehat{J}_k^{u_k,u_l}(t, \hat{x}_k)$ as follows: $\forall\ (\hat{x}_k, t) \in \mathbb{R} \times [0, T]$, $k, l \in \{1, 2\}$, $k \neq l$,

$$\widehat{J}_k^{u_k, u_l}(t, \hat{x}_k) := \sup_{u_k \in \mathcal{U}_k} \left\{ \mathbb{E}_{t, \hat{x}_k}^{\mathbb{P}} \left[\widehat{X}_k^{u_k, u_l}(T) \right] - \frac{m_k}{2} \mathrm{Var}_{t, \hat{x}_k}^{\mathbb{P}} \left[\widehat{X}_k^{u_k, u_l}(T) \right] \right\}. \tag{3.2.6}$$

Under this optimization criterion, we formulate the two insurers' strategic interaction as follows:

Problem 1: The classical non-zero-sum stochastic differential game between two competing ANIs under the mean-variance criterion is to find a Nash equilibrium $(u_1^*, u_2^*) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that for any $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, we have

$$\widehat{J}_1^{u_1^*, u_2^*}(t, \hat{x}_1) \geq \widehat{J}_1^{u_1, u_2^*}(t, \hat{x}_1),$$
$$\widehat{J}_2^{u_1^*, u_2^*}(t, \hat{x}_2) \geq \widehat{J}_2^{u_1^*, u_2}(t, \hat{x}_2).$$

The objective function in (3.2.6) ignores model uncertainty and **Problem 1** solves the competing insurers' decision making problems under the reference measure, which may bring about misspecified decisions. Next, we will incorporate the concepts of ambiguity aversion into **Problem 1**. Under this case, each insurer distrusts the veracity of the reference model $\mathbb{P}$ and selects $\mathbb{Q}_k$ from $\mathcal{Q}$ as an alternative model. In fact, the insurers aim to solve the mean-variance optimization problems under the worst-case scenario of the alternative measures. The objective function of insurer k in the robust optimization problem becomes

$$\sup_{u_k \in \mathcal{U}_k} \inf_{\mathbb{Q}_k \in \mathcal{Q}} \left\{ \mathbb{E}_{t, \hat{x}_k}^{\mathbb{Q}_k} \left[\widehat{X}_k^{u_k, u_l}(T) \right] - \frac{m_k}{2} \mathrm{Var}_{t, \hat{x}_k}^{\mathbb{Q}_k} \left[\widehat{X}_k^{u_k, u_l}(T) \right] + \mathbb{E}_{t, \hat{x}_k}^{\mathbb{Q}_k} \left[P_k(\mathbb{P} \| \mathbb{Q}_k) \right] \right\}, \tag{3.2.7}$$

where $P_k(\mathbb{P} \| \mathbb{Q}_k) \geq 0$ is a penalty function measuring the divergence of $\mathbb{Q}_k$ from $\mathbb{P}$. In the case of $P_k(\mathbb{P} \| \mathbb{Q}_k) \to \infty$, the robust optimization problem (3.2.7) reverts to the traditional optimization problem (3.2.6). On the other hand, $P_k(\mathbb{P} \| \mathbb{Q}_k) \to 0$ implies that the decision-maker is extremely ambiguous. Accordingly, we can modify **Problem 1** as the following robust optimization problem for two competing AAIs:

Problem 2: The robust non-zero-sum stochastic differential game between two competing AAIs under the mean-variance criterion is to find a Nash equilibrium $(u_1^*, u_2^*) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that for any $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, we have

$$\inf_{\mathbb{Q}_1 \in \mathcal{Q}} \widetilde{J}_1^{\mathbb{Q}_1, u_1^*, u_2^*}(t, \hat{x}_1) \geq \inf_{\mathbb{Q}_1 \in \mathcal{Q}} \widetilde{J}_1^{\mathbb{Q}_1, u_1, u_2^*}(t, \hat{x}_1),$$
$$\inf_{\mathbb{Q}_2 \in \mathcal{Q}} \widetilde{J}_2^{\mathbb{Q}_2, u_1^*, u_2^*}(t, \hat{x}_2) \geq \inf_{\mathbb{Q}_2 \in \mathcal{Q}} \widetilde{J}_2^{\mathbb{Q}_2, u_1^*, u_2}(t, \hat{x}_2),$$

where

$$\widetilde{J}_k^{\mathbb{Q}_k,u_k,u_l}(t,\hat{x}_k) = \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\widehat{X}_k^{u_k,u_l}(T)\right] - \frac{m_k}{2}\mathrm{Var}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\widehat{X}_k^{u_k,u_l}(T)\right] + \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[P_k(\mathbb{P}\|\mathbb{Q}_k)\right], \tag{3.2.8}$$

and for notational ease in the later paragraphs, here we define

$$J_k^{u_k,u_l}(t,\hat{x}_k) := \inf_{\mathbb{Q}_k\in\mathcal{Q}} \widetilde{J}_k^{\mathbb{Q}_k,u_k,u_l}(t,\hat{x}_k). \tag{3.2.9}$$

In Appendix D, we have shown that the increase in relative entropy from t to $t+\mathrm{d}t$ equals

$$\left[\frac{1}{2}\phi_{k1}^2(t) + \lambda_k\left(\phi_{k2}(t)\ln\phi_{k2}(t) + 1 - \phi_{k2}(t)\right) + \lambda\left(\phi_{k3}(s)\ln\phi_{k3}(s) + 1 - \phi_{k3}(s)\right)\right]\mathrm{d}t. \tag{3.2.10}$$

It should be noted that the first term in (3.2.10) increases relative entropy due to the diffusion component of the model, while the remaining terms give the increase due to the jump components.

To solve **Problem 2**, we consider a penalty function of the following form used in Maenhout (2004):

$$P_k(\mathbb{P}\|\mathbb{Q}_k) = \int_t^T \Psi_k\left(s,\phi_k(s),\widehat{X}_k^{u_k,u_l^*}(s)\right)\mathrm{d}s,$$

and define the value function of insurer k as follows:

$$\begin{aligned} V_k(t,\hat{x}_k) &:= \sup_{u_k\in\mathcal{U}_k}\inf_{\mathbb{Q}_k\in\mathcal{Q}}\Bigg\{\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\widehat{X}_k^{u_k,u_l^*}(T)\right] - \frac{m_k}{2}\mathrm{Var}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\widehat{X}_k^{u_k,u_l^*}(T)\right] \\ &\qquad + \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\int_t^T \Psi_k\left(s,\phi_k(s),\widehat{X}_k^{u_k,u_l^*}(s)\right)\mathrm{d}s\right]\Bigg\} \qquad (3.2.11) \\ &= \sup_{u_k\in\mathcal{U}_k} J_k^{u_k,u_l^*}(t,\hat{x}_k), \end{aligned}$$

where

$$\begin{aligned}\Psi_k\left(s,\phi_k(s),\widehat{X}_k^{u_k,u_l^*}(s)\right) &= \frac{\phi_{k1}^2(s)}{2\psi_{k1}\left(s,\widehat{X}_k^{u_k,u_l^*}(s)\right)} + \frac{\lambda_k\left(\phi_{k2}(s)\ln\phi_{k2}(s)+1-\phi_{k2}(s)\right)}{\psi_{k2}\left(s,\widehat{X}_k^{u_k,u_l^*}(s)\right)} \\ &\quad + \frac{\lambda\left(\phi_{k3}(s)\ln\phi_{k3}(s)+1-\phi_{k3}(s)\right)}{\psi_{k3}\left(s,\widehat{X}_k^{u_k,u_l^*}(s)\right)}.\end{aligned}$$

For $i\in\{1,2,3\}$, $\psi_{ki}\left(s,\widehat{X}_k^{u_k,u_l^*}(s)\right)$ are strictly positive deterministic functions. The larger $\psi_{ki}\left(s,\widehat{X}_k^{u_k,u_l^*}(s)\right)$ are, the less deviation from the reference model is penalized, then the AAI has less faith in the reference model and so has more tendency to consider alternative models. Therefore, the degree of the AAI's ambiguity aversion is increasing with respect to the function $\psi_{ki}\left(s, \widehat{X}_k^{u_k,u_l^*}(s)\right)$. For analytical tractability, following Zeng et al. (2016), we assume that ψ_{ki}, for $k \in \{1,2\}$ and $i \in \{1,2,3\}$, is fixed and state-independent function by putting

$$\psi_{ki}(t, \hat{x}_k) = \beta_{ki},$$

where β_{ki} are non-negative parameters, and we call them ambiguity aversion coefficients of insurer k representing the degree of his ambiguity aversion attitudes with respect to the diffusion risk and jump risk. We interpret β_{k1} as ambiguity aversion about the stock dynamics, β_{k2} and β_{k3} as ambiguity aversion about the claim process corresponding to $N_k(t)$ and $N(t)$, respectively. When $\beta_{ki} = 0$, insurer k is ambiguity-neutral about the corresponding kind of risk.

It is known that the mean-variance optimization problems in **Problem 2** are time-inconsistent. Under this circumstance, we cannot obtain time-consistent solutions of the reinsurance-investment optimization problems for the insurers since the classical Bellman's optimality principle cannot be applied directly any more. To overcome this difficulty, we follow the approach used in Björk et al. (2014) and Kronborg and Steffensen (2015). Basically, they dealt with the decision-maker's optimization problem as a non-cooperative game. This game theoretic approach for the time-inconsistent problems have been applied to many optimal reinsurance-investment problems. For example, Zeng and Li (2011) pioneered the study on optimal time-consistent investment and reinsurance problems for the insurers who had mean-variance preference. Li et al. (2015b) constructed an objective function as a weighted sum of the insurer's and reinsurer's mean-variance criteria to derive their optimal reinsurance and investment strategies when the price process of the risky asset followed CEV model. Li et al. (2016) formulated alpha-maxmin mean-variance criterion and derived the equilibrium reinsurance and investment strategies in the presence of model uncertainty.

The equilibrium strategies and the equilibrium value functions to **Problem 2** are defined as follows:

Definition 3.2.2. *For an admissible reinsurance-investment strategy of insurer k, denoted as $u_k^*(t) = (q_k^*(t), \pi_k^*(t))$, with any fixed chosen initial state $(\hat{x}_k, t) \in \mathbb{R} \times [0, T]$, when the optimal strategy u_l^* of the competing insurer is known, we define the following perturbed strategy*

$$u_k^{\epsilon}(s) := \begin{cases} \tilde{u}_k, & t \leq s < t + \varepsilon, \\ u_k^*(s), & t + \varepsilon \leq s \leq T, \end{cases}$$

where $\tilde{u}_k = (\tilde{q}_k, \tilde{\pi}_k)$ and $\varepsilon \in \mathbb{R}^+$. If $\forall\ \tilde{u}_k = (\tilde{q}_k, \tilde{\pi}_k) \in \mathbb{R}^+ \times \mathbb{R}^+$, we have

$$\liminf_{\varepsilon \to 0} \frac{J_k^{u_k^*, u_l^*}(t, \hat{x}_k) - J_k^{u_k^{\varepsilon}, u_l^*}(t, \hat{x}_k)}{\varepsilon} \geq 0,$$

then $u_k^*(t)$ is called an equilibrium strategy and the equilibrium value function of insurer k is given by

$$V_k(t, \hat{x}_k) = J_k^{u_k^*, u_l^*}(t, \hat{x}_k).$$

Note that there exist two games in our model setting. Specifically, the first one is the game between two competing insurers steming from relative performance concerns. The other game can be regarded as a non-cooperative game between each insurer and his future incarnations at different time points. The equilibrium strategy in Definition 3.2.2 is time-consistent, which implies that the optimal strategy derived at any time t agrees with the optimal strategy derived at time $t + \Delta t$. Hereafter, we call the equilibrium strategy solving **Problem 2** and satisfying Definition 3.2.2 as the robust optimal time-consistent strategy, and the equilibrium value function satisfying Definition 3.2.2 is called the optimal value function.

3.3 Nash equilibrium in compound Poisson risk model

In this section, we first present the verification theorem and then derive the robust Nash equilibrium reinsurance-investment strategy under the compound Poisson risk model. For notational convenience, we first define $C^{1,2}([0, T] \times \mathbb{R}) := \{f(t, x) | f(t, x)$ is continuously differentiable for $t \in [0, T]$ and twice continuously differentiable for $x \in \mathbb{R}\}$, $D_p^{1,2}([0, T] \times \mathbb{R}) := \{f(t, x) | f(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$ and all first order partial derivatives satisfy the polynomial growth condition on $\mathbb{R}\}$.

We suppress the arguments of the control policies and the density generators for notational simplicity in the following paragraphs. $\forall\ (t,\hat{x}_k) \in [0,T]\times\mathbb{R}$, $\forall\ W_k(t,\hat{x}_k) \in C^{1,2}([0,T]\times\mathbb{R})$, we define an operator $\mathcal{L}$ on $W_k(t,\hat{x}_k)$ as follows:

$$\begin{aligned}
\mathcal{L}^{u_k,u_l,\phi_k,\phi_l}W_k(t,\hat{x}_k) :=& \frac{\partial W_k(t,\hat{x}_k)}{\partial t} + \Big[r\hat{x}_k + (\mu-r)(\pi_k - n_k\pi_l) + \sigma\pi_k\phi_{k1} \\
&+ \mu_k(\lambda_k+\lambda)(\gamma_k + (1+\eta_k)q_k) - n_k\mu_l(\lambda_l+\lambda)(\gamma_l+(1+\eta_l)q_l) \\
&- \sigma n_k\pi_l\phi_{l1}\Big]\frac{\partial W_k(t,\hat{x}_k)}{\partial \hat{x}_k} + \frac{1}{2}\sigma^2\left(\pi_k^2 + n_k^2\pi_l^2 - 2n_k\pi_k\pi_l\right)\frac{\partial^2 W_k(t,\hat{x}_k)}{\partial \hat{x}_k^2} \\
&+ (\lambda_k\phi_{k2} + \lambda\phi_{k3})\mathbb{E}^{\mathbb{Q}_k}\left[W_k(t,\hat{x}_k - q_k z_k) - W_k(t,\hat{x}_k)\right] \\
&+ (\lambda_l\phi_{l2} + \lambda\phi_{l3})\mathbb{E}^{\mathbb{Q}_l}\left[W_k(t,\hat{x}_k + n_k q_l z_l) - W_k(t,\hat{x}_k)\right].
\end{aligned} \tag{3.3.12}$$

Theorem 3.3.1. *(Verification Theorem) For* ***Problem 2****, if there exist real-valued functions $W_k(t,\hat{x}_k)$ and $g_k(t,\hat{x}_k)$ lie in $D_p^{1,2}([0,T]\times\mathbb{R})$ satisfying the following extended HJB system of equations: $\forall\ (t,\hat{x}_k)\in[0,T]\times\mathbb{R}$,*

$$\begin{aligned}
\sup_{u_k\in\mathcal{U}_k}\inf_{\phi_k\in\Sigma_k}\Bigg\{&\mathcal{L}^{u_k,u_l^*,\phi_k,\phi_l^*}W_k(t,\hat{x}_k) - \mathcal{L}^{u_k,u_l^*,\phi_k,\phi_l^*}\frac{m_k}{2}g_k^2(t,\hat{x}_k) \\
&+ m_k g_k(t,\hat{x}_k)\mathcal{L}^{u_k,u_l^*,\phi_k,\phi_l^*}g_k(t,\hat{x}_k) + \frac{\phi_{k1}^2}{2\beta_{k1}} \\
&+ \frac{\lambda_k(\phi_{k2}\ln\phi_{k2} + 1 - \phi_{k2})}{\beta_{k2}} + \frac{\lambda(\phi_{k3}\ln\phi_{k3}+1-\phi_{k3})}{\beta_{k3}}\Bigg\} = 0,
\end{aligned} \tag{3.3.13}$$

$$W_k(T,\hat{x}_k) = \hat{x}_k, \tag{3.3.14}$$

$$g_k(T,\hat{x}_k) = \hat{x}_k, \tag{3.3.15}$$

$$\mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*}g_k(t,\hat{x}_k) = 0, \tag{3.3.16}$$

where

$$\begin{aligned}
(u_k^*,\phi_k^*) := \arg\sup_{u_k\in\mathcal{U}_k}\inf_{\phi_k\in\Sigma_k}\Bigg\{&\mathcal{L}^{u_k,u_l^*,\phi_k,\phi_l^*}W_k(t,\hat{x}_k) - \mathcal{L}^{u_k,u_l^*,\phi_k,\phi_l^*}\frac{m_k}{2}g_k^2(t,\hat{x}_k) \\
&+ m_k g_k(t,\hat{x}_k)\mathcal{L}^{u_k,u_l^*,\phi_k,\phi_l^*}g_k(t,\hat{x}_k) + \frac{\phi_{k1}^2}{2\beta_{k1}} \\
&+ \frac{\lambda_k(\phi_{k2}\ln\phi_{k2} + 1 - \phi_{k2})}{\beta_{k2}} + \frac{\lambda(\phi_{k3}\ln\phi_{k3}+1-\phi_{k3})}{\beta_{k3}}\Bigg\},
\end{aligned}$$

then $W_k(t,\hat{x}_k) = V_k(t,\hat{x}_k)$, $\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}^{u_k^*,u_l^*}(T)\right] = g_k(t,\hat{x}_k)$, and u_k^* is the robust equilibrium reinsurance-investment strategy of insurer k.

Proof. The proof of this theorem is similar to those of Theorem 4.1 in Björk and Murgoci (2010) and Theorem 3.1 in Li et al. (2016), and hence we move it to Appendix E. □

In the following theorem, we provide the equation satisfied by the insurer's equilibrium retention level of the claims and the explicit expressions for equilibrium investment strategy and equilibrium value function of each insurer.

Theorem 3.3.2. *For the robust game between two AAIs with mean-variance cost functionals described in* ***Problem 2****, the time-consistent Nash equilibrium reinsurance strategy of insurer k, for $k \in \{1,2\}$, can be obtained by solving the following equation:*

$$q_k^* = \frac{1}{m_k \sigma_k^2 e^{r(T-t)}} \left[\frac{\mu_k(\lambda_k+\lambda)(1+\eta_k)}{\lambda_k e^{\beta_{k2} X_k^*(t)} + \lambda e^{\beta_{k3} X_k^*(t)}} - \mu_k \right], \tag{3.3.17}$$

and the time-consistent Nash equilibrium investment strategy of each insurer is given by:

$$\begin{cases} \pi_1^* = \dfrac{(\mu-r)(\beta_{21}+m_2+m_1 n_1)}{\sigma^2 e^{r(T-t)}\left[(\beta_{11}+m_1)(\beta_{21}+m_2) - m_1 m_2 n_1 n_2\right]}, \\ \pi_2^* = \dfrac{(\mu-r)(\beta_{11}+m_1+m_2 n_2)}{\sigma^2 e^{r(T-t)}\left[(\beta_{11}+m_1)(\beta_{21}+m_2) - m_1 m_2 n_1 n_2\right]}. \end{cases} \tag{3.3.18}$$

Additionally, the optimal value function of insurer k is given by

$$\begin{aligned} V_k(t,\hat{x}_k) = {} & \hat{x}_k e^{r(T-t)} + \frac{n_k \mu_l \gamma_l(\lambda_l+\lambda) - \mu_k \gamma_k(\lambda_k+\lambda)}{r}\left(1-e^{r(T-t)}\right) \\ & + \int_t^T b_{k1}(s)\mathrm{d}s + \int_t^T b_{k2}(s)\mathrm{d}s, \end{aligned}$$

where $k,l \in \{1,2\}$, $k \neq l$, and

$$\begin{cases} b_{k1}(s) = & \Big[\mu_k(\lambda_k+\lambda)(1+\eta_k)q_k^* - n_k\mu_l(\lambda_l+\lambda)(1+\eta_l)q_l^* + n_k\mu_l q_l^* \\ & \times \left(\lambda_l e^{\beta_{l2}X_l^*(s)} + \lambda e^{\beta_{l3}X_l^*(s)}\right)\Big] e^{r(T-s)} - \dfrac{m_k e^{2r(T-s)}}{2} n_k^2 \sigma_l^2 (q_l^*)^2 \\ & \times \left(\lambda_l e^{\beta_{l2}X_l^*(s)} + \lambda e^{\beta_{l3}X_l^*(s)}\right) + \dfrac{\lambda_k}{\beta_{k2}}\left(1-e^{\beta_{k2}X_k^*(s)}\right) + \dfrac{\lambda}{\beta_{k3}}\left(1-e^{\beta_{k3}X_k^*(s)}\right), \\ b_{k2}(s) = & \left[(\mu-r)(\pi_k^* - n_k\pi_l^*) + n_k\sigma^2\beta_{l1}(\pi_l^*)^2 e^{r(T-s)}\right] e^{r(T-s)} \\ & - \dfrac{m_k e^{2r(T-s)}}{2}\sigma^2\left((\pi_k^*)^2 + n_k^2(\pi_l^*)^2 - 2n_k\pi_k^*\pi_l^*\right) - \dfrac{\beta_{k1}\sigma^2(\pi_k^*)^2}{2}e^{2r(T-s)}, \end{cases}$$

with

$$\begin{cases} X_k^*(t) = \mu_k q_k^* e^{r(T-t)} + \frac{m_k\sigma_k^2(q_k^*)^2 e^{2r(T-t)}}{2}, \\ X_l^*(t) = \mu_l q_l^* e^{r(T-t)} + \frac{m_l\sigma_l^2(q_l^*)^2 e^{2r(T-t)}}{2}. \end{cases}$$

The expected value of each insurer's relative surplus at terminal time is given by

$$\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}^{u_k^*,u_l^*}(T)\right] = e^{r(T-t)}\hat{x}_k + \frac{n_k\mu_l\gamma_l(\lambda_l+\lambda) - \mu_k\gamma_k(\lambda_k+\lambda)}{r}\left(1-e^{r(T-t)}\right)$$
$$+ \int_t^T \tilde{b}_{k1}(s)\mathrm{d}s + \int_t^T \tilde{b}_{k2}(s)\mathrm{d}s,$$

where

$$\begin{cases} \tilde{b}_{k1}(s) = [\mu_k(\lambda_k+\lambda)(1+\eta_k)q_k^* - n_k\mu_l(\lambda_l+\lambda)(1+\eta_l)q_l^*]e^{r(T-s)} \\ \qquad\quad -\left[\mu_k q_k^*\left(\lambda_k e^{\beta_{k2}X_k^*(s)} + \lambda e^{\beta_{k3}X_k^*(s)}\right) - n_k\mu_l q_l^*\left(\lambda_l e^{\beta_{l2}X_l^*(s)} + \lambda e^{\beta_{l3}X_l^*(s)}\right)\right]e^{r(T-s)}, \\ \tilde{b}_{k2}(s) = (\mu - r)(\pi_k^* - n_k\pi_l^*)e^{r(T-s)} - \sigma^2\left(\beta_{k1}(\pi_k^*)^2 - n_k\beta_{l1}(\pi_l^*)^2\right)e^{2r(T-s)}. \end{cases}$$

Finally, the worst-case density generator $\phi_k^* := (\phi_{k1}^*, \phi_{k2}^*, \phi_{k3}^*)$ of insurer k is

$$\begin{cases} \phi_{k1}^* = -\beta_{k1}\sigma\pi_k^* e^{r(T-t)}, \\ \phi_{k2}^* = e^{\beta_{k2}X_k^*(t)}, \\ \phi_{k3}^* = e^{\beta_{k3}X_k^*(t)}. \end{cases}$$

Proof. See Appendix F. □

Proposition 3.3.1. *The robust equilibrium reinsurance strategy q_k^* of insurer k, $k \in \{1,2\}$, can be uniquely determined by Equation* (3.3.17)*, i.e., there exists a unique $q_k^* > 0$ satisfying Equation* (3.3.17)*.*

Proof. We rearrange Equation (3.3.17) as

$$\left(\mu_k + m_k\sigma_k^2 e^{r(T-t)}q_k^*\right)\left(\lambda_k e^{\beta_{k2}X_k^*(t)} + \lambda e^{\beta_{k3}X_k^*(t)}\right) = \mu_k(\lambda_k+\lambda)(1+\eta_k),$$

and define

$$f(q_k) := \left(\mu_k + m_k\sigma_k^2 e^{r(T-t)}q_k\right)\left(\lambda_k e^{\beta_{k2}X_k(t)} + \lambda e^{\beta_{k3}X_k(t)}\right) - \mu_k(\lambda_k+\lambda)(1+\eta_k).$$

Accordingly, we have

$$f'(q_k) = m_k\sigma_k^2 e^{r(T-t)}\left(\lambda_k e^{\beta_{k2}X_k(t)} + \lambda e^{\beta_{k3}X_k(t)}\right)$$
$$+ e^{r(T-t)}\left(\mu_k + m_k\sigma_k^2 e^{r(T-t)}q_k\right)^2\left(\lambda_k\beta_{k2}e^{\beta_{k2}X_k(t)} + \lambda\beta_{k3}e^{\beta_{k3}X_k(t)}\right) > 0,$$

which implies that $f(q_k)$ is a strictly increasing function with respect to q_k over $[0,\infty)$. It is obvious that $f(0) = -\mu_k\eta_k(\lambda_k+\lambda) < 0$, and hence Equation (3.3.17) has a unique positive root. This completes the proof. □

3.4 Nash equilibrium in diffusion approximated model

Having obtained the robust equilibrium reinsurance-investment strategies for both insurers in the case of classical compound Poisson risk process, in this section we investigate the robust reinsurance-investment game problem when the compound Poisson risk process of each insurer is approximated by a diffusion model. Specifically, according to Grandell (1991), the aggregate claim process $\sum_{i=1}^{N_k(t)+N(t)} Z_{ki}$ in (3.2.1) could be approximated by a Brownian motion with drift, that is

$$\sum_{i=1}^{N_k(t)+N(t)} Z_{ki} \approx (\lambda_k + \lambda)\mu_k t - \sigma_k \sqrt{\lambda_k + \lambda} B_k(t), \tag{3.4.19}$$

where $\{B_k(t)\}_{t\in[0,T]}$ denotes another standard one-dimensional Brownian motion under the probability measure $\mathbb{P}$. Here, we assume that the two Brownian motions $\{B(t)\}_{t\in[0,T]}$ and $\{B_k(t)\}_{t\in[0,T]}$ are mutually independent under probability measure $\mathbb{P}$, and the correlation coefficient of $\{B_1(t)\}_{t\in[0,T]}$ and $\{B_2(t)\}_{t\in[0,T]}$ is $\rho \in (-1, 1)$, i.e., $\mathrm{Cov}(\mathrm{d}B_1(t), \mathrm{d}B_2(t)) = \rho \mathrm{d}t$. Under the proportional reinsurance contract $q_k(t)$, the surplus process of insurer k in (3.2.2) becomes

$$\mathrm{d}U_k(t) = \mu_k(\lambda_k + \lambda)\left(\gamma_k + \eta_k q_k(t)\right)\mathrm{d}t + \sigma_k q_k(t)\sqrt{\lambda_k + \lambda}\,\mathrm{d}B_k(t), \qquad k \in \{1, 2\}.$$

After considering the investment opportunities, the surplus process of insurer k becomes

$$\begin{aligned}\mathrm{d}X_k^{u_k}(t) =& \left[rX_k^{u_k}(t) + (\mu - r)\pi_k(t) + \mu_k(\lambda_k + \lambda)(\gamma_k + \eta_k q_k(t))\right]\mathrm{d}t + \sigma\pi_k(t)\mathrm{d}B(t)\\ &+ \sigma_k q_k(t)\sqrt{\lambda_k + \lambda}\,\mathrm{d}B_k(t),\end{aligned}$$

with initial condition $X_k^{u_k}(0) = x_k^0$.

Under the alternative probability measure $\mathbb{Q}_k$ which is parameterized by $\phi_k(t) = (\phi_{k1}(t), \phi_{k2}(t))$, the processes $\{B^{\mathbb{Q}_k}(t)\}_{t\in[0,T]}$ and $\{B_k^{\mathbb{Q}_k}(t)\}_{t\in[0,T]}$ are standard Brownian motions, and they satisfy

$$\begin{aligned}\mathrm{d}B^{\mathbb{Q}_k}(t) &= \mathrm{d}B(t) - \phi_{k1}(t)\mathrm{d}t,\\ \mathrm{d}B_k^{\mathbb{Q}_k}(t) &= \mathrm{d}B_k(t) - \phi_{k2}(t)\mathrm{d}t.\end{aligned}$$

The dynamics of the relative performance process $\widehat{X}_k^{u_k,u_l}(t)$ of insurer k under $\mathbb{Q}_k$

are governed by the following SDE:

$$\begin{aligned}\mathrm{d}\widehat{X}_k^{u_k,u_l}(t) =& \Big[r\widehat{X}_k^{u_k,u_l}(t) + (\mu - r)(\pi_k(t) - n_k\pi_l(t)) + \mu_k(\lambda_k + \lambda)(\gamma_k + \eta_k q_k(t)) \\ & - n_k\mu_l(\lambda_l + \lambda)(\gamma_l + \eta_l q_l(t)) + \sigma\pi_k(t)\phi_{k1}(t) - n_k\sigma\pi_l(t)\phi_{l1}(t) \\ & + \sigma_k q_k\sqrt{\lambda_k + \lambda}\phi_{k2}(t) - n_k\sigma_l q_l\sqrt{\lambda_l + \lambda}\phi_{l2}(t)\Big]\mathrm{d}t + \sigma\pi_k(t)\mathrm{d}B^{\mathbb{Q}_k}(t) \\ & - n_k\sigma\pi_l(t)\mathrm{d}B^{\mathbb{Q}_l}(t) + \sigma_k q_k\sqrt{\lambda_k + \lambda}\mathrm{d}B_k^{\mathbb{Q}_k}(t) - n_k\sigma_l q_l\sqrt{\lambda_l + \lambda}\mathrm{d}B_l^{\mathbb{Q}_l}(t).\end{aligned} \tag{3.4.20}$$

The value function of insurer k becomes:

$$\begin{aligned}V_k(t,\hat{x}_k) := \sup_{u_k\in\mathcal{U}_k}\inf_{\mathbb{Q}_k\in\mathcal{Q}} \Bigg\{ & \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\widehat{X}_k^{u_k,u_l^*}(T)\right] - \frac{m_k}{2}\mathrm{Var}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\widehat{X}_k^{u_k,u_l^*}(T)\right] \\ & + \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\int_t^T \left(\frac{\phi_{k1}^2(s)}{2\beta_{k1}} + \frac{\phi_{k2}^2(s)}{2\beta_{k2}}\right)\mathrm{d}s\right]\Bigg\},\end{aligned} \tag{3.4.21}$$

where $\widehat{X}_k^{u_k,u_l^*}(T)$ is the relative performance process of insurer k, whose dynamics are given in (3.4.20), at terminal time T when the optimal reinsurance-investment strategy u_l^* of the competing insurer is given. In other words, knowing the competing insurer chooses the optimal control policies, insurer k aims to find an admissible time-consistent reinsurance-investment strategy that solves the optimization problem (3.4.21).

We define an operator $\mathcal{L}_1$ as follows, $\forall\ (t,\hat{x}_k) \in [0,T]\times\mathbb{R}$, $\forall\ W_k(t,\hat{x}_k) \in C^{1,2}([0,T]\times\mathbb{R})$,

$$\begin{aligned}\mathcal{L}_1^{u_k,u_l,\phi_k,\phi_l}W_k(t,\hat{x}_k) : = & \frac{\partial W_k(t,\hat{x}_k)}{\partial t} + \Big[r\hat{x}_k + (\mu - r)(\pi_k - n_k\pi_l) + \sigma\pi_k\phi_{k1} - \sigma n_k\pi_l\phi_{l1} \\ & + \mu_k(\lambda_k+\lambda)(\gamma_k + \eta_k q_k) - n_k\mu_l(\lambda_l+\lambda)(\gamma_l+\eta_l q_l) \\ & + \sigma_k q_k\sqrt{\lambda_k+\lambda}\phi_{k2} - n_k\sigma_l q_l\sqrt{\lambda_l+\lambda}\phi_{l2}\Big]\frac{\partial W_k(t,\hat{x}_k)}{\partial \hat{x}_k} \\ & + \frac{1}{2}\Big(\sigma^2\pi_k^2 + n_k^2\sigma^2\pi_l^2 + \sigma_k^2 q_k^2(\lambda_k+\lambda) + n_k^2\sigma_l^2 q_l^2(\lambda_l+\lambda) \\ & - 2\sigma^2 n_k\pi_k\pi_l - 2\rho n_k\sigma_k\sigma_l q_k q_l\sqrt{(\lambda_k+\lambda)(\lambda_l+\lambda)}\Big)\frac{\partial^2 W_k(t,\hat{x}_k)}{\partial \hat{x}_k^2}.\end{aligned}$$

Accordingly, we provide the following verification theorem whose proof would be similar to that of Theorem 3.3.1, and so we omit it here.

Theorem 3.4.1. *(Verification Theorem) If there exist real-valued functions $W_k(t,\hat{x}_k)$ and $g_k(t,\hat{x}_k) \in D_p^{1,2}([0,T]\times\mathbb{R})$ satisfying the following extended HJB system of* *equations:* $\forall\ (t,\hat{x}_k)\in[0,T]\times\mathbb{R}$,

$$\begin{aligned}\sup_{u_k\in\mathcal{U}_k}\inf_{\phi_k\in\Sigma_k}\Bigg\{&\mathcal{L}_1^{u_k,u_l^*,\phi_k,\phi_l^*}W_k(t,\hat{x}_k)-\mathcal{L}_1^{u_k,u_l^*,\phi_k,\phi_l^*}\frac{m_k}{2}g_k^2(t,\hat{x}_k)\\&+m_kg_k(t,\hat{x}_k)\mathcal{L}_1^{u_k,u_l^*,\phi_k,\phi_l^*}g_k(t,\hat{x}_k)+\frac{\phi_{k1}^2}{2\beta_{k1}}+\frac{\phi_{k2}^2}{2\beta_{k2}}\Bigg\}=0,\end{aligned}\tag{3.4.22}$$

with $W_k(T,\hat{x}_k)=\hat{x}_k$, *and*

$$g_k(T,\hat{x}_k)=\hat{x}_k,\ \ \mathcal{L}_1^{u_k^*,u_l^*,\phi_k^*,\phi_l^*}g_k(t,\hat{x}_k)=0,\tag{3.4.23}$$

where

$$\begin{aligned}(u_k^*,\phi_k^*):=\arg\sup_{u_k\in\mathcal{U}_k}\inf_{\phi_k\in\Sigma_k}\Bigg\{&\mathcal{L}_1^{u_k,u_l^*,\phi_k,\phi_l^*}W_k(t,\hat{x}_k)-\mathcal{L}_1^{u_k,u_l^*,\phi_k,\phi_l^*}\frac{m_k}{2}g_k^2(t,\hat{x}_k)\\&+m_kg_k(t,\hat{x}_k)\mathcal{L}_1^{u_k,u_l^*,\phi_k,\phi_l^*}g_k(t,\hat{x}_k)+\frac{\phi_{k1}^2}{2\beta_{k1}}+\frac{\phi_{k2}^2}{2\beta_{k2}}\Bigg\},\end{aligned}$$

then $W_k(t,\hat{x}_k)=V_k(t,\hat{x}_k)$, $\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k(u_k^*,u_l^*)\right]=g_k(t,\hat{x}_k)$, *and* u_k^* *is the robust equilibrium reinsurance-investment strategy of insurer* k.

Next, we are going to solve the equilibrium control policy and value function by applying similar procedure as that in Section 3.3.

Equation (3.4.22) is equivalent to

$$\begin{aligned}\sup_{u_k\in\mathcal{U}_k}\inf_{\phi_k\in\Sigma_k}&\Bigg\{\frac{\partial W_k(t,\hat{x}_k)}{\partial t}+\Big[r\hat{x}_k+(\mu-r)(\pi_k-n_k\pi_l^*)+\mu_k(\lambda_k+\lambda)(\gamma_k+\eta_kq_k)\\&-n_k\mu_l(\lambda_l+\lambda)(\gamma_l+\eta_lq_l^*)+\sigma\pi_k\phi_{k1}-\sigma n_k\pi_l^*\phi_{l1}^*+\sigma_kq_k\sqrt{\lambda_k+\lambda}\phi_{k2}\\&-n_k\sigma_lq_l^*\sqrt{\lambda_l+\lambda}\phi_{l2}^*\Big]\frac{\partial W_k(t,\hat{x}_k)}{\partial\hat{x}_k}+\frac{1}{2}\Big(\sigma^2\pi_k^2+n_k^2\sigma^2(\pi_l^*)^2+\sigma_k^2q_k^2(\lambda_k+\lambda)\\&+n_k^2\sigma_l^2(q_l^*)^2(\lambda_l+\lambda)-2\sigma^2n_k\pi_k\pi_l^*-2\rho n_k\sigma_k\sigma_lq_kq_l^*\sqrt{(\lambda_k+\lambda)(\lambda_l+\lambda)}\Big)\\&\times\left(\frac{\partial^2W_k(t,\hat{x}_k)}{\partial\hat{x}_k^2}-m_k\left(\frac{\partial g_k(t,\hat{x}_k)}{\partial\hat{x}_k}\right)^2\right)+\frac{\phi_{k1}^2}{2\beta_{k1}}+\frac{\phi_{k2}^2}{2\beta_{k2}}\Bigg\}=0.\end{aligned}\tag{3.4.24}$$

In order to solve (3.4.23) and (3.4.24), we start from

$$\begin{aligned}W_k(t,\hat{x}_k)&=C_k(t)\hat{x}_k+D_k(t),\\g_k(t,\hat{x}_k)&=\widetilde{C}_k(t)\hat{x}_k+\widetilde{D}_k(t),\end{aligned}$$

with boundary conditions

$$C_k(T)=1,\qquad D_k(T)=0,\qquad \widetilde{C}_k(T)=1,\qquad \widetilde{D}_k(T)=0.$$

Putting the corresponding partial derivatives of $W_k(t,\hat{x}_k)$ into (3.4.24), we obtain

$$\begin{aligned}\sup_{u_k\in\mathcal{U}_k}\inf_{\phi_k\in\Sigma_k}&\bigg\{C_k'\hat{x}_k+D_k'+\Big[r\hat{x}_k+(\mu-r)(\pi_k-n_k\pi_l^*)+\mu_k(\lambda_k+\lambda)(\gamma_k+\eta_k q_k)\\&-n_k\mu_l(\lambda_l+\lambda)(\gamma_l+\eta_l q_l^*)+\sigma\pi_k\phi_{k1}-\sigma n_k\pi_l^*\phi_{l1}^*+\sigma_k q_k\sqrt{\lambda_k+\lambda}\phi_{k2}\\&-n_k\sigma_l q_l^*\phi_{l2}^*\sqrt{\lambda_l+\lambda}\Big]C_k-\frac{m_k\widetilde{C}_k^2}{2}\Big[\sigma^2\pi_k^2+n_k^2\sigma^2(\pi_l^*)^2-2n_k\sigma^2\pi_k\pi_l^*\\&+(\lambda_k+\lambda)q_k^2\sigma_k^2+(\lambda_l+\lambda)n_k^2(q_l^*)^2\sigma_l^2-2\rho n_k\sigma_k\sigma_l q_k q_l^*\sqrt{(\lambda_k+\lambda)(\lambda_l+\lambda)}\Big]\\&+\frac{\phi_{k1}^2}{2\beta_{k1}}+\frac{\phi_{k2}^2}{2\beta_{k2}}\bigg\}=0.\end{aligned}\tag{3.4.25}$$

Fixing u_k and the first-order optimality conditions over ϕ_k yields the infimum point ϕ_k^* as follows:

$$\begin{cases}\phi_{k1}^*=-\beta_{k1}\sigma\pi_k C_k,\\ \phi_{k2}^*=-\sigma_k q_k C_k\beta_{k2}\sqrt{\lambda_k+\lambda}.\end{cases}\tag{3.4.26}$$

Inserting (3.4.26) into (3.4.25), we then have

$$\begin{aligned}\sup_{u_k\in\mathcal{U}_k}&\bigg\{C_k'\hat{x}_k+D_k'+\Big[r\hat{x}_k+(\mu-r)(\pi_k-n_k\pi_l^*)+\mu_k(\lambda_k+\lambda)(\gamma_k+\eta_k q_k)\\&-n_k\mu_l(\lambda_l+\lambda)(\gamma_l+\eta_l q_l^*)-n_k\sigma\pi_l^*\phi_{l1}^*-n_k\sigma_l q_l^*\phi_{l2}^*\sqrt{\lambda_l+\lambda}\Big]C_k\\&-\frac{m_k\widetilde{C}_k^2}{2}\Big[\sigma^2\pi_k^2+n_k^2\sigma^2(\pi_l^*)^2-2\sigma^2 n_k\pi_k\pi_l^*+\sigma_k^2q_k^2(\lambda_k+\lambda)+n_k^2\sigma_l^2(q_l^*)^2(\lambda_l+\lambda)\\&-2\rho n_k\sigma_k\sigma_l q_k q_l^*\sqrt{(\lambda_k+\lambda)(\lambda_l+\lambda)}\Big]-\frac{\beta_{k1}\sigma^2\pi_k^2C_k^2}{2}-\frac{\sigma_k^2q_k^2C_k^2\beta_{k2}(\lambda_k+\lambda)}{2}\bigg\}=0,\end{aligned}\tag{3.4.27}$$

Similarly, according to the first-order conditions for u_k, we have

$$\begin{cases}\pi_k^*=\dfrac{1}{\beta_{k1}C_k^2+m_k\widetilde{C}_k^2}\left[\dfrac{(\mu-r)C_k}{\sigma^2}+m_k n_k\pi_l^*\widetilde{C}_k^2\right],\\ q_k^*=\dfrac{\mu_k\eta_k C_k\sqrt{\lambda_k+\lambda}+\rho\widetilde{C}_k^2m_k n_k\sigma_k\sigma_l q_l^*\sqrt{\lambda_l+\lambda}}{\sigma_k^2\sqrt{\lambda_k+\lambda}(m_k\widetilde{C}_k^2+C_k^2\beta_{k2})}.\end{cases}\tag{3.4.28}$$

If we plug (3.4.28) into (3.4.27), we obtain

$$
\begin{aligned}
&(C_k' + rC_k)\,\hat{x}_k + D_k' + \Big[(\mu - r)(\pi_k^* - n_k\pi_l^*) + \mu_k\gamma_k(\lambda_k+\lambda) + \mu_k\eta_k(\lambda_k+\lambda)q_k^* \\
&\quad - n_k\mu_l\gamma_l(\lambda_l+\lambda) - n_k\mu_l\eta_l(\lambda_l+\lambda)q_l^* + n_k\sigma^2(\pi_l^*)^2C_l\beta_{l1} - \sigma_k^2(q_k^*)^2C_k\beta_{k2}(\lambda_k+\lambda) \\
&\quad + n_k\sigma_l^2(q_l^*)^2C_l\beta_{l2}(\lambda_l+\lambda)\Big]C_k - \frac{m_k\widetilde{C}_k^2}{2}\Big[\sigma^2(\pi_k^*)^2 + n_k^2\sigma^2(\pi_l^*)^2 - 2\sigma^2 n_k\pi_k^*\pi_l^* \\
&\quad + \sigma_k^2(q_k^*)^2(\lambda_k+\lambda) + n_k^2\sigma_l^2(q_l^*)^2\,(\lambda_l+\lambda) - 2\rho n_k\sigma_k\sigma_l q_k^* q_l^*\sqrt{(\lambda_k+\lambda)(\lambda_l+\lambda)}\Big] \\
&\quad - \frac{\beta_{k1}\sigma^2(\pi_k^*)^2C_k^2}{2} + \frac{\sigma_k^2(q_k^*)^2C_k^2\beta_{k2}(\lambda_k+\lambda)}{2} = 0.
\end{aligned}
$$

Substituting (3.4.26) and (3.4.28) into (3.4.23) yields

$$
\begin{aligned}
&\Big(\widetilde{C}_k' + r\widetilde{C}_k\Big)\,\hat{x}_k + \widetilde{D}_k' + \Big[(\mu - r)(\pi_k^* - n_k\pi_l^*) + \mu_k\gamma_k(\lambda_k+\lambda) + \mu_k\eta_k(\lambda_k+\lambda)q_k^* \\
&\quad - n_k\mu_l\gamma_l(\lambda_l+\lambda) - n_k\mu_l\eta_l(\lambda_l+\lambda)q_l^* - \sigma^2(\pi_k^*)^2C_k\beta_{k1} + n_k\sigma^2(\pi_l^*)^2C_l\beta_{l1} \\
&\quad - (\lambda_k+\lambda)\sigma_k^2(q_k^*)^2C_k\beta_{k2} + n_k(\lambda_l+\lambda)\sigma_l^2(q_l^*)^2C_l\beta_{l2}\Big]\widetilde{C}_k = 0.
\end{aligned}
$$

Therefore, we have to solve the following system of ODEs:

$$
\begin{cases}
C_k' + rC_k = 0, \qquad \widetilde{C}_k' + r\widetilde{C}_k = 0, \\
D_k' + \Big[(\mu - r)(\pi_k^* - n_k\pi_l^*) + \mu_k\gamma_k(\lambda_k+\lambda) + \mu_k\eta_k(\lambda_k+\lambda)q_k^* - n_k\mu_l\gamma_l(\lambda_l+\lambda) \\
\quad - n_k\mu_l\eta_l(\lambda_l+\lambda)q_l^* + n_k\sigma^2(\pi_l^*)^2C_l\beta_{l1} - \sigma_k^2(q_k^*)^2C_k\beta_{k2}(\lambda_k+\lambda) \\
\quad + n_k\sigma_l^2(q_l^*)^2C_l\beta_{l2}(\lambda_l+\lambda)\Big]C_k - \frac{m_k\widetilde{C}_k^2}{2}\Big[\sigma^2(\pi_k^*)^2 + n_k^2\sigma^2(\pi_l^*)^2 - 2\sigma^2 n_k\pi_k^*\pi_l^* \\
\quad + \sigma_k^2(q_k^*)^2(\lambda_k+\lambda) + n_k^2\sigma_l^2(q_l^*)^2\,(\lambda_l+\lambda) - 2\rho n_k\sigma_k\sigma_l q_k^* q_l^*\sqrt{(\lambda_k+\lambda)(\lambda_l+\lambda)}\Big] \\
\quad - \frac{\beta_{k1}\sigma^2(\pi_k^*)^2C_k^2}{2} + \frac{\sigma_k^2(q_k^*)^2C_k^2\beta_{k2}(\lambda_k+\lambda)}{2} = 0, \\
\widetilde{D}_k' + \Big[(\mu - r)(\pi_k^* - n_k\pi_l^*) + \mu_k\gamma_k(\lambda_k+\lambda) + \mu_k\eta_k(\lambda_k+\lambda)q_k^* \\
\quad - n_k\mu_l\gamma_l(\lambda_l+\lambda) - n_k\mu_l\eta_l(\lambda_l+\lambda)q_l^* - \sigma^2(\pi_k^*)^2C_k\beta_{k1} + n_k\sigma^2(\pi_l^*)^2C_l\beta_{l1} \\
\quad - (\lambda_k+\lambda)\sigma_k^2(q_k^*)^2C_k\beta_{k2} + n_k(\lambda_l+\lambda)\sigma_l^2(q_l^*)^2C_l\beta_{l2}\Big]\widetilde{C}_k = 0.
\end{cases}
$$

Recalling the boundary conditions, we can obtain

$$
\widetilde{C}_k(t) = e^{r(T-t)}, \qquad C_k(t) = e^{r(T-t)},
$$

$$
\widetilde{D}_k(t) = \frac{n_k\mu_l\gamma_l(\lambda_l+\lambda) - \mu_k\gamma_k(\lambda_k+\lambda)}{r}\left(1 - e^{r(T-t)}\right) + \int_t^T \tilde{d}_{k1}(s)\mathrm{d}s + \int_t^T \tilde{d}_{k2}(s)\mathrm{d}s,
$$

$$
D_k(t) = \frac{n_k\mu_l\gamma_l(\lambda_l+\lambda) - \mu_k\gamma_k(\lambda_k+\lambda)}{r}\left(1 - e^{r(T-t)}\right) + \int_t^T d_{k1}(s)\mathrm{d}s + \int_t^T d_{k2}(s)\mathrm{d}s,
$$

where

$$
\begin{cases}
\tilde{d}_{k1}(s) = \Big[\mu_k\eta_k(\lambda_k+\lambda)q_k^* - n_k\mu_l\eta_l(\lambda_l+\lambda)q_l^* - \beta_{k2}\sigma_k^2(\lambda_k+\lambda)(q_k^*)^2e^{r(T-s)} \\
\qquad + n_k\beta_{l2}\sigma_l^2(\lambda_l+\lambda)(q_l^*)^2e^{r(T-s)}\Big]e^{r(T-s)}, \\
\tilde{d}_{k2}(s) = \Big[(\mu-r)(\pi_k^* - n_k\pi_l^*) - \sigma^2\beta_{k1}(\pi_k^*)^2e^{r(T-s)} + n_k\sigma^2\beta_{l1}(\pi_l^*)^2e^{r(T-s)}\Big]e^{r(T-s)}, \\
d_{k1}(s) = \Big[\mu_k\eta_k(\lambda_k+\lambda)q_k^* - n_k\mu_l\eta_l(\lambda_l+\lambda)q_l^* - \frac{1}{2}\sigma_k^2\beta_{k2}(\lambda_k+\lambda)e^{r(T-s)}(q_k^*)^2 \\
\qquad + n_k\sigma_l^2\beta_{l2}(\lambda_l+\lambda)e^{r(T-s)}(q_l^*)^2\Big]e^{r(T-s)} - \frac{m_ke^{2r(T-s)}}{2}\Big[\sigma_k^2(\lambda_k+\lambda)(q_k^*)^2 \\
\qquad + n_k^2\sigma_l^2\left(\lambda_l+\lambda\right)(q_l^*)^2 - 2\rho n_k\sigma_k\sigma_l q_k^* q_l^*\sqrt{(\lambda_k+\lambda)(\lambda_l+\lambda)}\Big], \\
d_{k2}(s) = \left[(\mu-r)(\pi_k^*-n_k\pi_l^*) + n_k\sigma^2\beta_{l1}e^{r(T-s)}(\pi_l^*)^2\right]e^{r(T-s)} \\
\qquad - \frac{m_ke^{2r(T-s)}\sigma^2}{2}\left[(\pi_k^*)^2 + n_k^2(\pi_l^*)^2 - 2n_k\pi_k^*\pi_l^*\right] - \frac{\beta_{k1}\sigma^2(\pi_k^*)^2e^{2r(T-s)}}{2}.
\end{cases}
$$

Moreover, the time-consistent Nash equilibrium reinsurance-investment strategy of insurer k should satisfy

$$
\begin{cases}
\pi_k^* = \dfrac{1}{\sigma^2(\beta_{k1}+m_k)e^{r(T-t)}}\left[\mu - r + m_kn_k\sigma^2\pi_l^*e^{r(T-t)}\right], \\
q_k^* = \dfrac{\mu_k\eta_k\sqrt{\lambda_k+\lambda} + \rho m_kn_k\sigma_k\sigma_l q_l^*\sqrt{\lambda_l+\lambda}e^{r(T-t)}}{\sigma_k^2\sqrt{\lambda_k+\lambda}(m_k+\beta_{k2})e^{r(T-t)}}.
\end{cases} \tag{3.4.29}
$$

Explicit expressions for the equilibrium reinsurance strategy of each insurer can be obtained by solving the following system of equations:

$$
\begin{cases}
q_1^* = \dfrac{\mu_1\eta_1\sqrt{\lambda_1+\lambda} + m_1\rho n_1\sigma_1\sigma_2q_2^*\sqrt{\lambda_2+\lambda}e^{r(T-t)}}{\sigma_1^2\sqrt{\lambda_1+\lambda}(m_1+\beta_{12})e^{r(T-t)}}, \\
q_2^* = \dfrac{\mu_2\eta_2\sqrt{\lambda_2+\lambda} + m_2\rho n_2\sigma_1\sigma_2q_1^*\sqrt{\lambda_1+\lambda}e^{r(T-t)}}{\sigma_2^2\sqrt{\lambda_2+\lambda}(m_2+\beta_{22})e^{r(T-t)}},
\end{cases}
$$

and explicit expressions for the equilibrium investment strategy of each insurer can be obtained by solving the following system of equations:

$$
\begin{cases}
\pi_1^* = \dfrac{1}{\sigma^2(\beta_{11}+m_1)e^{r(T-t)}}\left[\mu - r + m_1n_1\sigma^2\pi_2^*e^{r(T-t)}\right], \\
\pi_2^* = \dfrac{1}{\sigma^2(\beta_{21}+m_2)e^{r(T-t)}}\left[\mu - r + m_2n_2\sigma^2\pi_1^*e^{r(T-t)}\right].
\end{cases}
$$

Based on the above derivation, we have the following two propositions.

Proposition 3.4.1. *The reinsurance-investment strategy $u_k^* = (q_k^*, \pi_k^*)$ derived by the first-order optimality conditions and given in* (3.4.29) *is definitely the optimal reinsurance and investment strategies of insurer k.*

Proof. The proof is similar to that in Theorem 3.3.2, and hence we don't repeat it here. □

Proposition 3.4.2. *For the robust mean-variance competition problem for insurers in diffusion approximation risk processes, the time-consistent Nash equilibrium reinsurance strategy of each insurer is given by*

$$\begin{cases} q_1^* = \dfrac{\mu_1\eta_1\sqrt{\lambda_1+\lambda}\sigma_2(m_2+\beta_{22})+\rho m_1 n_1\sigma_1\mu_2\eta_2\sqrt{\lambda_2+\lambda}}{\sqrt{\lambda_1+\lambda}e^{r(T-t)}\sigma_1^2\sigma_2\left[(m_1+\beta_{12})(m_2+\beta_{22})-\rho^2 m_1m_2n_1n_2\right]}, \\ q_2^* = \dfrac{\mu_2\eta_2\sqrt{\lambda_2+\lambda}\sigma_1(m_1+\beta_{12})+\rho m_2 n_2\sigma_2\mu_1\eta_1\sqrt{\lambda_1+\lambda}}{\sqrt{\lambda_2+\lambda}e^{r(T-t)}\sigma_2^2\sigma_1\left[(m_1+\beta_{12})(m_2+\beta_{22})-\rho^2 m_1m_2n_1n_2\right]}, \end{cases}$$

and the time-consistent Nash equilibrium investment strategy of each insurer is given by

$$\begin{cases} \pi_1^* = \dfrac{(\mu-r)(\beta_{21}+m_2+m_1n_1)}{\sigma^2 e^{r(T-t)}\left[(\beta_{11}+m_1)(\beta_{21}+m_2)-m_1m_2n_1n_2\right]}, \\ \pi_2^* = \dfrac{(\mu-r)(\beta_{11}+m_1+m_2n_2)}{\sigma^2 e^{r(T-t)}\left[(\beta_{11}+m_1)(\beta_{21}+m_2)-m_1m_2n_1n_2\right]}. \end{cases}$$

The equilibrium value function of insurer k, for $k,l\in\{1,2\}$, $k\neq l$, is given by

$$\begin{aligned} V_k(t,\hat{x}_k) = {} & \hat{x}_k e^{r(T-t)} + \frac{n_k\mu_l\gamma_l(\lambda_l+\lambda)-\mu_k\gamma_k(\lambda_k+\lambda)}{r}\left(1-e^{r(T-t)}\right) \\ & + \int_t^T d_{k1}(s)\mathrm{d}s + \int_t^T d_{k2}(s)\mathrm{d}s, \end{aligned}$$

where

$$\begin{cases} d_{k1}(s) = \Big[\mu_k\eta_k(\lambda_k+\lambda)q_k^* - n_k\mu_l\eta_l(\lambda_l+\lambda)q_l^* - \dfrac{1}{2}\sigma_k^2\beta_{k2}(\lambda_k+\lambda)e^{r(T-s)}(q_k^*)^2 \\ \qquad + n_k\sigma_l^2\beta_{l2}(\lambda_l+\lambda)e^{r(T-s)}(q_l^*)^2\Big]e^{r(T-s)} - \dfrac{m_k e^{2r(T-s)}}{2}\Big[\sigma_k^2(\lambda_k+\lambda)(q_k^*)^2 \\ \qquad + n_k^2\sigma_l^2\left(\lambda_l+\lambda\right)(q_l^*)^2 - 2\rho n_k\sigma_k\sigma_l q_k^* q_l^*\sqrt{(\lambda_k+\lambda)(\lambda_l+\lambda)}\Big], \\ d_{k2}(s) = \left[(\mu-r)(\pi_k^*-n_k\pi_l^*)+n_k\sigma^2\beta_{l1}e^{r(T-s)}(\pi_l^*)^2\right]e^{r(T-s)} \\ \qquad - \dfrac{m_k e^{2r(T-s)}}{2}\left[\sigma^2(\pi_k^*)^2+n_k^2\sigma^2(\pi_l^*)^2-2\sigma^2 n_k\pi_k^*\pi_l^*\right]-\dfrac{\beta_{k1}\sigma^2(\pi_k^*)^2e^{2r(T-s)}}{2}. \end{cases}$$

The expected value of each insurer's relative terminal surplus is given by

$$\begin{aligned} \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}^{u_k^*,u_l^*}(T)\right] = {} & e^{r(T-t)}\hat{x}_k + \frac{n_k\mu_l\gamma_l(\lambda_l+\lambda)-\mu_k\gamma_k(\lambda_k+\lambda)}{r}\left(1-e^{r(T-t)}\right) \\ & + \int_t^T \tilde{d}_{k1}(s)\mathrm{d}s + \int_t^T \tilde{d}_{k2}(s)\mathrm{d}s, \end{aligned}$$

where

$$\begin{cases} \tilde{d}_{k1}(s) = \Big[\mu_k\eta_k(\lambda_k+\lambda)q_k^* - n_k\mu_l\eta_l(\lambda_l+\lambda)q_l^* - \beta_{k2}\sigma_k^2(\lambda_k+\lambda)(q_k^*)^2e^{r(T-s)} \\ \qquad\qquad + n_k\beta_{l2}\sigma_l^2(\lambda_l+\lambda)(q_l^*)^2e^{r(T-s)}\Big]e^{r(T-s)}, \\ \tilde{d}_{k2}(s) = \Big[(\mu-r)(\pi_k^*-n_k\pi_l^*) - \sigma^2\beta_{k1}(\pi_k^*)^2e^{r(T-s)} + n_k\sigma^2\beta_{l1}(\pi_l^*)^2e^{r(T-s)}\Big]e^{r(T-s)}. \end{cases}$$

The worst-case density generator $\phi_k^* := (\phi_{k1}^*, \phi_{k2}^*)$ of insurer k is

$$\begin{cases} \phi_{k1}^* = -\beta_{k1}\sigma\pi_k^*e^{r(T-t)}, \\ \phi_{k2}^* = -\sigma_k q_k^* e^{r(T-t)}\beta_{k2}\sqrt{\lambda_k+\lambda}. \end{cases}$$

Remark 3.4.1. *Comparing the results in Theorem* 3.3.2 *and Proposition* 3.4.2*, it can be noted that the robust equilibrium investment strategy π_k^* of insurer k under the diffusion approximation risk process is the same with that under the classical Cramér-Lundberg risk process. However, different from that in the classical risk process, the k-th insurer's robust optimal retained proportion q_k^* of claims in Proposition* 3.4.2 *has an explicit form which is expressed in terms of his competitor's robust optimal reinsurance strategy q_l^*. This indicates that the diffusion-approximated model is more tractable compared with the compound Poisson model under the non-zero-sum game framework. Moreover, we find that the insurer's robust equilibrium reinsurance strategies under these two cases are independent of the parameters for risky asset.*

3.5 Numerical examples

In this section, we conduct some numerical experiments to provide sensitivity analyses for the robust equilibrium reinsurance and investment strategies derived in Section 3.3 and Section 3.4. The model parameters as our benchmark are shown in Table 3.1. The values of some parameters are borrowed from Bensoussan et al. (2014), Zeng et al. (2016), Siu et al. (2017) and Deng et al. (2018) to make our analyses reasonable. According to (3.4.19), we select the values of λ_k, μ_k and σ_k, for $k \in \{1, 2\}$, to make these two insurers' aggregate claim distributions are similar, which makes considering their competition seem interesting. Specifically, Insurer 2 has larger intensity of incoming claims, but these claims have smaller mean value and variance, and so the drift and diffusion coefficients in (3.4.19) will not differentiate too much from those of Insurer 1. In each of the following figures, we vary the value of one parameter and study the sensitivity of robust equilibrium reinsurance and/or investment strategies with respect to the change of that parameter.

Table 3.1: Summary of parameter values

Common parameters									
t	T	r	μ	σ	λ	ρ			
0	10	0.05	0.1	0.6	2	0.5			
Insurer 1									
λ_1	m_1	θ_1	n_1	β_{11}	β_{12}	β_{13}	η_1	μ_1	σ_1
1	0.5	0.15	0.4	0.2	0.4	0.6	0.2	2.5	5
Insurer 2									
λ_2	m_2	θ_2	n_2	β_{21}	β_{22}	β_{23}	η_2	μ_2	σ_2
4	0.9	0.2	0.8	0.5	0.6	0.7	0.3	2	3

Figures 3.1-3.2 illustrate the effects of ambiguity aversion and risk aversion on the robust equilibrium reinsurance strategies of two insurers obtained in Section 3.3 under the compound Poisson model. Firstly, we can observe that both insurers decrease their respective optimal retention level $q_k^*(0)$, for $k \in \{1, 2\}$, as the ambiguity aversion parameters corresponding to the jump risk become larger. Noting that $\beta_{ki} = 0$, for $k \in \{1, 2\}$, $i \in \{2, 3\}$, corresponds to an ANI and $\beta_{ki} > 0$ corresponds to an AAI, we find that an AAI chooses a more conservative reinsurance strategy than an ANI by ceding more insurance risk to the reinsurer. These results coincide with our intuition in the sense that each insurer is prone to purchasing more reinsurance when his levels of ambiguity aversion increase in order to offset the adverse effects of model misspecification. Moreover, for a fixed ambiguity aversion level, the equilibrium retention level $q_k^*(0)$, for $k \in \{1, 2\}$, decreases as the risk aversion parameter m_k increases. This is due to the fact that the more risk-averse an insurer is, the less the insurance risk he would like to bear. Finally, we note that the insurer's sensitivity parameter n_k to the competition takes no effects on the robust optimal reinsurance strategy given in Theorem 3.3.2, and so the response of $q_k^*(0)$ to n_k is not studied in this case.

In Figure 3.3, we show the effects of ambiguity aversion and competition on the robust equilibrium reinsurance strategies of two insurers under the diffusion-approximated model established in Section 3.4. Explanations for the effects of the ambiguity aversion parameter corresponding to the diffusion risk on the optimal retained proportion of the claims are similar to those in Figures 3.1-3.2. Thus, we don't repeat them here. For a fixed level of ambiguity aversion, we can see that the equilibrium retention level $q_k^*(0)$, for $k \in \{1, 2\}$, increases with the growth of n_k. Particularly, the case with no relative performance concerns is highlighted by $n_k = 0$. Under this circumstance, the insurer tends to purchase the top level of reinsurance protection. In other words, competition makes the insurers more risk-seeking. One possible explanation for this phenomenon would be that the insurer with greater competition sensitivity parameter n_k is more concerned about his expected terminal surplus relative to that of his competitor, and hence he tends to retain more risks, so less reinsurance premium would be paid out and more capital could be accumulated at terminal time.

Figure 3.4 displays the effects of the ambiguity aversion coefficient β_{k1} and the competition sensitivity parameter n_k on the robust equilibrium investment strategy $\pi_k^*(0)$, for $k \in \{1, 2\}$. From Figure 3.4, we find that if the AAI has a higher level of ambiguity aversion, he would reduce the amount invested in the risky asset. Intuitively, this conclusion is reasonable because the insurer would invest less wealth in an asset that he has less information or confidence to mitigate financial risk. This conclusion also indicates that an AAI is more conservative to financial risk than an ANI, which is reflected in the decrement in the amount invested in the risky asset. Additionally, for a fixed ambiguity aversion parameter, the robust equilibrium investment strategy increases as the relative performance concerns parameter n_k increases. That is to say, the presence of competition makes the AAIs more risk-seeking and the insurer with larger competition parameter n_k would increase his amount invested in the stock market. This is because the chance of generating greater wealth than his competitor at terminal time would be enhanced if the insurer increases his exposure on the risky asset.

Figures 3.5-3.6 study the effects of the rate of return μ and the volatility σ of the risky asset on the robust equilibrium investment strategy $\pi_k^*(0)$, for $k \in \{1, 2\}$. As is shown in Figure 3.5, $\pi_k^*(0)$ is a linear increasing function of the rate of return for a given value of volatility, i.e., the higher the rate of return, the more the amount that the insurer invests in the risky asset, which is in line with intuition. Figure 3.6 shows that $\pi_k^*(0)$ is a decreasing function with respect to σ. This can be explained by the fact that a larger σ implies that the stock market becomes more volatile, which induces the insurer to decrease the amount invested in the risky asset. Moreover, Figures 3.5-3.6 capture the effect of the risk aversion parameter m_k on the robust equilibrium investment strategy $\pi_k^*(0)$. We can see that $\pi_k^*(0)$ decreases as m_k increases for fixed μ and σ. This is because the greater m_k is, the more risk-averse insurer k is. Therefore, he tends to reduce the amount invested in the stock market to avoid risks. Finally, we note that $\pi_2^*(0)$ shows relatively less sensitivity with respect to risk aversion parameter m_2 compared with that of insurer 1. One possible reason may be that insurer 2 has a higher level of ambiguity aversion than insurer 1. Hence, even if insurer 2 becomes less risk-averse, he tends to select a conservative and cautious investment strategy and refrains himself from increasing the amount invested in the risky asset dramatically.

Figure 3.7 shows how the time horizon and decision time impact the robust optimal investment strategies. From this figure, we note that $\pi_k^*(0)$ decreases with respect to the time horizon T. This may be attributed to the fact that the decision-makers face more model uncertainty when T becomes larger, and hence they tend to allocate less wealth in the risky asset to reduce the effects of misspecification in the stock dynamics. On the other hand, for a fixed T, we observe that the equilibrium investment strategies $\pi_k^*(t)$ are increasing functions of decision time t. This is because, as time goes by, the decision-makers accumulate more surpluses and can invest a larger proportion of their wealth in the risky asset. These conclusions are consistent with the theoretical results obtained from Equation (3.3.18) in Theorem 3.3.2 and coincide with those in the prior studies, see, for example, Gu et al. (2020), Guan and Wang (2020) and Zhu et al. (2020).

3.6 Concluding remarks

In this chapter, we study a robust non-zero-sum reinsurance-investment game between two competing AAIs who take into account model uncertainty and intend to seek robust equilibrium reinsurance and investment strategies. We formulate

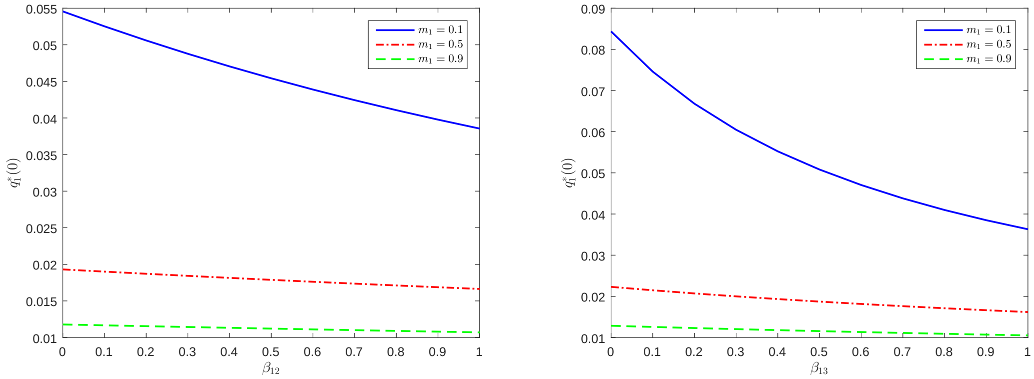


Figure 3.1: Effects of β_{12} and β_{13} on the robust equilibrium reinsurance strategy (Compound Poisson model) of insurer 1.

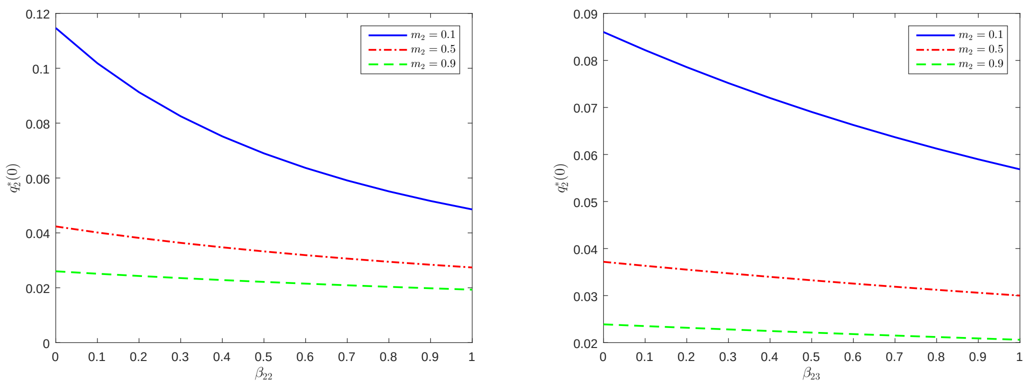


Figure 3.2: Effects of β_{22} and β_{23} on the robust equilibrium reinsurance strategy (Compound Poisson model) of insurer 2.

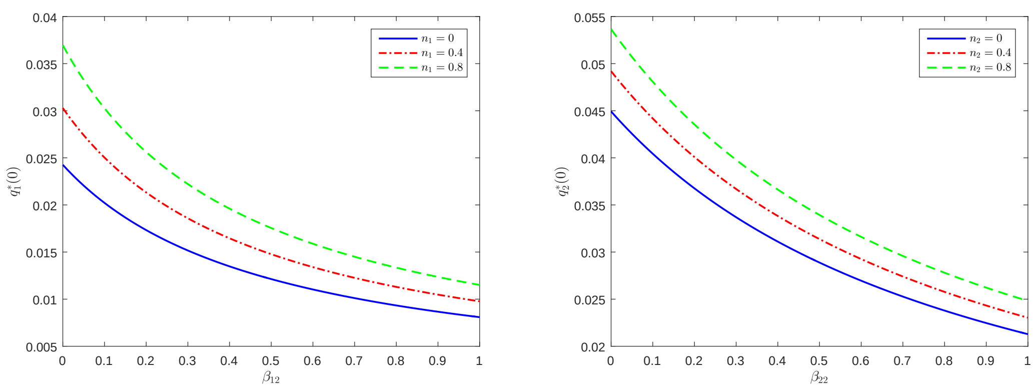


Figure 3.3: Effect of β_{k2} on the robust equilibrium reinsurance strategy (Diffusion-approximated model) of insurer k, for $k \in \{1, 2\}$.

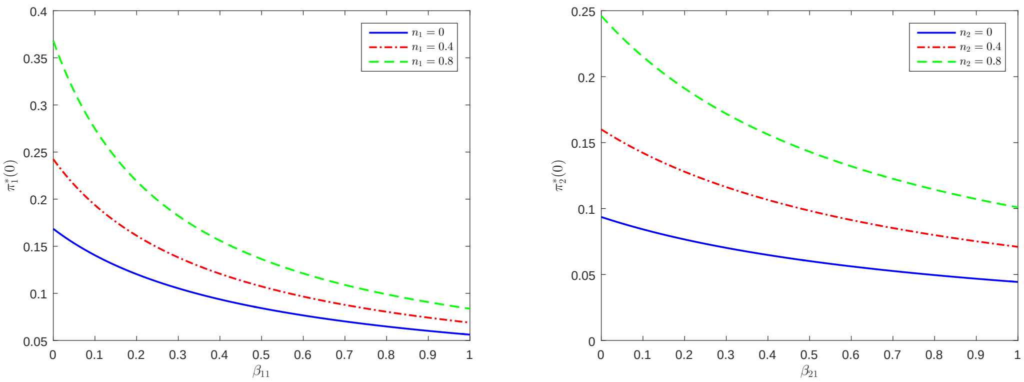


Figure 3.4: Effect of β_{k1} on the robust equilibrium investment strategy of insurer k, for $k \in \{1, 2\}$.

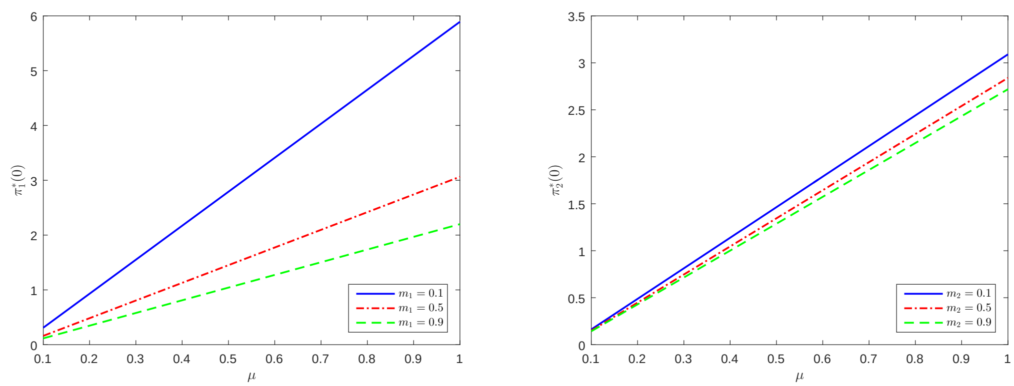


Figure 3.5: Effect of μ on the robust equilibrium investment strategy of insurer k, for $k \in \{1, 2\}$.

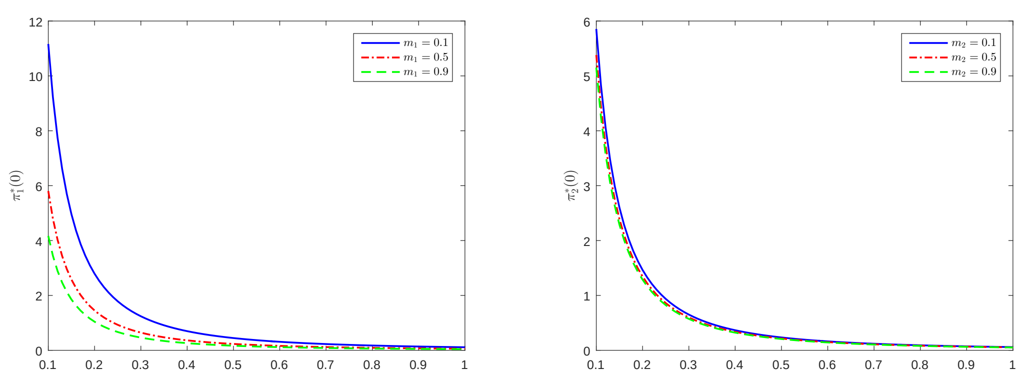


Figure 3.6: Effect of σ on the robust equilibrium investment strategy of insurer k, for $k \in \{1, 2\}$.

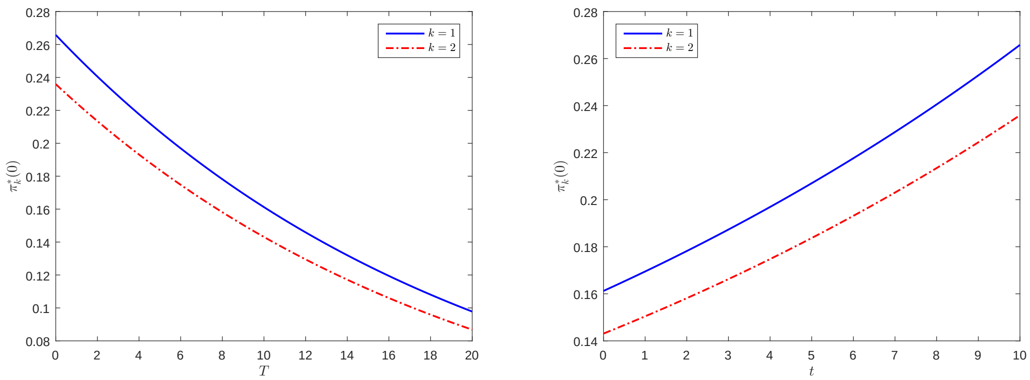


Figure 3.7: Effects of time horizon T and decision time t on the robust equilibrium investment strategy of insurer k, for $k \in \{1, 2\}$.

the competition between these two insurers by assuming that they concern about their relative performance and aim to outperform one another at terminal time. In addition, the mutual dependency between these two AAIs is described by a common Poisson process in their aggregate claim processes. Under the mean-variance criteria, we obtain the time-consistent robust equilibrium reinsurance-investment strategies and equilibrium value functions when the surplus process of each insurer follows the classical Cramér-Lundberg risk model and its diffusion approximation. Numerical examples are provided to illustrate the economic implications of our results. The results show that the insurer who is more concerned about the terminal performance relative to the competing insurer tends to cede less insurance risk to the reinsurer and invest more wealth in the risky asset, which may indicate that competition makes insurers more risk-seeking. Additionally, we find that ambiguity aversion attitudes have significant effects on the robust equilibrium reinsurance-investment strategies. An AAI with a higher level of ambiguity aversion would select more conservative strategies, which is reflected in transferring more claim risks to the reinsurer and decreasing the amount invested in the risky asset.

Chapter 4

Robust Reinsurance Contracts with Mean-variance Criteria

4.1 Introduction

It is quite common that insurance companies manage their assets through investing in a financial market and reduce their risk exposures through purchasing reinsurance protection. This seems to partly motivate the study of optimal reinsurance and investment problems of an insurer in the actuarial science literature. Some commonly used optimality criteria include, for example, the expected utility maximization, the ruin probability minimization and the mean-variance criterion. Using techniques in stochastic optimal control, Zeng et al. (2013) studied an investment and reinsurance optimization problem for mean-variance insurers in a dynamic setting. Meng and Zhang (2013) showed that an excess-of-loss reinsurance contract was better than any other reinsurance forms under their model settings by minimizing the insurer's ruin probability. Zhao et al. (2013) determined the optimal reinsurance-investment strategy by maximizing the expected exponential utility of the insurer's terminal wealth. Further investigation to the optimal investment and reinsurance problems can be found in Promislow and Young (2005), Liang and Yuen (2016), Bi and Cai (2019), and the relevant references therein.

In the aforementioned works, it does not seem that the effects of model ambiguity on optimal reinsurance-investment strategies have been very well-explored. However, it may be noted that some parameters in the financial models, for example, the appreciation rates of risky assets, could be difficult to predict or estimate precisely in the long run, see, for example, Merton (1980). This partly motivates the incorporation of model ambiguity or uncertainty into the reinsurance-investment optimization problems. One popular approach to describe model ambiguity was proposed by Anderson et al. (2003), who studied asset pricing problems in a stochastic continuous-time model setup by incorporating the investor's concerns about model misspecification. Basically, it is a "penalty" approach to robust control, and they considered equivalent priors, which are given by probability measures equivalent to a given reference probability measure, as alternatives and formulated the robust stochastic optimal control problems in a maximin manner. In the past two decades or so, there has been increasing interest of applying the method in Anderson et al. (2003) to study robust optimal investment and reinsurance problems. For example, Maenhout (2004) obtained an optimal portfolio decision for an investor with ambiguity aversion attitudes. Zhang and Siu (2009) studied an optimal reinsurance and investment problem under the expected utility criterion and the survival probability criterion with model uncertainty. Wang and Li (2018) incorporated ambiguity aversion into an optimal investment problem for a defined contribution (DC) pension plan, where stochastic interest rate and stochastic volatility were introduced to model a financial market consisting of a risk-free asset, a rolling bond and a stock. Li et al. (2018) investigated a robust excess-of-loss reinsurance and investment optimization problem for an ambiguity-averse insurer (AAI), where the price process of the risky asset was described by a jump-diffusion model. Pun (2018) constructed a modelling framework for the time-inconsistent stochastic control problems under the consideration of model uncertainty and used a portfolio selection problem to illustrate an application of the framework. Wang et al. (2019b) discussed a robust non-zero-sum stochastic differential game between two competitive insurers who were ambiguity-averse in deriving optimal investment and reinsurance strategies under mean-variance criteria. One may refer to other relevant papers, including Yi et al. (2013), Yi et al. (2015), Sun et al. (2017), Gu et al. (2018) and Feng et al. (2020), to mention a few.

It appears that the optimal reinsurance problems considered in most of the existing literature were discussed from the insurer's point of view and the interests of the reinsurer were not well-explored. However, a reinsurance contract may be thought of as a mutual agreement between an insurer and a reinsurer. Consequently, it seems to be natural to analyze the reinsurance problems from the perspectives of both an insurer and a reinsurer. Some works have been done along this direction. For example, in a discrete-time single-period setting, Cai et al. (2013) designed optimal quota-share and stop-loss reinsurance policies by maximizing the joint survival and profitable probabilities of the insurer and the reinsurer under different premium principles. Li et al. (2016) obtained an optimal reinsurance strategy which maximized the expected exponential utility of the weighted average of the insurer's and reinsurer's wealth at the terminal time. Zhang et al. (2018) developed the optimal quota-share reinsurance agreements using the optimization criteria and utility increment constraints reflecting mutual beneficiary. Another strand of literature applies game theory to model the strategic interaction between an insurer and a reinsurer in optimal reinsurance design. Borch (1960) seemed to be the first to discuss the optimal reinsurance contract problems within the context of bargaining games. Indeed there have been many interesting developments in applications of game theory to reinsurance design problems. For instance, Chen and Shen (2018, 2019) considered the joint interests of the insurer and the reinsurer in a reinsurance contract design problem through modeling the two negotiating parties using a stochastic leader-follower differential game framework. Jiang et al. (2019) derived the Pareto-optimal reinsurance contracts under the two-person cooperative game framework. Chen et al. (2019) studied an optimal risk-sharing problem in a stochastic differential game theoretic framework, where the insurer's objective was to minimize the ruin probability and the main goal of the reinsurer was to maximize her profits up to the time when the insurer's bankruptcy occurred.

On the other hand, the two-party agreement nature of reinsurance suggests that it is natural to adopt the methods in economic contract theory and model the relationship between the insurer and the reinsurer as a principal-agent relationship, where the insurer is the agent and the reinsurer is the principal. Along the direction of a principal-agent framework, Hu et al. (2018a) studied optimal excess-of-loss and proportional reinsurance contracts when the reinsurer was ambiguity-averse and the surplus of the insurer was described by a classical Cramér-Lundberg model. Their objectives were to maximize the expected exponential utility of their terminal wealth in the worst-case scenario over a family of alternative measures. Hu et al. (2018b) investigated optimal proportional reinsurance contracts when the reinsurer had robust preferences and the insurer's claim process was approximated by a diffusion model. Hu and Wang (2019) designed the robust proportional and excess-of-loss reinsurance contracts when both the principal and agent were ambiguity-averse under the classical Cramér-Lundberg model for insurance claims. Gu et al. (2020) discussed an optimal excess-of-loss reinsurance contracting problem when the insurer and the reinsurer were ambiguity-averse. They also supposed that both the insurer and the reinsurer can invest in a financial market consisting of one risk-free asset and one risky asset. In a recent paper by Wang and Siu (2020), robust optimal reinsurance contracting was studied in a principal-agent modeling framework in the presence of risk constraints described by VaR. The principal-agent problems in the aforementioned papers assumed that the principal and the agent shared the same information. However, in reality, the principal can only gain partial information from the agent, and this information asymmetry crucially determines what kind of contract is optimal. Two distinct types of these problems are moral hazard and adverse selection. When the action of the agent is hidden to the principal, we come to moral hazard problems. Seminal works such as Shavell (1979) and Holmstrom (1979) provided a foundation for optimal insurance contracting problems under moral hazard. Doherty and Smetters (2005) developed a two-period principal-agent model and provided empirical evidence of moral hazard in the reinsurance market. For a more recent review about this topic, the readers may refer to Winter (2013) and the references therein. When some key characteristics of the agent are hidden, we may establish adverse selection problems. Following the celebrated works of Rothschild and Stiglitz (1976) and Stiglitz (1977), various models have been proposed to study adverse selection in insurance contracting. Examples include Crocker and Snow (1985), Crocker and Snow (2008), Cohen and Siegelman (2010), Spinnewijn (2017) and Cheung et al. (2019), just to name a few.

(Robust) optimal reinsurance and investment problems under mean-variance criteria have been studied using time-consistent controls in the literature. In the traditional mean-variance optimization problems, it appears that a considerable amount of literature may obtain the pre-commitment strategy which could be time-inconsistent and only optimal at the initial time. However, time-consistency of the optimal strategies is a basic requirement for a rational decision-maker. To articulate this issue, Björk and Murgoci (2010), Björk et al. (2014) and Kronborg and Steffensen (2015) proposed an approach to derive time-consistent investment strategies. A key feature of this approach is to tackle the problem within a non-cooperative game theoretic framework, where the players are the future incarnations of the decision-maker at different time points. In the contexts of insurance, this approach was applied by Li et al. (2015a) to derive the time-consistent reinsurance and investment strategies when the insurer could purchase a proportional reinsurance contract and invest the insurance surplus in a financial market consisting of one risk-free asset, one risky asset, one zero-coupon bond and Treasury Inflation Protected Securities. Lin and Qian (2016) obtained the time-consistent reinsurance-investment strategy for an insurer whose surplus process was governed by a compound Poisson model, and a constant elasticity of variance (CEV) model was used to describe the time-varying volatility of the risky asset. Zeng et al. (2016) studied the robust reinsurance-investment optimization problem for a mean-variance insurer who is concerned about model uncertainty and obtained robust equilibrium strategies when the price process of the risky asset was described by a jump-diffusion model. More literature about the application of this approach can be found in, for example, Zeng et al. (2013), Li et al. (2015b), Guan et al. (2018), Chen and Shen (2019), Wang et al. (2019b) and Wang et al. (2019).

However, it does not seem that the robust reinsurance problems with mean-variance criteria from the perspective of a principal-agent problem have been well-explored. The purpose of this study is to investigate the interaction between an insurer and a reinsurer who are both ambiguity-averse. Suppose that the decision-makers aim to maximize the expected return of the surplus and minimize the corresponding risk. In this case, we apply the mean-variance criteria to formulate the objective functions of the insurer and the reinsurer, where the expected returns and the risks are measured by the expected values and the variances of their terminal surpluses, respectively. Following Hu et al. (2018a,b) and Hu and Wang (2019), we allow the insurer to purchase a proportional reinsurance treaty and extend the safety loading factor of the reinsurer in the expected value premium principle to be time-varying, which is regarded as the choice variable of the reinsurer. Here, we also suppose that the insurer and the reinsurer invest their surpluses in the financial market consisting of one risk-free asset and one risky asset. Additionally, we assume that both the insurer and the reinsurer concern about and are averse to model uncertainty. Specifically, the ambiguity-averse decision-makers may regard the claim process and the financial market's dynamics as reference models and aim to obtain robust strategies under the worst-case scenario over a family of alternative models. The main contribution of this chapter is threefold. First, different from the techniques used in Hu et al. (2018a,b), Hu and Wang (2019) and Gu et al. (2020), where the expected utility maximization criteria were applied, we shall embed non-cooperative games into the principal-agent framework and establish two systems of extended HJB equations to derive the time-consistent optimal reinsurance contract and investment strategies of the insurer and the reinsurer. Another difference between the current chapter and the papers by Hu et al. (2018a,b) and Hu and Wang (2019) is that we allow the contracting parties to invest their surpluses into a risky asset with a view to enhancing their profits. Finally, though Chen and Shen (2019) considered the Stackelberg differential game between the insurer and the reinsurer under mean-variance criteria, they assumed that the decision-makers are ambiguity-neutral.

The remaining parts of this chapter are structured as follows. Section 4.2 presents the formulation of the model. In Section 4.3, we derive the explicit expression for the robust optimal strategies and the equilibrium value functions of the insurer and the reinsurer. We thereafter analyze the decision-makers' utility losses in Section 4.4. Numerical examples are provided to illustrate the effects of some key parameters on the equilibrium reinsurance-investment strategies and the utility losses of the insurer and the reinsurer in Section 4.5. Finally, some concluding remarks are provided in Section 4.6.

4.2 Problem formulation

The model setup considered here resembles to that in Wang and Siu (2020). To describe uncertainties, as it is usual, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a reference probability measure under which a reference model is specified. The time horizon of the model for investment and reinsurance is given by a finite horizon $[0, T]$, where $T < \infty$. The resolution of uncertainties over the horizon $[0, T]$ is described by a $\mathbb{P}$-augmented filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$. The classical Cramér-Lundberg model is considered, where the risk process of the insurer is described as:

$$S(t) = x_0 + pt - \sum_{i=1}^{N(t)} Z_i,$$

where $x_0 \geq 0$ is the initial surplus, p is the constant insurance premium rate, the claim arrival process $\{N(t)\}_{t \in [0,T]}$ is a Poisson process with a constant intensity $\lambda > 0$, and the claim sizes $Z_i, i = 1, 2, \ldots$, are independent and identically distributed (i.i.d.) random variables which are supposed to be independent of $N(t)$ under the measure $\mathbb{P}$. Suppose that the claim size has finite mean and second moment, which are denoted as μ and σ^2, respectively. The constant insurance premium rate p is, for simplicity, supposed to be determined by the expected value principle, i.e., $p = (1 + \theta)\lambda\mu$, where $\theta > 0$, and it is the relative safety loading factor of the insurer.

It is further assumed that an insurance company can purchase proportional reinsurance contracts or acquire new businesses to transfer and manage insurance risks. Though reinsurance policies may take more complicated forms than proportional reinsurance in practice, the consideration of proportional reinsurance here may render the problem more tractable and may hopefully throw light on certain theoretical aspects on the optimal reinsurance and investment problem under the principal-agent modelling framework. Denote by $q(t) : [0, T] \to [0, \infty)$ the level of risk exposure for the insurer at time t. When $q(t) \in [0, 1]$, $q(t)$ is the risk retention level of the insurer at time t. In this case, the insurer must allocate parts of the premium incomes at a rate of $p^q(t)$ at time t to the reinsurer. To simplify the analysis, the reinsurance premium is also evaluated using the expected value principle. In contrast with some existing studies where the relative safety loading factor of the reinsurance is a given positive constant, we assume that the reinsurer's safety loading could be adjusted according to the reinsurance demand, i.e.,

$$p^q(t) = (1 + \eta(t))\mathbb{E}^{\mathbb{P}}(\,\cdot\,). \tag{4.2.1}$$

See also Hu et al. (2018a,b), Hu and Wang (2019) and Wang and Siu (2020), where the same assumption was imposed. Unlike charging the same premium per unit of risk exposure per unit time as in the traditional expected value principle, the assumption (4.2.1) may allow the flexibility in modelling the bargaining process between the insurer and the reinsurer. Following Hu and Wang (2019), we refer to $\eta = \{\eta(t) \geq \theta : 0 \leq t \leq T\}$ as the reinsurance price, and in this chapter we consider non-cheap reinsurance only. Thus, we have that the reinsurance premium payable to the reinsurer is given by:

$$p^q(t) = \lambda\mu(1 + \eta(t))(1 - q(t)).$$

After taking account of reinsurance protection, the insurer's surplus process becomes:

$$U(t) = x_0 + \int_0^t [(1+\theta)\lambda\mu - \lambda\mu(1+\eta(s))(1-q(s))]\,\mathrm{d}s - \sum_{i=1}^{N(t)} q(T_i)Z_i, \tag{4.2.2}$$

where T_i denotes the arrival time of the i-th claim. Using the diffusion approximation to an insurance surplus process in Grandell (1991), the dynamics of $U(t)$ in (4.2.2) can be approximated by the following diffusion process:

$$\mathrm{d}U(t) \approx \lambda\mu(\theta - \eta(t) + q(t)\eta(t))\mathrm{d}t + \sigma\sqrt{\lambda}q(t)\mathrm{d}B(t),$$

where $\{B(t)\}_{t\in[0,T]}$ is a standard one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Similarly, the dynamics of the reinsurer's surplus process can be approximated by the following diffusion process:

$$\mathrm{d}V(t) \approx \lambda\mu\eta(t)(1 - q(t))\mathrm{d}t + \sigma\sqrt{\lambda}(1 - q(t))\mathrm{d}B(t).$$

As in Promislow and Young (2005) and Li et al. (2015a), we assume that the ratio $\sqrt{\lambda}\mu/\sigma$ is large enough (e.g., $\sqrt{\lambda}\mu/\sigma > 3$) to guarantee that the probability of realizing a negative claim is small in any periods of time.

In addition, we assume that both the insurer and the reinsurer invest their surpluses in the financial market consisting of one risk-free asset and one risky asset. The price process $\{S_0(t)\}_{t\in[0,T]}$ of the risk-free asset is given by the following ordinary differential equation (ODE):

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t,$$

where r is the risk-free, instantaneous interest rate, and $S_0(0) = s_0 > 0$. The price process $\{S_1(t)\}_{t\in[0,T]}$ of the risky asset follows a geometric Brownian motion:

$$\mathrm{d}S_1(t) = S_1(t)\left[\tilde{\mu}\mathrm{d}t + \tilde{\sigma}\mathrm{d}\widetilde{B}(t)\right],$$

where $\{\widetilde{B}(t)\}_{t\in[0,T]}$ is another standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, which is supposed to be independent of $\{B(t)\}_{t\in[0,T]}$, $\tilde{\mu} > r$ is the appreciation rate, $\tilde{\sigma} > 0$ denotes the volatility, and $S_1(0) = s_1 > 0$. Note that investments in the risky share by the insurer and the reinsurer were not discussed in Hu et al. (2018a,b), Hu and Wang (2019) or Wang and Siu (2020).

For all $t \in [0, T]$, we denote $\pi(t)$ as the dollar amounts invested by the insurer in the risky asset at time t. The outstanding amount of the surplus, $X^{u,v}(t) - \pi(t)$, is invested in the risk-free asset, where $X^{u,v}(t)$ is the surplus process of the insurer controlled by the reinsurance-investment strategy $u(t) := (q(t), \pi(t))$ and $v(t)$ is the control policy of the reinsurer which will be described later. Hence, under the probability measure $\mathbb{P}$, the surplus process $\{X^{u,v}(t)\}_{t\in[0,T]}$ of the insurer with investments in the financial market is governed by:

$$\begin{aligned}\mathrm{d}X^{u,v}(t) = {} & [rX^{u,v}(t) + (\tilde{\mu} - r)\pi(t) + \lambda\mu(\theta - \eta(t) + q(t)\eta(t))]\mathrm{d}t + \sigma\sqrt{\lambda}q(t)\mathrm{d}B(t) \\ & + \tilde{\sigma}\pi(t)\mathrm{d}\widetilde{B}(t),\end{aligned} \tag{4.2.3}$$

where $X^{u,v}(0) = x_0$ is the initial surplus of the insurer.

Similarly, $\forall\ t \in [0, T]$, we suppose that the reinsurer invests $\tilde{\pi}(t)$ in the risky asset, and the rest of her surplus would be invested in the risk-free asset. Taking account of the reinsurance-investment strategy $v(t) := (\eta(t), \tilde{\pi}(t))$, the surplus process of the reinsurer is expressed as follows:

$$\begin{aligned}\mathrm{d}Y^{u,v}(t) = {} & [rY^{u,v}(t) + (\tilde{\mu} - r)\tilde{\pi}(t) + \lambda\mu\eta(t)(1 - q(t))]\mathrm{d}t + \sigma\sqrt{\lambda}(1 - q(t))\mathrm{d}B(t) \\ & + \tilde{\sigma}\tilde{\pi}(t)\mathrm{d}\widetilde{B}(t),\end{aligned} \tag{4.2.4}$$

where $Y^{u,v}(0) = y_0$ is the reinsurer's initial surplus.

In practice, model uncertainty or ambiguity prevails in financial and insurance modelling. Consequently, it may be of some interest to investigate how the insurer and the reinsurer having ambiguity aversion attitudes make their investment and reinsurance decisions consistently. In this current chapter, we take model uncertainty or ambiguity into account by considering an ambiguity-averse insurer (AAI) and an ambiguity-averse reinsurer (AAR). From the perspectives of the AAI and AAR, the probability measure $\mathbb{P}$ is taken as a reference measure, and they are interested in considering a family of alternative probability measures surrounding the reference measure in a certain sense to be described in the sequel. A class of probability measures which are equivalent to $\mathbb{P}$ is defined. That is,

$$\mathcal{Q} := \{\mathbb{Q} | \mathbb{Q} \sim \mathbb{P}\},$$

where $\mathbb{Q}$ is to be defined in what follows.

For the ease of reference, we define a variable $k \in \{1, 2\}$, where $k = 1$ refers to the insurer and $k = 2$ corresponds to the reinsurer. Define, for each $k \in \{1, 2\}$, an exponential process $\{\Lambda^{\phi_k}(t)\}_{t\in[0,T]}$ by putting:

$$\Lambda^{\phi_k}(t) = \exp\left\{ \int_0^t \phi_{k1}(s) \mathrm{d}B(s) - \frac{1}{2} \int_0^t \phi_{k1}^2(s) \mathrm{d}s + \int_0^t \phi_{k2}(s) \mathrm{d}\widetilde{B}(s) - \frac{1}{2} \int_0^t \phi_{k2}^2(s) \mathrm{d}s \right\}, \tag{4.2.5}$$

where $\{\phi_k(t)\}_{t\in[0,T]}$ is a measurable adapted process and it is defined by $\phi_k(t) := (\phi_{k1}(t), \phi_{k2}(t))'$ for each $t \in [0, T]$.

Assumption 4.2.1. *Suppose that, for each $k \in \{1, 2\}$, the density generator process $\{\phi_k(t)\}_{t\in[0,T]}$ satisfies the following two conditions:*

1. $\{\phi_k(t)\}_{t\in[0,T]}$ *is* $\{\mathcal{F}_t\}_{t\in[0,T]}$*-adapted;*
2. $\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \|\phi_k(t)\|^2 \mathrm{d}t\right)\right] < \infty$ *with* $\|\phi_k(t)\|^2 = \phi_{k1}^2(t) + \phi_{k2}^2(t)$*. This condition is called Novikov's condition.*

We denote Σ_k as the space of all such processes $\{\phi_k(t)\}_{t\in[0,T]}$.

Under Assumption 4.2.1, we know that the exponential process $\{\Lambda^{\phi_k}(t)\}_{t\in[0,T]}$ is a $(\{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$-martingale, for each $k \in \{1, 2\}$. This then implies that, for each

$k \in \{1,2\}$, $\mathbb{E}^{\mathbb{P}}[\Lambda^{\phi_k}(T)] = 1$. Consequently, for each $k \in \{1,2\}$, a new probability measure $\mathbb{Q}_k \sim \mathbb{P}$ on $\mathcal{F}_T$ can be defined by putting:

$$\left.\frac{\mathrm{d}\mathbb{Q}_k}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} := \Lambda^{\phi_k}(T).$$

By the standard Girsanov's theorem for Brownian motions, under an alternative probability measure $\mathbb{Q}_k$, the processes $\{B^{\mathbb{Q}_k}(t)\}_{t\in[0,T]}$, $\{\widetilde{B}^{\mathbb{Q}_k}(t)\}_{t\in[0,T]}$ are real-valued standard Brownian motions, and for each $\{\phi_k(t)\}_{t\in[0,T]} \in \Sigma_k$, we have:

$$\begin{cases} \mathrm{d}B^{\mathbb{Q}_k}(t) = \mathrm{d}B(t) - \phi_{k1}(t)\mathrm{d}t, \\ \mathrm{d}\widetilde{B}^{\mathbb{Q}_k}(t) = \mathrm{d}\widetilde{B}(t) - \phi_{k2}(t)\mathrm{d}t. \end{cases}$$

Accordingly, the insurer's surplus process under the alternative measure $\mathbb{Q}_1$ is given by:

$$\begin{aligned} \mathrm{d}X^{u,v}(t) = \Big[& rX^{u,v}(t) + (\tilde{\mu} - r)\pi(t) + \lambda\mu(\theta - \eta(t) + q(t)\eta(t)) + \sigma\sqrt{\lambda}q(t)\phi_{11}(t) \\ & + \tilde{\sigma}\pi(t)\phi_{12}(t)\Big]\mathrm{d}t + \sigma\sqrt{\lambda}q(t)\mathrm{d}B^{\mathbb{Q}_1}(t) + \tilde{\sigma}\pi(t)\mathrm{d}\widetilde{B}^{\mathbb{Q}_1}(t), \end{aligned} \tag{4.2.6}$$

and the reinsurer's surplus process under the alternative measure $\mathbb{Q}_2$ is governed by the following stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}Y^{u,v}(t) = \Big[& rY^{u,v}(t) + (\tilde{\mu} - r)\tilde{\pi}(t) + \lambda\mu\eta(t)(1 - q(t)) + \sigma\sqrt{\lambda}(1 - q(t))\phi_{21}(t) \\ & + \tilde{\sigma}\tilde{\pi}(t)\phi_{22}(t)\Big]\mathrm{d}t + \sigma\sqrt{\lambda}(1 - q(t))\mathrm{d}B^{\mathbb{Q}_2}(t) + \tilde{\sigma}\tilde{\pi}(t)\mathrm{d}\widetilde{B}^{\mathbb{Q}_2}(t). \end{aligned} \tag{4.2.7}$$

Next, we first define the admissible set of reinsurance and investment strategies for the insurer and the reinsurer in the following two definitions.

Definition 4.2.1. *A reinsurance-investment strategy $u(t) := (q(t), \pi(t))$ is said to be admissible for the insurer, if*

1. *$q(t), \pi(t) \in [0, \infty)$, $\forall\ t \in [0,T]$, that is, the insurer can acquire reinsurance or new business and short-selling for the share is not allowed;*

2. *$\{u(t)\}_{t\in[0,T]}$ is a progressively measurable process with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ and it satisfies that $\mathbb{E}^{\mathbb{Q}_1^*}_{t,x}\left[\int_0^T \|u(t)\|^2\mathrm{d}t\right] < \infty$, where $\|u(t)\|^2 = q^2(t) + \pi^2(t)$, $\mathbb{E}^{\mathbb{Q}_1^*}_{t,x}[\,\cdot\,] = \mathbb{E}^{\mathbb{Q}_1^*}\left[\,\cdot\,\middle|X^{u,v}(t) = x\right]$ and $\mathbb{Q}_1^*$ is an optimal probability measure corresponding to the worst-case scenario to be chosen by the insurer, and it will be determined later;*

3. *For all* $(t,x) \in [0,T] \times \mathbb{R}$, *the SDE* (4.2.3) *has a unique strong solution* $\{X^{u,v}(t)\}_{t\in[0,T]}$, $\mathbb{P}$*-almost surely.*

Let $\mathcal{U}$ denote the set of all admissible strategies for the insurer.

Definition 4.2.2. *A pricing (or reinsurance premium)-investment strategy* $v(t) := (\eta(t), \tilde{\pi}(t))$ *is said to be admissible for the reinsurer, if*

1. $\eta(t), \tilde{\pi}(t) \in [0,\infty)$, $\forall\, t \in [0,T]$, *which indicates that short-selling in the risky share is also not allowed for the reinsurer;*

2. $\{v(t)\}_{t\in[0,T]}$ *is a progressively measurable process with respect to the filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$ *and it satisfies that* $\mathbb{E}^{\mathbb{Q}_2^*}_{t,y}\left[\int_0^T \|v(t)\|^2 \mathrm{d}t\right] < \infty$, *where* $\|v(t)\|^2 = \eta^2(t) + \tilde{\pi}^2(t)$, $\mathbb{E}^{\mathbb{Q}_2^*}_{t,y}[\,\cdot\,] = \mathbb{E}^{\mathbb{Q}_2^*}\left[\,\cdot\,\middle|Y^{u,v}(t) = y\right]$ *and* $\mathbb{Q}_2^*$ *is an optimal probability measure corresponding to the worst-case scenario to be selected by the reinsurer, and it will be determined later;*

3. $\forall\ (t,y) \in [0,T] \times \mathbb{R}$, *the SDE given by* (4.2.4) *has a unique strong solution* $\{Y^{u,v}(t)\}_{t\in[0,T]}$, $\mathbb{P}$*-almost surely.*

Let $\mathcal{V}$ denote the set of all admissible strategies for the reinsurer.

In practice, regulations may prevent insurers and reinsurers from short-selling risky assets. This may partly motivate the assumptions for no short-selling of the risky share in the admissible investment strategies for the insurer and the reinsurer.

In this chapter, both the insurer and the reinsurer are assumed to have a mean-variance preference. Note that the mean-variance preference may be related to a quadratic utility function. When the insurer and the reinsurer are ambiguity-neutral, some of the existing papers derive their optimal reinsurance-investment strategies by considering the optimality of the solution at the initial time, where the corresponding value functions are defined by:

$$\begin{cases} \check{J}_1^v(0,x_0) := \sup\limits_{u\in\mathcal{U}} \left\{\mathbb{E}^{\mathbb{P}}_{0,x_0}\left[X^{u,v}(T)\right] - \frac{m_1}{2}\mathrm{Var}^{\mathbb{P}}_{0,x_0}\left[X^{u,v}(T)\right]\right\}, \\ \check{J}_2^u(0,y_0) := \sup\limits_{v\in\mathcal{V}} \left\{\mathbb{E}^{\mathbb{P}}_{0,y_0}\left[Y^{u,v}(T)\right] - \frac{m_2}{2}\mathrm{Var}^{\mathbb{P}}_{0,y_0}\left[Y^{u,v}(T)\right]\right\}, \end{cases} \tag{4.2.8}$$

where

$$\begin{cases} \mathbb{E}^{\mathbb{P}}_{t,x}[\,\cdot\,] = \mathbb{E}^{\mathbb{P}}\left[\,\cdot\,\middle|X^{u,v}(t) = x\right], \\ \mathbb{E}^{\mathbb{P}}_{t,y}[\,\cdot\,] = \mathbb{E}^{\mathbb{P}}\left[\,\cdot\,\middle|Y^{u,v}(t) = y\right], \\ \mathrm{Var}^{\mathbb{P}}_{t,x}[\,\cdot\,] = \mathrm{Var}^{\mathbb{P}}\left[\,\cdot\,\middle|X^{u,v}(t) = x\right], \\ \mathrm{Var}^{\mathbb{P}}_{t,y}[\,\cdot\,] = \mathrm{Var}^{\mathbb{P}}\left[\,\cdot\,\middle|Y^{u,v}(t) = y\right], \end{cases}$$

and $m_k > 0$, for $k \in \{1, 2\}$, is the risk-averse coefficient of the insurer and the reinsurer. It is obvious that we can only obtain the strategies that are optimal at time zero by solving the optimization problem (4.2.8). As in, for example, Björk et al. (2014) and Kronborg and Steffensen (2015), we aim to establish time-consistent reinsurance-investment strategies by defining time-varying (indirect) value functions for the insurer and the reinsurer as follows: $\forall\ (x, t) \in \mathbb{R} \times [0, T]$ and $\forall\ (y, t) \in \mathbb{R} \times [0, T]$,

$$\begin{cases} \widehat{J}_1^v(t,x) := \sup\limits_{u\in\mathcal{U}} \left\{\mathbb{E}_{t,x}^{\mathbb{P}}\left[X^{u,v}(T)\right] - \frac{m_1}{2}\mathrm{Var}_{t,x}^{\mathbb{P}}\left[X^{u,v}(T)\right]\right\}, \\ \widehat{J}_2^u(t,y) := \sup\limits_{v\in\mathcal{V}} \left\{\mathbb{E}_{t,y}^{\mathbb{P}}\left[Y^{u,v}(T)\right] - \frac{m_2}{2}\mathrm{Var}_{t,y}^{\mathbb{P}}\left[Y^{u,v}(T)\right]\right\}. \end{cases} \tag{4.2.9}$$

Next, we are going to incorporate ambiguity aversion into (4.2.9). The rationale of incorporating ambiguity aversion is that both the insurer and the reinsurer may distrust the veracity of the reference model $\mathbb{P}$ and select an alternative measure $\mathbb{Q}_k$ from $\mathcal{Q}$. Using a robust approach to ambiguity, the insurer and the reinsurer aim at solving the mean-variance optimization problems under the worst-case scenario over a family of alternative probability measures. The objective functions of the insurer and the reinsurer in the robust optimization problems are, respectively, given by:

$$\begin{cases} J_1^v(t,x) := \sup\limits_{u\in\mathcal{U}} \inf\limits_{\mathbb{Q}_1\in\mathcal{Q}} \left\{\mathbb{E}_{t,x}^{\mathbb{Q}_1}\left[X^{u,v}(T)\right] - \frac{m_1}{2}\mathrm{Var}_{t,x}^{\mathbb{Q}_1}\left[X^{u,v}(T)\right] + \mathbb{E}_{t,x}^{\mathbb{Q}_1}\left[P_1(\mathbb{P}\|\mathbb{Q}_1)\right]\right\}, \\ J_2^u(t,y) := \sup\limits_{v\in\mathcal{V}} \inf\limits_{\mathbb{Q}_2\in\mathcal{Q}} \left\{\mathbb{E}_{t,y}^{\mathbb{Q}_2}\left[Y^{u,v}(T)\right] - \frac{m_2}{2}\mathrm{Var}_{t,y}^{\mathbb{Q}_2}\left[Y^{u,v}(T)\right] + \mathbb{E}_{t,y}^{\mathbb{Q}_2}\left[P_2(\mathbb{P}\|\mathbb{Q}_2)\right]\right\}, \end{cases} \tag{4.2.10}$$

where $P_k(\mathbb{P}\|\mathbb{Q}_k) \geq 0$, for $k \in \{1, 2\}$, is a penalty function measuring the divergence of $\mathbb{Q}_k$ from $\mathbb{P}$. Here we allow that the insurer and the reinsurer apply different penalty functions, where P_1 and P_2 are the penalty functions adopted by the insurer and the reinsurer, respectively. The interpretations of the penalty functions would be similar to those in Chapters 2 and 3, so we omit them here.

Throughout this chapter, under a principal-agent modelling framework, we call the insurer (resp., the reinsurer) and the agent (resp., the principal) interchangeably. The robust optimization problems for the insurer and the reinsurer under the principal-agent framework with ambiguity and the mean-variance criterion are presented in the following definitions.

Definition 4.2.3. *The robust mean-variance optimization problem of the insurer is the following stochastic optimization problem:*

$$\begin{cases} \sup\limits_{u\in\mathcal{U}} \inf\limits_{\mathbb{Q}_1\in\mathcal{Q}} \widetilde{J}_1^{\mathbb{Q}_1,u,v}(t,x) \\ := \sup\limits_{u\in\mathcal{U}} \inf\limits_{\mathbb{Q}_1\in\mathcal{Q}} \left\{\mathbb{E}_{t,x}^{\mathbb{Q}_1}\left[X^{u,v}(T)\right] - \frac{m_1}{2}\mathrm{Var}_{t,x}^{\mathbb{Q}_1}\left[X^{u,v}(T)\right] + \mathbb{E}_{t,x}^{\mathbb{Q}_1}\left[P_1(\mathbb{P}\|\mathbb{Q}_1)\right]\right\}, \\ \textit{subject to that } X^{u,v}(t) \textit{ satisfies } (4.2.6), \textit{ for any } v\in\mathcal{V}. \end{cases} \tag{4.2.11}$$

Here, we define

$$J_1^{u,v}(t,x) := \inf_{\mathbb{Q}_1\in\mathcal{Q}} \widetilde{J}_1^{\mathbb{Q}_1,u,v}(t,x).$$

Definition 4.2.4. *The robust mean-variance optimization problem of the reinsurer is the following stochastic optimization problem:*

$$\begin{cases} \sup\limits_{v\in\mathcal{V}} \inf\limits_{\mathbb{Q}_2\in\mathcal{Q}} \widetilde{J}_2^{\mathbb{Q}_2,u^*,v}(t,y) \\ := \sup\limits_{v\in\mathcal{V}} \inf\limits_{\mathbb{Q}_2\in\mathcal{Q}} \left\{\mathbb{E}_{t,y}^{\mathbb{Q}_2}\left[Y^{u^*,v}(T)\right] - \frac{m_2}{2}\mathrm{Var}_{t,y}^{\mathbb{Q}_2}\left[Y^{u^*,v}(T)\right] + \mathbb{E}_{t,y}^{\mathbb{Q}_2}\left[P_2(\mathbb{P}\|\mathbb{Q}_2)\right]\right\}, \\ \textit{subject to that } Y^{u^*,v}(t) \textit{ satisfies } (4.2.7) \textit{ and } u^* \textit{ is an optimal solution to Problem} \\ (4.2.11). \end{cases} \tag{4.2.12}$$

Here, we define

$$J_2^{u,v}(t,y) := \inf_{\mathbb{Q}_2\in\mathcal{Q}} \widetilde{J}_2^{\mathbb{Q}_2,u,v}(t,y).$$

According to the approach in Maenhout (2004), it is easy to show that an increase in the relative entropy in the infinitesimal period from t to $t+\mathrm{d}t$ equals:

$$\frac{1}{2}\left(\phi_{k1}^2(t)+\phi_{k2}^2(t)\right)\mathrm{d}t.$$

To solve Problem (4.2.11), we consider a penalty function of the following form which was used by Maenhout (2004), for example, for the insurer:

$$P_1(\mathbb{P}\|\mathbb{Q}_1) = \int_t^T \Psi_1\left(s,\phi_1(s),X^{u,v}(s)\right)\mathrm{d}s,$$

and define the value function of the insurer as follows:

$$\begin{aligned} V_1(t,x,v) := \sup_{u\in\mathcal{U}} \inf_{\mathbb{Q}_1\in\mathcal{Q}} &\left\{ \mathbb{E}_{t,x}^{\mathbb{Q}_1}\left[X^{u,v}(T)\right] - \frac{m_1}{2}\mathrm{Var}_{t,x}^{\mathbb{Q}_1}\left[X^{u,v}(T)\right] \right.\\ &\left. + \mathbb{E}_{t,x}^{\mathbb{Q}_1}\left[\int_t^T \Psi_1\left(s,\phi_1(s),X^{u,v}(s)\right)\mathrm{d}s\right]\right\} \\ = \sup_{u\in\mathcal{U}} J_1^{u,v}(t,x), & \end{aligned}$$

where

$$\Psi_1\left(s, \phi_1(s), X^{u,v}(s)\right) = \frac{\phi_{11}^2(s)}{2\psi_{11}\left(s, X^{u,v}(s)\right)} + \frac{\phi_{12}^2(s)}{2\psi_{12}\left(s, X^{u,v}(s)\right)}.$$

For each $j \in \{1,2\}$, $\psi_{1j}\left(s, X^{u,v}(s)\right)$ is a strictly positive deterministic function in (s,x). The larger $\psi_{1j}\left(s, X^{u,v}(s)\right)$ is, the less deviation from the reference model is penalized. Consequently, this reflects that the AAI is less confident about the reference model and has an incentive to consider alternative models. In other words, the degree of the AAI's ambiguity aversion increases with respect to the function $\psi_{1j}\left(s, X^{u,v}(s)\right)$. For analytical tractability, as in, for example, Zeng et al. (2016), we assume that ψ_{1j}, for each $j \in \{1,2\}$, is a given state-independent function by putting:

$$\psi_{1j}(t,x) = \beta_{1j},$$

where β_{1j} is the ambiguity aversion coefficient of the insurer with respect to the diffusion risk and $\beta_{1j} \geq 0$. When $\beta_{1j} = 0$, the insurer is ambiguity-neutral for the diffusion risk. Similarly, for the reinsurer's robust optimization problem presented in (4.2.12), the following penalty function is adopted:

$$P_2(\mathbb{P}\|\mathbb{Q}_2) = \int_t^T \Psi_2\left(s, \phi_2(s), Y^{u,v}(s)\right) \mathrm{d}s,$$

and the value function of the reinsurer is defined as:

$$\begin{aligned} V_2(t,y) &:= \sup_{v \in \mathcal{V}} \inf_{\mathbb{Q}_2 \in \mathcal{Q}} \Bigg\{ \mathbb{E}_{t,y}^{\mathbb{Q}_2}\left[Y^{u,v}(T)\right] - \frac{m_2}{2} \mathrm{Var}_{t,y}^{\mathbb{Q}_2}\left[Y^{u,v}(T)\right] \\ &\qquad + \mathbb{E}_{t,y}^{\mathbb{Q}_2}\left[\int_t^T \Psi_2\left(s, \phi_2(s), Y^{u,v}(s)\right) \mathrm{d}s\right] \Bigg\} \\ &= \sup_{v \in \mathcal{V}} J_2^{u,v}(t,y), \end{aligned}$$

where

$$\Psi_2\left(s, \phi_2(s), Y^{u,v}(s)\right) = \frac{\phi_{21}^2(s)}{2\psi_{21}\left(s, Y^{u,v}(s)\right)} + \frac{\phi_{22}^2(s)}{2\psi_{22}\left(s, Y^{u,v}(s)\right)}.$$

For each $j \in \{1,2\}$, it is also supposed that ψ_{2j} is a fixed and state-independent function by setting:

$$\psi_{2j}(t,y) = \beta_{2j},$$

where β_{2j} is the ambiguity aversion parameter of the AAR with respect to the diffusion risk and $\beta_{2j} \geq 0$. The reinsurer is ambiguity-neutral for the diffusion risk when $\beta_{2j} = 0$.

To articulate the time-inconsistency issue in the principal-agent problem given in (4.2.11) and (4.2.12), we follow the approach in, for example, Björk and Murgoci (2010), Björk et al. (2014) and Kronborg and Steffensen (2015). Basically, they formulated the decision-maker's optimization problem with time-inconsistency as a non-cooperative game and sought a subgame perfect Nash equilibrium. This approach has been adopted in Chapter 3 to study the competition between two insurance companies who have mean-variance preference.

The equilibrium strategies and the equilibrium value functions for the optimization problems (4.2.11) and (4.2.12) are defined below. These two definitions appear to be standard, see, for example, Björk et al. (2014) and Kronborg and Steffensen (2015).

Definition 4.2.5. *For any given reinsurance price $\eta(t)$ and any initial states $(t, x) \in [0, T] \times \mathbb{R}$, let $u^*(t) = (q^*(t), \pi^*(t)) = (q^*(t, \eta(t)), \pi^*(t))$ be an admissible strategy of the insurer, and we define the following (perturbed) reinsurance-investment strategy:*

$$u^{\epsilon}(s) := \begin{cases} \hat{u}, & t \leq s < t + \varepsilon, \\ u^*(s), & t + \varepsilon \leq s < T, \end{cases}$$

where $\hat{u} = (\hat{q}, \hat{\pi})$ and $\varepsilon \in \mathbb{R}^+$. If $\forall\ \hat{u} \in \mathbb{R}^+ \times \mathbb{R}^+$, we have

$$\liminf_{\varepsilon \to 0} \frac{J_1^{u^*,v}(t,x) - J_1^{u^{\varepsilon},v}(t,x)}{\varepsilon} \geq 0,$$

then $u^*(t)$ is called an equilibrium reinsurance-investment strategy of the insurer and the equilibrium value function of the insurer is given by:

$$V_1(t, x, v) = J_1^{u^*,v}(t, x),$$

where $J_1^{u^*,v}(t,x)$ was defined in Definition 4.2.3.

Definition 4.2.6. *For any initial states $(t, y) \in [0, T] \times \mathbb{R}$, let $v^*(t) = (\eta^*(t), \tilde{\pi}^*(t))$ be an admissible strategy of the reinsurer, and we define a perturbed strategy as follows:*

$$v^{\epsilon}(s) := \begin{cases} \bar{v}, & t \leq s < t + \varepsilon, \\ v^*(s), & t + \varepsilon \leq s < T, \end{cases}$$

where $\bar{v} = (\bar{\eta}, \bar{\pi})$ *and* $\varepsilon \in \mathbb{R}^+$. *If* $\forall\ \bar{v} \in \mathbb{R}^+ \times \mathbb{R}^+$, *we have*

$$\liminf_{\varepsilon \to 0} \frac{J_2^{u^*,v^*}(t,y) - J_2^{u^*,v^\varepsilon}(t,y)}{\varepsilon} \geq 0,$$

then $v^*(t)$ *is called an equilibrium reinsurance-investment strategy of the reinsurer and the equilibrium value function of the reinsurer is given by:*

$$V_2(t,y) = J_2^{u^*,v^*}(t,y),$$

where $J_2^{u^*,v^*}(t,y)$ *was defined in Definition* 4.2.4. *Furthermore, when there is no risk of confusion, we write*

$$V_1(t,y) := J_1^{u^*,v^*}(t,y).$$

Note that there are two game theoretic problems in our model setting. More precisely, the first one is the game problem between the insurer and the reinsurer from the principal-agent perspective. The other game problem can be regarded as a non-cooperative game between each decision-maker at time t and future incarnations of himself/herself, which is introduced to articulate the time-inconsistency of the optimization problems with mean-variance criteria. Specifically, the equilibrium strategies in the definitions in Definition 4.2.5 and Definition 4.2.6 are time-consistent. Hereafter, the equilibrium strategy solving (4.2.11) and satisfying Definition 4.2.5 is called the robust optimal time-consistent strategy of the insurer; the equilibrium strategy solving (4.2.12) and satisfying Definition 4.2.6 is called the robust optimal time-consistent strategy of the reinsurer; the corresponding equilibrium value functions satisfying Definitions 4.2.5 and 4.2.6 are called the optimal value functions of the insurer and the reinsurer, respectively.

4.3 Solution to the robust reinsurance contract

In this section, we shall present the verification theorems and derive the robust equilibrium reinsurance-investment strategies of the insurer and the reinsurer. Let $C^{1,2}([0,T]\times\mathbb{R})$ denote the space of functions $f(t,x)$ which are continuously differentiable in $t \in [0,T]$ and twice continuously differentiable in $x \in \mathbb{R}$, respectively. Write $D_p^{1,2}([0,T]\times\mathbb{R})$ for the space of functions $f(t,x) \in C^{1,2}([0,T]\times\mathbb{R})$ such that all of its first-order partial derivatives satisfy the polynomial growth conditions.

4.3.1 The insurer's problem

We suppress the arguments of the functions for notational simplicity in the following paragraphs. For all $(t,x) \in [0,T] \times \mathbb{R}$, we define the infinitesimal generator $\mathcal{L}_1$ acting on $W_1(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$ as follows:

$$\mathcal{L}_1^{u,v,\phi_1,\phi_2} W_1(t,x) := \frac{\partial W_1(t,x)}{\partial t} + \Big[rx + (\tilde{\mu} - r)\pi + \lambda\mu(\theta - \eta) + \lambda\mu\eta q + \sigma\sqrt{\lambda}\phi_{11} q \\ + \tilde{\sigma}\phi_{12}\pi \Big] \frac{\partial W_1(t,x)}{\partial x} + \frac{1}{2}\left(\lambda\sigma^2 q^2 + \tilde{\sigma}^2\pi^2\right) \frac{\partial^2 W_1(t,x)}{\partial x^2}.$$

Theorem 4.3.1. *(Verification Theorem for the insurer's optimization problem) For Problem* (4.2.11)*, if there exist real-valued functions $W_1(t,x)$ and $g_1(t,x) \in D_p^{1,2}([0,T] \times \mathbb{R})$ satisfying the following extended HJB system of equations: $\forall\ (t,x) \in [0,T] \times \mathbb{R}$,*

$$\sup_{u \in \mathcal{U}} \inf_{\phi_1 \in \Sigma_1} \Bigg\{ \mathcal{L}_1^{u,v,\phi_1,\phi_2} W_1(t,x) - \mathcal{L}_1^{u,v,\phi_1,\phi_2} \frac{m_1}{2} g_1^2(t,x) \\ m_1 g_1(t,x) \mathcal{L}_1^{u,v,\phi_1,\phi_2} g_1(t,x) + \sum_{j=1}^{2} \frac{\phi_{1j}^2}{2\beta_{1j}} \Bigg\} = 0, \tag{4.3.13}$$

$$W_1(T,x) = x, \tag{4.3.14}$$

$$g_1(T,x) = x, \tag{4.3.15}$$

$$\mathcal{L}_1^{u^*,v,\phi_1^*,\phi_2} g_1(t,x) = 0, \tag{4.3.16}$$

where

$$(u^*, \phi_1^*) := \arg\sup_{u \in \mathcal{U}} \inf_{\phi_1 \in \Sigma_1} \Bigg\{ \mathcal{L}_1^{u,v,\phi_1,\phi_2} W_1(t,x) - \mathcal{L}_1^{u,v,\phi_1,\phi_2} \frac{m_1}{2} g_1^2(t,x) \\ + m_1 g_1(t,x) \mathcal{L}_1^{u,v,\phi_1,\phi_2} g_1(t,x) + \sum_{j=1}^{2} \frac{\phi_{1j}^2}{2\beta_{1j}} \Bigg\},$$

then $W_1(t,x) = V_1(t,x)$, $\mathbb{E}_{t,x}^{\mathbb{Q}_1^*}\left[X^{u^*,v}(T)\right] = g_1(t,x)$, and u^* is the robust equilibrium reinsurance-investment strategy of the insurer; ϕ_1^* is the worst-case scenario density generator of the insurer.

Proof. The proof of this theorem is similar to that of Theorem 4.1 in Björk and Murgoci (2010), and so we do not repeat it here. □

Simplifying Equation (4.3.13) in Theorem 4.3.1, we obtain:

$$\sup_{u\in\mathcal{U}}\inf_{\phi_1\in\Sigma_1}\Bigg\{\frac{\partial W_1(t,x)}{\partial t}+\Big[rx+(\tilde{\mu}-r)\pi+\lambda\mu(\theta-\eta)+\lambda\mu\eta q+\sigma\sqrt{\lambda}\phi_{11}q$$
$$+\tilde{\sigma}\phi_{12}\pi\Big]\frac{\partial W_1(t,x)}{\partial x}+\frac{1}{2}\left(\lambda\sigma^2q^2+\tilde{\sigma}^2\pi^2\right)\left(\frac{\partial^2 W_1(t,x)}{\partial x^2}-m_1\left(\frac{\partial g_1(t,x)}{\partial x}\right)^2\right)$$
$$+\frac{\phi_{11}^2}{2\beta_{11}}+\frac{\phi_{12}^2}{2\beta_{12}}\Bigg\}=0. \tag{4.3.17}$$

To solve (4.3.16) and (4.3.17), it is conjectured that the solutions have the following separated affine forms:

$$W_1(t,x)=A_1(t)x+B_1(t),\ A_1(T)=1,\ B_1(T)=0,$$

$$g_1(t,x)=\widetilde{A}_1(t)x+\widetilde{B}_1(t),\ \widetilde{A}_1(T)=1,\ \widetilde{B}_1(T)=0.$$

where the terminal conditions for A_1, B_1, $\widetilde{A}_1$ and $\widetilde{B}_1$ are determined from the terminal conditions for W_1 and g_1 in (4.3.14) and (4.3.15). These functions are supposed to be sufficiently smooth.
Differentiating W_1 and g_1 with respect to t and state variables gives:

$$\frac{\partial W_1(t,x)}{\partial t}=A_1'(t)x+B_1'(t),\qquad \frac{\partial W_1(t,x)}{\partial x}=A_1(t),\qquad \frac{\partial^2 W_1(t,x)}{\partial x^2}=0, \tag{4.3.18}$$
$$\frac{\partial g_1(t,x)}{\partial t}=\widetilde{A}_1'(t)x+\widetilde{B}_1'(t),\qquad \frac{\partial g_1(t,x)}{\partial x}=\widetilde{A}_1(t),\qquad \frac{\partial^2 g_1(t,x)}{\partial x^2}=0. \tag{4.3.19}$$

Substituting (4.3.18) and (4.3.19) into (4.3.17) yields:

$$\sup_{u\in\mathcal{U}}\inf_{\phi_1\in\Sigma_1}\Bigg\{A_1'x+B_1'+\Big[rx+(\tilde{\mu}-r)\pi+\lambda\mu(\theta-\eta)+\lambda\mu\eta q+\sigma\sqrt{\lambda}\phi_{11}q+\tilde{\sigma}\phi_{12}\pi\Big]$$
$$\times A_1-\frac{m_1\widetilde{A}_1^2}{2}\left(\lambda\sigma^2q^2+\tilde{\sigma}^2\pi^2\right)+\frac{\phi_{11}^2}{2\beta_{11}}+\frac{\phi_{12}^2}{2\beta_{12}}\Bigg\}=0. \tag{4.3.20}$$

For each fixed u, the first-order condition of the left-hand side of (4.3.20) with respect to ϕ_1 gives rise to the infimum point $\phi_1^*(t):=(\phi_{11}^*(t),\phi_{12}^*(t))$ as follows:

$$\begin{cases}\phi_{11}^*(t)=-\beta_{11}\sigma\sqrt{\lambda}A_1(t)q(t),\\ \phi_{12}^*(t)=-\beta_{12}\tilde{\sigma}A_1(t)\pi(t).\end{cases} \tag{4.3.21}$$

Next, we shall justify that ϕ_1^* given in (4.3.21) is the infimum point by evaluating the second-order derivatives. That is to check the convexity conditions. To this end, we gather the terms of ϕ_{1j}, for $j \in \{1,2\}$, in the left-hand side of (4.3.20) and define the following functions:

$$\begin{cases} f_1(\phi_{11}) := \sigma\sqrt{\lambda} q\phi_{11}A_1 + \frac{\phi_{11}^2}{2\beta_{11}}, \\ f_2(\phi_{12}) := \tilde{\sigma}\pi\phi_{12}A_1 + \frac{\phi_{12}^2}{2\beta_{12}}. \end{cases}$$

Accordingly, we have that:

$$f_j''(\phi_{1j}) = \frac{1}{\beta_{1j}} > 0, \qquad j \in \{1,2\},$$

which implies that the first-order optimality condition gives rise to the infimum point of the left-hand side of (4.3.20).

Putting (4.3.21) back into (4.3.20), we obtain:

$$\sup_{u\in\mathcal{U}} \Bigg\{ A_1'x + B_1' + \Big[rx + (\tilde{\mu} - r)\pi + \lambda\mu(\theta - \eta) + \lambda\mu\eta q - \beta_{11}\sigma^2\lambda A_1 q^2 - \beta_{12}\tilde{\sigma}^2 A_1\pi^2\Big] \\ \times A_1 - \frac{m_1\widetilde{A}_1^2}{2}\left(\lambda\sigma^2 q^2 + \tilde{\sigma}^2\pi^2\right) + \frac{\beta_{11}\lambda\sigma^2A_1^2q^2}{2} + \frac{\beta_{12}\tilde{\sigma}^2A_1^2\pi^2}{2}\Bigg\} = 0. \tag{4.3.22}$$

The first-order condition of the left-hand side of (4.3.22) with respect to u gives the optimal reinsurance-investment strategy $u^*(t) := (q^*(t), \pi^*(t))$ of the insurer as follows:

$$\begin{cases} q^*(t) = \dfrac{\mu\eta(t)A_1(t)}{\beta_{11}\sigma^2A_1^2(t) + m_1\sigma^2\widetilde{A}_1^2(t)}, & \text{(4.3.23a)} \\ \pi^*(t) = \dfrac{(\tilde{\mu} - r)A_1(t)}{m_1\tilde{\sigma}^2\widetilde{A}_1^2(t) + \beta_{12}\tilde{\sigma}^2A_1^2(t)}. & \text{(4.3.23b)} \end{cases}$$

In order to check that u^* is the maximum point, we define

$$h_1(\pi) := \left[(\tilde{\mu} - r)\pi - \beta_{12}\tilde{\sigma}^2A_1\pi^2\right]A_1 - \frac{\tilde{\sigma}^2\pi^2(m_1\widetilde{A}_1^2 - \beta_{12}A_1^2)}{2},$$

and then we have the following second-order condition:

$$h_1''(\pi) = -m_1\widetilde{A}_1^2\tilde{\sigma}^2 - \beta_{12}A_1^2\tilde{\sigma}^2 < 0.$$

Finally, we define the function involving the insurer's reinsurance strategy

$$h_2(q) := \lambda\mu q\eta A_1 - \frac{m_1\widetilde{A}_1^2}{2}\lambda\sigma^2q^2 - \frac{\beta_{11}\sigma^2\lambda q^2A_1^2}{2},$$

which leads to the following second-order condition:

$$h_2''(q) = -m_1\widetilde{A}_1^2\lambda\sigma^2 - \beta_{11}\sigma^2\lambda A_1^2 < 0.$$

Therefore, the reinsurance-investment strategy given in (4.3.23) is the maximizer of the left-hand side of (4.3.22).

Substituting q^* and π^* in (4.3.23) into (4.3.16) and (4.3.22), we obtain:

$$\begin{aligned}\left(\widetilde{A}_1' + r\widetilde{A}_1\right)x + \widetilde{B}_1' + \Big[(\tilde{\mu} - r)\pi^* + \lambda\mu(\theta-\eta) + \lambda\mu\eta q^* \\ - \beta_{11}\sigma^2\lambda A_1(q^*)^2 - \beta_{12}\tilde{\sigma}^2 A_1(\pi^*)^2\Big]\widetilde{A}_1 = 0,\end{aligned} \tag{4.3.24}$$

and

$$\begin{aligned}(A_1' + rA_1)x + B_1' + \Big[(\tilde{\mu} - r)\pi^* + \lambda\mu(\theta-\eta) + \lambda\mu\eta q^*\Big]A_1 \\ - \lambda\sigma^2(q^*)^2\left(\frac{m_1\widetilde{A}_1^2}{2} + \frac{\beta_{11}A_1^2}{2}\right) - \tilde{\sigma}^2(\pi^*)^2\left(\frac{m_1\widetilde{A}_1^2}{2} + \frac{\beta_{12}A_1^2}{2}\right) = 0.\end{aligned} \tag{4.3.25}$$

By separating the variables with and without x, respectively, we can obtain the following system of equations:

$$\begin{cases}\widetilde{A}_1' + r\widetilde{A}_1 = 0, \qquad A_1' + rA_1 = 0, \\ \widetilde{B}_1' + \Big[(\tilde{\mu} - r)\pi^* + \lambda\mu(\theta-\eta) + \lambda\mu\eta q^* - \beta_{11}\sigma^2\lambda A_1(q^*)^2 - \beta_{12}\tilde{\sigma}^2 A_1(\pi^*)^2\Big]\widetilde{A}_1 = 0, \\ B_1' + \Big[(\tilde{\mu} - r)\pi^* + \lambda\mu(\theta-\eta) + \lambda\mu\eta q^*\Big]A_1 \\ \quad - \lambda\sigma^2(q^*)^2\left(\frac{m_1\widetilde{A}_1^2}{2} + \frac{\beta_{11}A_1^2}{2}\right) - \tilde{\sigma}^2(\pi^*)^2\left(\frac{m_1\widetilde{A}_1^2}{2} + \frac{\beta_{12}A_1^2}{2}\right) = 0.\end{cases}$$

Solving the above equations with the respective boundary conditions gives:

$$\widetilde{A}_1(t) = e^{r(T-t)}, \qquad A_1(t) = e^{r(T-t)},$$

$$\widetilde{B}_1(t) = \int_t^T \lambda\mu(\theta-\eta)\mathrm{d}s + \int_t^T \tilde{b}_{11}(s)\mathrm{d}s + \int_t^T \tilde{b}_{12}(s)\mathrm{d}s,$$

$$B_1(t) = \int_t^T \lambda\mu(\theta-\eta)\mathrm{d}s + \int_t^T b_{11}(s)\mathrm{d}s + \int_t^T b_{12}(s)\mathrm{d}s,$$

where

$$\begin{cases}\tilde{b}_{11}(s) = \left[\lambda\mu\eta q^*(s) - \beta_{11}\sigma^2\lambda(q^*(s))^2 e^{r(T-s)}\right]e^{r(T-s)}, \\ \tilde{b}_{12}(s) = \left[(\tilde{\mu} - r)\pi^*(s) - \beta_{12}\tilde{\sigma}^2(\pi^*(s))^2 e^{r(T-s)}\right]e^{r(T-s)},\end{cases} \tag{4.3.26}$$

and

$$
\begin{cases}
b_{11}(s) = \left[\lambda\mu\eta q^*(s) - \dfrac{\lambda\sigma^2(q^*(s))^2}{2}(m_1+\beta_{11})e^{r(T-s)}\right]e^{r(T-s)}, \\
b_{12}(s) = \left[(\tilde{\mu}-r)\pi^*(s) - \dfrac{\tilde{\sigma}^2(\pi^*(s))^2}{2}(m_1+\beta_{12})e^{r(T-s)}\right]e^{r(T-s)}.
\end{cases}
\tag{4.3.27}
$$

It should be noted that the solution to the insurer's robust optimization problem in (4.2.11) is derived based on a given reinsurance price η, and the equilibrium reinsurance price η^* would be determined in the next subsection. A reinsurance contract (q, η) is said to be incentive compatible if and only if the insurer's risk retention level q and the reinsurance premium to be determined by the reinsurer, say η, satisfy (4.3.23a). From (4.3.23a), it can be seen that the insurer's optimal retained proportion q^* of insurance risk increases linearly with the given reinsurance price η. This conclusion is in line with the economic intuition that the insurer determines the optimal reinsurance demand when the insurer is given the information about the reinsurance price. Furthermore, the optimal reinsurance demand $1-q^*$ decreases as the reinsurance price increases. This appears to be consistent with the law of demand in economic theory. A similar conclusion was also drawn in Wang and Siu (2020), where a robust optimal reinsurance contract with risk constraint was derived.

4.3.2 The reinsurer's problem

The optimization problem of the reinsurer is discussed in this subsection. First, for all $(t, y) \in [0, T] \times \mathbb{R}$, we define an infinitesimal generator $\mathcal{L}_2$ acting on $W_2(t, y) \in C^{1,2}([0, T] \times \mathbb{R})$ as follows:

$$
\begin{aligned}
\mathcal{L}_2^{u,v,\phi_1,\phi_2} W_2(t,y) := & \frac{\partial W_2(t,y)}{\partial t} + \Big[ry + (\tilde{\mu}-r)\tilde{\pi} + \lambda\mu\eta(1-q) + \sigma\sqrt{\lambda}\phi_{21}(1-q) \\
& + \tilde{\sigma}\phi_{22}\tilde{\pi}\Big]\frac{\partial W_2(t,y)}{\partial y} + \frac{1}{2}\left(\lambda\sigma^2(1-q)^2 + \tilde{\sigma}^2\tilde{\pi}^2\right)\frac{\partial^2 W_2(t,y)}{\partial y^2}.
\end{aligned}
$$

The following verification theorem for the reinsurer is stated without giving the proof, which follows similarly from that of Theorem 4.1 in Björk and Murgoci (2010).

Theorem 4.3.2. *(Verification Theorem for the reinsurer's optimization problem) For Problem* (4.2.12)*, if there exist real-valued functions* $W_2(t, y)$ *and* $g_2(t, y) \in$

$D_p^{1,2}([0,T]\times\mathbb{R})$ *satisfying the following extended HJB system of equations:* $\forall\ (t,y)\in[0,T]\times\mathbb{R}$,

$$\begin{aligned}\sup_{v\in\mathcal{V}}\inf_{\phi_2\in\Sigma_2}\Bigg\{&\mathcal{L}_2^{u^*,v,\phi_1^*,\phi_2}W_2(t,y)-\mathcal{L}_2^{u^*,v,\phi_1^*,\phi_2}\frac{m_2}{2}g_2^2(t,y)\\&+m_2g_2(t,y)\mathcal{L}_2^{u^*,v,\phi_1^*,\phi_2}g_2(t,y)+\sum_{j=1}^{2}\frac{\phi_{2j}^2}{2\beta_{2j}}\Bigg\}=0,\end{aligned}\tag{4.3.28}$$

$$W_2(T,y)=y,\tag{4.3.29}$$

$$g_2(T,y)=y,\tag{4.3.30}$$

$$\mathcal{L}_2^{u^*,v^*,\phi_1^*,\phi_2^*}g_2(t,y)=0,\tag{4.3.31}$$

where

$$\begin{aligned}(v^*,\phi_2^*):=\arg\sup_{v\in\mathcal{V}}\inf_{\phi_2\in\Sigma_2}\Bigg\{&\mathcal{L}_2^{u^*,v,\phi_1^*,\phi_2}W_2(t,y)-\mathcal{L}_2^{u^*,v,\phi_1^*,\phi_2}\frac{m_2}{2}g_2^2(t,y)\\&+m_2g_2(t,y)\mathcal{L}_2^{u^*,v,\phi_1^*,\phi_2}g_2(t,y)+\sum_{j=1}^{2}\frac{\phi_{2j}^2}{2\beta_{2j}}\Bigg\},\end{aligned}$$

and u^* is the robust equilibrium reinsurance-investment strategy of the insurer; ϕ_1^* is the worst-case scenario density generator of the insurer in Theorem 4.3.1, then $W_2(t,y)=V_2(t,y)$, $\mathbb{E}_{t,y}^{\mathbb{Q}_2^*}\left[Y^{u^*,v^*}(T)\right]=g_2(t,y)$, and v^* is the robust equilibrium reinsurance-investment strategy of the reinsurer; ϕ_2^* is the worst-case scenario density generator of the reinsurer.

Equation (4.3.28) in Theorem 4.3.2 is equivalent to:

$$\begin{aligned}\sup_{v\in\mathcal{V}}\inf_{\phi_2\in\Sigma_2}\Bigg\{&\frac{\partial W_2(t,y)}{\partial t}+\left[ry+(\tilde{\mu}-r)\tilde{\pi}+\lambda\mu\eta(1-q^*)+\sigma\sqrt{\lambda}\phi_{21}(1-q^*)+\tilde{\sigma}\phi_{22}\tilde{\pi}\right]\\&\times\frac{\partial W_2(t,y)}{\partial y}+\frac{1}{2}\left(\lambda\sigma^2(1-q^*)^2+\tilde{\sigma}^2\tilde{\pi}^2\right)\left(\frac{\partial^2W_2(t,y)}{\partial y^2}-m_2\left(\frac{\partial g_2(t,y)}{\partial y}\right)^2\right)\\&+\frac{\phi_{21}^2}{2\beta_{21}}+\frac{\phi_{22}^2}{2\beta_{22}}\Bigg\}=0.\end{aligned}\tag{4.3.32}$$

To solve (4.3.31) and (4.3.32), the following trial solutions which are of affine forms are considered:

$$\begin{cases}W_2(t,y)=A_2(t)y+B_2(t),\\A_2(T)=1,\quad B_2(T)=0,\end{cases}$$

and

$$\begin{cases} g_2(t,y) = \widetilde{A}_2(t)y + \widetilde{B}_2(t), \\ \widetilde{A}_2(T) = 1, \quad \widetilde{B}_2(T) = 0, \end{cases}$$

Again, the terminal conditions for A_2, B_2, $\widetilde{A}_2$ and $\widetilde{B}_2$ are determined from the terminal conditions for W_2 and g_2. These functions are supposed to be sufficiently smooth.

Putting the corresponding partial derivatives of W_2 and g_2 into (4.3.32), we obtain:

$$\sup_{v\in\mathcal{V}} \inf_{\phi_2\in\Sigma_2} \Bigg\{ A_2'y + B_2' + \Big[ry + (\tilde{\mu}-r)\tilde{\pi} + \lambda\mu\eta(1-q^*) + \sigma\sqrt{\lambda}\phi_{21}(1-q^*) + \tilde{\sigma}\phi_{22}\tilde{\pi} \Big] A_2 - \frac{m_2\widetilde{A}_2^2}{2}\left(\lambda\sigma^2(1-q^*)^2 + \tilde{\sigma}^2\tilde{\pi}^2\right) + \frac{\phi_{21}^2}{2\beta_{21}} + \frac{\phi_{22}^2}{2\beta_{22}} \Bigg\} = 0. \tag{4.3.33}$$

For each fixed v, the first-order condition of the left-hand side of (4.3.33) with respect to ϕ_2 gives the minimum point $\phi_2^*(t) := (\phi_{21}^*(t), \phi_{22}^*(t))$ as follows:

$$\begin{cases} \phi_{21}^*(t) = -\beta_{21}\sigma\sqrt{\lambda}A_2(t)(1-q^*(t)), \\ \phi_{22}^*(t) = -\beta_{22}\tilde{\sigma}A_2(t)\tilde{\pi}(t). \end{cases} \tag{4.3.34}$$

We can follow the similar procedures as those in Subsection 4.3.1 to verify that ϕ_2^* given in (4.3.34) gives rise to the minimum point of the left-hand side of (4.3.33), so we do not repeat them here.

Substituting (4.3.34) into (4.3.33), we obtain:

$$\sup_{v\in\mathcal{V}} \Bigg\{ A_2'y + B_2' + \Big[ry + (\tilde{\mu}-r)\tilde{\pi} + \lambda\mu\eta(1-q^*) - \beta_{21}\sigma^2\lambda A_2(1-q^*)^2 - \beta_{22}\tilde{\sigma}^2 A_2\tilde{\pi}^2 \Big] \times A_2 - \frac{m_2\widetilde{A}_2^2}{2}\left(\lambda\sigma^2(1-q^*)^2 + \tilde{\sigma}^2\tilde{\pi}^2\right) + \frac{\beta_{21}\lambda\sigma^2 A_2^2(1-q^*)^2}{2} + \frac{\beta_{22}\tilde{\sigma}^2 A_2^2\tilde{\pi}^2}{2} \Bigg\} = 0. \tag{4.3.35}$$

Substituting q^* in (4.3.23a) into (4.3.35), we obtain:

$$\sup_{v\in\mathcal{V}} \Bigg\{ A_2'y + B_2' + \left[ry + (\tilde{\mu}-r)\tilde{\pi} + \lambda\mu\eta - \frac{\lambda\mu^2\eta^2 A_1}{\beta_{11}\sigma^2 A_1^2 + m_1\sigma^2\widetilde{A}_1^2} \right] A_2 - \lambda\sigma^2 \times \left(\frac{m_2\widetilde{A}_2^2}{2} + \frac{\beta_{21}A_2^2}{2} \right) \left(1 - \frac{2\mu\eta A_1}{\beta_{11}\sigma^2 A_1^2 + m_1\sigma^2\widetilde{A}_1^2} + \frac{\mu^2\eta^2 A_1^2}{\left(\beta_{11}\sigma^2 A_1^2 + m_1\sigma^2\widetilde{A}_1^2\right)^2} \right) \Bigg\} = 0. \tag{4.3.36}$$

Similarly, the first-order condition of the left-hand side of (4.3.36) with respect to v gives the maximum point $v^*(t) := (\eta^*(t), \tilde{\pi}^*(t))$ as follows:

$$
\begin{cases}
\eta^*(t) = \dfrac{A_2(t)\sigma^2(\beta_{11}A_1^2(t) + m_1\widetilde{A}_1^2(t)) + A_1(t)\sigma^2(\beta_{21}A_2^2(t) + m_2\widetilde{A}_2^2(t))}{2\mu A_1(t)A_2(t) + \frac{\mu A_1^2(t)(\beta_{21}A_2^2(t)+m_2\widetilde{A}_2^2(t))}{\beta_{11}A_1^2(t)+m_1\widetilde{A}_1^2(t)}}, & \text{(4.3.37a)} \\
\tilde{\pi}^*(t) = \dfrac{(\tilde{\mu}-r)A_2(t)}{\tilde{\sigma}^2(\beta_{22}A_2^2(t) + m_2\widetilde{A}_2^2(t))}. & \text{(4.3.37b)}
\end{cases}
$$

Putting η^* and $\tilde{\pi}^*$ in (4.3.37) back into (4.3.31) and (4.3.35) gives:

$$
\left(\widetilde{A}_2' + r\widetilde{A}_2\right)y + \widetilde{B}_2' + \Big[(\tilde{\mu}-r)\tilde{\pi}^* + \lambda\mu\eta^*(1-q^*) - \beta_{21}\sigma^2\lambda A_2(1-q^*)^2 - \beta_{22}\tilde{\sigma}^2A_2(\tilde{\pi}^*)^2\Big]\widetilde{A}_2 = 0,
$$

and

$$
(A_2' + rA_2)y + B_2' + \Big[(\tilde{\mu}-r)\tilde{\pi}^* + \lambda\mu\eta^*(1-q^*)\Big]A_2 - \frac{\lambda\sigma^2(1-q^*)^2}{2}(m_2\widetilde{A}_2^2 + \beta_{21}A_2^2) - \frac{\tilde{\sigma}^2(\tilde{\pi}^*)^2}{2}(m_2\widetilde{A}_2^2 + \beta_{22}A_2^2) = 0,
$$

Therefore, by the method of separation of variables, we obtain the following ODEs:

$$
\begin{cases}
\widetilde{A}_2' + r\widetilde{A}_2 = 0, \qquad A_2' + rA_2 = 0, \\
\widetilde{B}_2' + \Big[(\tilde{\mu}-r)\tilde{\pi}^* + \lambda\mu\eta^*(1-q^*) - \beta_{21}\sigma^2\lambda A_2(1-q^*)^2 - \beta_{22}\tilde{\sigma}^2A_2(\tilde{\pi}^*)^2\Big]\widetilde{A}_2 = 0, \\
B_2' + \Big[(\tilde{\mu}-r)\tilde{\pi}^* + \lambda\mu\eta^*(1-q^*)\Big]A_2 \\
\quad - \dfrac{\lambda\sigma^2(1-q^*)^2}{2}(m_2\widetilde{A}_2^2 + \beta_{21}A_2^2) - \dfrac{\tilde{\sigma}^2(\tilde{\pi}^*)^2}{2}(m_2\widetilde{A}_2^2 + \beta_{22}A_2^2) = 0,
\end{cases}
$$

Using the boundary conditions, we obtain:

$$
\begin{aligned}
\widetilde{A}_2(t) &= e^{r(T-t)}, \qquad A_2(t) = e^{r(T-t)}, \\
\widetilde{B}_2(t) &= \int_t^T \tilde{b}_{21}(s)\mathrm{d}s + \int_t^T \tilde{b}_{22}(s)\mathrm{d}s, \\
B_2(t) &= \int_t^T b_{21}(s)\mathrm{d}s + \int_t^T b_{22}(s)\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{cases}
\tilde{b}_{21}(s) = \left[\lambda\mu\eta^*(s)(1-q^*(s)) - \beta_{21}\sigma^2\lambda(1-q^*(s))^2e^{r(T-s)}\right]e^{r(T-s)}, \\
\tilde{b}_{22}(s) = \left[(\tilde{\mu}-r)\tilde{\pi}^*(s) - \beta_{22}\tilde{\sigma}^2(\tilde{\pi}^*(s))^2e^{r(T-s)}\right]e^{r(T-s)}, \\
b_{21}(s) = \left[\lambda\mu\eta^*(s)(1-q^*(s)) - \dfrac{\lambda\sigma^2(1-q^*(s))^2}{2}(m_2+\beta_{21})e^{r(T-s)}\right]e^{r(T-s)}, \\
b_{22}(s) = \left[(\tilde{\mu}-r)\tilde{\pi}^*(s) - \dfrac{\tilde{\sigma}^2(\tilde{\pi}^*(s))^2}{2}(m_2+\beta_{22})e^{r(T-s)}\right]e^{r(T-s)}.
\end{cases}
\tag{4.3.38}
$$

Based on the above derivations, we summarize the main results of this chapter in the following theorems. In Theorem 4.3.3, we provide the explicit expressions for the insurer's equilibrium retention level of the claims and the reinsurer's optimal reinsurance price, and we present the analytical expressions for the equilibrium investment strategies and the value functions of the insurer and the reinsurer. In Theorem 4.3.4, we give the expected values of the insurer's and the reinsurer's terminal surpluses and the worst-case density generators of the insurer and the reinsurer.

We first impose the following assumption.

Assumption 4.3.1. *Suppose the following conditions are satisfied:*

$$\begin{cases} \sqrt{\lambda}\mu > 3\sigma, \\ \dfrac{\sigma^2(\beta_{11}+m_1)^2 e^{r(T-t)} + \sigma^2(\beta_{11}+m_1)(\beta_{21}+m_2)e^{r(T-t)}}{2\mu(\beta_{11}+m_1)+\mu(\beta_{21}+m_2)} \geq \theta. \end{cases}$$

Theorem 4.3.3. *Under Assumption* 4.3.1*, the insurer's robust optimal retained proportion of the claims and the reinsurer's robust optimal reinsurance price are, respectively, given by:*

$$q^*(t) = \frac{\beta_{11}+m_1+\beta_{21}+m_2}{2(\beta_{11}+m_1)+\beta_{21}+m_2}, \tag{4.3.39}$$

and

$$\eta^*(t) = \frac{\sigma^2(\beta_{11}+m_1)^2 e^{r(T-t)} + \sigma^2(\beta_{11}+m_1)(\beta_{21}+m_2)e^{r(T-t)}}{2\mu(\beta_{11}+m_1)+\mu(\beta_{21}+m_2)}. \tag{4.3.40}$$

Furthermore, the robust equilibrium investment strategies of the insurer and the reinsurer are, respectively, given by:

$$\pi^*(t) = \frac{\tilde{\mu}-r}{(m_1+\beta_{12})\tilde{\sigma}^2 e^{r(T-t)}}, \tag{4.3.41}$$

and

$$\tilde{\pi}^*(t) = \frac{\tilde{\mu}-r}{(m_2+\beta_{22})\tilde{\sigma}^2 e^{r(T-t)}}. \tag{4.3.42}$$

Finally, the equilibrium value functions of the insurer and the reinsurer are, respectively, given by the following integral representations:

$$V_1(t,x) = xe^{r(T-t)} + \int_t^T \lambda\mu(\theta-\eta^*(s))\mathrm{d}s + \int_t^T b_{11}(s)\mathrm{d}s + \int_t^T b_{12}(s)\mathrm{d}s,$$

$$V_2(t,y) = ye^{r(T-t)} + \int_t^T b_{21}(s)\mathrm{d}s + \int_t^T b_{22}(s)\mathrm{d}s,$$

where b_{1i}, *for* $i \in \{1,2\}$, *were given by* (4.3.27) *with* η *substituted for* η^* *and* b_{2i}, *for* $i \in \{1,2\}$, *were given by* (4.3.38).

Proof. It was derived that

$$A_1(t) = A_2(t) = \widetilde{A}_1(t) = \widetilde{A}_2(t) = e^{r(T-t)}. \tag{4.3.43}$$

The explicit solution to the optimal reinsurance price in (4.3.40) is obtained by substituting (4.3.43) into η^* in (4.3.37a). Inserting (4.3.40) into q^* in (4.3.23a), we can obtain the optimal reinsurance retention level of the insurer given by (4.3.39). Similarly, if we put (4.3.43) back into (4.3.23b) and (4.3.37b), we can obtain the robust equilibrium investment strategies of the insurer and the reinsurer presented in (4.3.41) and (4.3.42), respectively. This completes the proof. □

Remark 4.3.1. *The insurer's robust optimal retention level of the claims in* (4.3.39) *lies in the interval* $(0,1)$. *Consequently, we do not have to consider the cases at the boundary points, say* $q^* = 0$ *or* $q^* = 1$ *which correspond, respectively, to the cases where the insurer purchases a full reinsurance coverage and where the insurer has no reinsurance demand at all.*

Remark 4.3.2. *The results in Theorem* 4.3.3 *indicate that the robust optimal reinsurance contract* $(q^*(t), \eta^*(t))$ *is independent of the ambiguity levels on the stock return. This may stem from the assumption that the random shocks in the stock price and the claim process are independent.*

In the following theorem, we provide the expectation of the terminal surpluses associated with the robust equilibrium strategies of the insurer and the reinsurer and determine the worst-case scenario density generators. Plugging the expressions of $\widetilde{A}_i(t)$ and $\widetilde{B}_i(t)$, for $i \in \{1,2\}$, in the preceding paragraphs to the trial solutions of $g_i(t,x)$, the results in this theorem can be directly obtained.

Theorem 4.3.4. *The expected values of the insurer's and the reinsurer's terminal surpluses are, respectively, given by:*

$$\begin{aligned}\mathbb{E}_{t,x}^{\mathbb{Q}_1}\left[X^{u^*,v^*}(T)\right] &= g_1(t,x)\\ &= xe^{r(T-t)} + \int_t^T \lambda\mu(\theta - \eta^*(s))\mathrm{d}s + \int_t^T \tilde{b}_{11}(s)\mathrm{d}s + \int_t^T \tilde{b}_{12}(s)\mathrm{d}s,\end{aligned}$$

$$\mathbb{E}^{\mathbb{Q}_2}_{t,y}\left[Y^{u^*,v^*}(T)\right] = g_2(t,y) = ye^{r(T-t)} + \int_t^T \tilde{b}_{21}(s)\mathrm{d}s + \int_t^T \tilde{b}_{22}(s)\mathrm{d}s,$$

where $\tilde{b}_{1i}$*, for* $i \in \{1,2\}$*, were given by* (4.3.26) *with* η *substituted for* η^* *and* $\tilde{b}_{2i}$*, for* $i \in \{1,2\}$*, were given by* (4.3.38)*. The worst-case density generator* $\phi_1^*(t) := (\phi_{11}^*(t), \phi_{12}^*(t))$ *of the insurer is given by:*

$$\begin{cases} \phi_{11}^*(t) = -\beta_{11}\sigma\sqrt{\lambda}q^*(t)e^{r(T-t)}, \\ \phi_{12}^*(t) = -\beta_{12}\tilde{\sigma}\pi_1^*(t)e^{r(T-t)}. \end{cases}$$

The reinsurer's worst-case density generator $\phi_2^*(t) := (\phi_{21}^*(t), \phi_{22}^*(t))$ *is given by:*

$$\begin{cases} \phi_{21}^*(t) = -\beta_{21}\sigma\sqrt{\lambda}(1-q^*(t))e^{r(T-t)}, \\ \phi_{22}^*(t) = -\beta_{22}\tilde{\sigma}\tilde{\pi}_1^*(t)e^{r(T-t)}. \end{cases}$$

In the above expressions, $\eta^*(t)$*,* $q^*(t)$*,* $\tilde{\pi}^*(t)$ *and* $\pi^*(t)$ *were given in Theorem* 4.3.3*.*

When the insurer (or the reinsurer) completely trusts the reference model under the reference probability measure $\mathbb{P}$, the respective ambiguity aversion coefficients would be identical to zero. In this case, the robust optimization problem (4.2.10) would reduce to the traditional optimization problem (4.2.9). Consequently, setting the ambiguity aversion parameters of the insurer in Theorem 4.3.3 to be zero would yield the robust reinsurance contract and the robust equilibrium investment strategies of an ANI and an AAR, respectively; similarly, putting the ambiguity aversion parameters of the reinsurer in Theorem 4.3.3 to be zero would yield the robust reinsurance contract and the robust equilibrium investment strategies of an AAI and an ANR, respectively. These two results are presented in the following corollaries.

Corollary 4.3.1. *The equilibrium value functions of the ANI and the AAR are, respectively, given by the following integral representations:*

$$\widehat{V}_1(t,x) = xe^{r(T-t)} + \int_t^T \lambda\mu(\theta - \hat{\eta}^*(s))\mathrm{d}s + \int_t^T \hat{b}_{11}(s)\mathrm{d}s + \int_t^T \hat{b}_{12}(s)\mathrm{d}s,$$

$$\widehat{V}_2(t,y) = ye^{r(T-t)} + \int_t^T \hat{b}_{21}(s)\mathrm{d}s + \int_t^T b_{22}(s)\mathrm{d}s,$$

where

$$\begin{cases} \hat{b}_{11}(s) = \left[\lambda\mu\hat{\eta}^*(s)\hat{q}^*(s) - \frac{\lambda\sigma^2(\hat{q}^*(s))^2}{2}m_1e^{r(T-s)}\right]e^{r(T-s)}, \\ \hat{b}_{12}(s) = \left[(\tilde{\mu}-r)\hat{\pi}^*(s) - \frac{\tilde{\sigma}^2(\hat{\pi}^*(s))^2}{2}m_1e^{r(T-s)}\right]e^{r(T-s)}, \end{cases}$$

and

$$\begin{cases} \hat{b}_{21}(s) = \left[\lambda\mu\hat{\eta}^*(s)(1-\hat{q}^*(s)) - \frac{\lambda\sigma^2(1-\hat{q}^*(s))^2}{2}(m_2+\beta_{21})e^{r(T-s)}\right]e^{r(T-s)}, \\ b_{22}(s) = \left[(\tilde{\mu}-r)\tilde{\pi}^*(s) - \frac{\tilde{\sigma}^2(\tilde{\pi}^*(s))^2}{2}(m_2+\beta_{22})e^{r(T-s)}\right]e^{r(T-s)}, \end{cases}$$

The ANI's robust optimal retained proportion and the AAR's robust optimal reinsurance price (or premium) are, respectively, given by:

$$\hat{q}^*(t) = \frac{m_1+\beta_{21}+m_2}{2m_1+\beta_{21}+m_2},$$

and

$$\hat{\eta}^*(t) = \frac{\sigma^2 m_1 e^{r(T-t)}(m_1+\beta_{21}+m_2)}{\mu(2m_1+\beta_{21}+m_2)}.$$

Furthermore, the robust equilibrium investment strategy of the ANI is given by:

$$\hat{\pi}^*(t) = \frac{\tilde{\mu}-r}{m_1\tilde{\sigma}^2 e^{r(T-t)}}.$$

The robust equilibrium investment strategy of the AAR remain the same as that in (4.3.42).

Corollary 4.3.2. *The equilibrium value functions of the AAI and the ANR are, respectively, given by the following integral representations:*

$$\check{V}_1(t,x) = xe^{r(T-t)} + \int_t^T \lambda\mu(\theta-\check{\eta}^*(s))\mathrm{d}s + \int_t^T \check{b}_{11}(s)\mathrm{d}s + \int_t^T b_{12}(s)\mathrm{d}s,$$

$$\check{V}_2(t,y) = ye^{r(T-t)} + \int_t^T \check{b}_{21}(s)\mathrm{d}s + \int_t^T \check{b}_{22}(s)\mathrm{d}s,$$

where

$$\begin{cases} \check{b}_{11}(s) = \left[\lambda\mu\check{\eta}^*(s)\check{q}^*(s) - \frac{\lambda\sigma^2(\check{q}^*(s))^2}{2}(m_1+\beta_{11})e^{r(T-s)}\right]e^{r(T-s)}, \\ b_{12}(s) = \left[(\tilde{\mu}-r)\pi^*(s) - \frac{\tilde{\sigma}^2(\pi^*(s))^2}{2}(m_1+\beta_{12})e^{r(T-s)}\right]e^{r(T-s)}, \end{cases}$$

and

$$\begin{cases} \check{b}_{21}(s) = \left[\lambda\mu\check{\eta}^*(s)(1-\check{q}^*(s)) - \frac{\lambda\sigma^2(1-\check{q}^*(s))^2}{2}m_2 e^{r(T-s)}\right]e^{r(T-s)}, \\ \check{b}_{22}(s) = \left[(\tilde{\mu}-r)\check{\pi}^*(s) - \frac{\tilde{\sigma}^2(\check{\pi}^*(s))^2}{2}m_2 e^{r(T-s)}\right]e^{r(T-s)}, \end{cases}$$

The AAI's robust optimal retained proportion and the ANR's robust optimal reinsurance price (or premium) are, respectively, given by:

$$\check{q}^*(t) = \frac{\beta_{11}+m_1+m_2}{2\beta_{11}+2m_1+m_2},$$

and

$$\check{\eta}^*(t) = \frac{\sigma^2(\beta_{11}+m_1)e^{r(T-t)}(\beta_{11}+m_1+m_2)}{\mu(2\beta_{11}+2m_1+m_2)}.$$

Furthermore, the robust equilibrium investment strategy of the ANR is given by:

$$\check{\pi}^*(t) = \frac{\tilde{\mu}-r}{m_2\tilde{\sigma}^2 e^{r(T-t)}}.$$

The robust equilibrium investment strategy of the AAI remain the same as that in (4.3.41).

Remark 4.3.3. *The robust optimal reinsurance contracts derived in Corollary* 4.3.1 *and Corollary* 4.3.2 *imply that the optimal retention level of an ANI and the optimal reinsurance premium of an ANR are influenced by the ambiguity aversion coefficients of their counterparties. This may be attributed to the strategic interaction between the insurer and the reinsurer implied by the principal-agent framework.*

4.4 Utility loss of the suboptimal reinsurance and investment strategies

In this section, we shall discuss the utility losses of an AAI and an AAR. To this end, it is assumed that the insurer and the reinsurer are ambiguous about the insurance and financial risks. It is supposed, however, that they do not adopt the robust optimal reinsurance-investment strategies $u^* = (q^*, \pi^*)$ and $v^* = (\eta^*, \tilde{\pi}^*)$ given in Theorem 4.3.3. Instead, they make their decisions as if they were ambiguity-neutral. Say they follow the strategies given in Corollary 4.3.1 and Corollary 4.3.2, respectively. Under these circumstances, the insurer's suboptimal value function is defined by:

$$\widetilde{V}_1(t,x) := \inf_{\mathbb{Q}_1\in\mathcal{Q}}\Bigg\{\mathbb{E}^{\mathbb{Q}_1}_{t,x}\left[X^{\hat{u}^*,\hat{v}^*}(T)\right] - \frac{m_1}{2}\mathrm{Var}^{\mathbb{Q}_1}_{t,x}\left[X^{\hat{u}^*,\hat{v}^*}(T)\right] + \mathbb{E}^{\mathbb{Q}_1}_{t,x}\left[\int_t^T\left(\frac{\phi^2_{11}(s)}{2\beta_{11}}+\frac{\phi^2_{12}(s)}{2\beta_{12}}\right)\mathrm{d}s\right]\Bigg\},$$

and the reinsurer's suboptimal value function is defined by:

$$\widetilde{V}_2(t,y) := \inf_{\mathbb{Q}_2\in\mathcal{Q}} \Bigg\{ \mathbb{E}_{t,y}^{\mathbb{Q}_2}\left[Y^{\hat{u}^*,\hat{v}^*}(T)\right] - \frac{m_2}{2}\mathrm{Var}_{t,y}^{\mathbb{Q}_2}\left[Y^{\hat{u}^*,\hat{v}^*}(T)\right] + \mathbb{E}_{t,y}^{\mathbb{Q}_2}\left[\int_t^T \left(\frac{\phi_{21}^2(s)}{2\beta_{21}} + \frac{\phi_{22}^2(s)}{2\beta_{22}}\right)\mathrm{d}s\right]\Bigg\}.$$

It should be noted that the equilibrium reinsurance-investment strategies of the insurer and the reinsurer are now pre-specified, by which the worst-case alternative measures $\mathbb{Q}_k$, for $k\in\{1,2\}$, would be endogenously determined. As in, for example, Zhao et al. (2019), we define the (relative) utility losses of the insurer and the reinsurer under the suboptimal reinsurance-investment strategies as follows:

$$UL_1(t) := 1 - \frac{\widetilde{V}_1(t,x)}{V_1(t,x)},$$

and

$$UL_2(t) := 1 - \frac{\widetilde{V}_2(t,y)}{V_2(t,y)},$$

where $V_1(t,x)$ and $V_2(t,y)$ are the robust optimal value functions of the insurer and the reinsurer given in Theorem 4.3.3, respectively. Note that the expected utility loss of a decision-maker with respect to a suboptimal reinsurance contract was considered in, for example, Hu et al. (2018a,b), Li et al. (2018) and Wang and Siu (2020).

The suboptimal value function $\widetilde{V}_1(t,x)$ of the insurer associated with the suboptimal reinsurance contract $(\hat{q}^*(t),\hat{\eta}^*(t))$ and the suboptimal investment strategy $\hat{\pi}^*(t)$ solves the following minimization problem:

$$\inf_{\phi_1\in\Sigma_1}\Bigg\{\frac{\partial\widetilde{W}_1(t,x)}{\partial t} + \left[rx + (\tilde{\mu}-r)\hat{\pi}^* + \lambda\mu(\theta-\hat{\eta}^*) + \lambda\mu\hat{\eta}^*\hat{q}^* + \sigma\sqrt{\lambda}\tilde{\phi}_{11}\hat{q}^* + \tilde{\sigma}\tilde{\phi}_{12}\hat{\pi}^*\right] \times \frac{\partial\widetilde{W}_1(t,x)}{\partial x} + \frac{1}{2}\left(\lambda\sigma^2(\hat{q}^*)^2 + \tilde{\sigma}^2(\hat{\pi}^*)^2\right)\left(\frac{\partial^2\widetilde{W}_1(t,x)}{\partial x^2} - m_1\left(\frac{\partial\tilde{g}_1(t,x)}{\partial x}\right)^2\right) + \frac{\tilde{\phi}_{11}^2}{2\beta_{11}} + \frac{\tilde{\phi}_{12}^2}{2\beta_{12}}\Bigg\} = 0. \tag{4.4.44}$$

The suboptimal value function $\widetilde{V}_2(t,y)$ of the reinsurer associated with the suboptimal reinsurance contract $(\check{q}^*(t),\check{\eta}^*(t))$ and the suboptimal investment strategy

$\breve{\pi}^*(t)$ solves the following minimization problem:

$$\inf_{\phi_2\in\Sigma_2}\left\{\frac{\partial \widetilde{W}_2(t,y)}{\partial t}+\left[ry+(\tilde{\mu}-r)\breve{\pi}^*+\lambda\mu\breve{\eta}^*(1-\breve{q}^*)+\sigma\sqrt{\lambda}\tilde{\phi}_{21}(1-\breve{q}^*)+\tilde{\sigma}\tilde{\phi}_{22}\breve{\pi}^*\right]\right.$$
$$\times\frac{\partial \widetilde{W}_2(t,y)}{\partial y}+\frac{1}{2}\left(\lambda\sigma^2(1-\breve{q}^*)^2+\tilde{\sigma}^2(\breve{\pi}^*)^2\right)\left(\frac{\partial^2 \widetilde{W}_2(t,y)}{\partial y^2}-m_2\left(\frac{\partial \tilde{g}_2(t,y)}{\partial y}\right)^2\right)$$
$$\left.+\frac{\tilde{\phi}_{21}^2}{2\beta_{21}}+\frac{\tilde{\phi}_{22}^2}{2\beta_{22}}\right\}=0. \tag{4.4.45}$$

Following the similar procedures for deriving the robust optimal value functions of the insurer and the reinsurer, the optimization problems (4.4.44) and (4.4.45) can be solved. The suboptimal value function of the insurer is given as follows:

$$\widetilde{V}_1(t,x)=xe^{r(T-t)}+\int_t^T\lambda\mu(\theta-\hat{\eta}^*(s))\mathrm{d}s+\int_t^T c_{11}(s)\mathrm{d}s+\int_t^T c_{12}(s)\mathrm{d}s,$$

and the suboptimal value function of the reinsurer is given by:

$$\widetilde{V}_2(t,y)=ye^{r(T-t)}+\int_t^T c_{21}(s)\mathrm{d}s+\int_t^T c_{22}(s)\mathrm{d}s,$$

with

$$\begin{cases} c_{11}(s)=\left[\lambda\mu\hat{\eta}^*(s)\hat{q}^*(s)-\frac{\lambda\sigma^2(\hat{q}^*(s))^2}{2}(m_1+\beta_{11})e^{r(T-s)}\right]e^{r(T-s)},\\ c_{12}(s)=\left[(\tilde{\mu}-r)\hat{\pi}^*(s)-\frac{\tilde{\sigma}^2(\hat{\pi}^*(s))^2}{2}(m_1+\beta_{12})e^{r(T-s)}\right]e^{r(T-s)},\end{cases}$$

and

$$\begin{cases} c_{21}(s)=\left[\lambda\mu\breve{\eta}^*(s)(1-\breve{q}^*(s))-\frac{\lambda\sigma^2(1-\breve{q}^*(s))^2}{2}(m_2+\beta_{21})e^{r(T-s)}\right]e^{r(T-s)},\\ c_{22}(s)=\left[(\tilde{\mu}-r)\breve{\pi}^*(s)-\frac{\tilde{\sigma}^2(\breve{\pi}^*(s))^2}{2}(m_2+\beta_{22})e^{r(T-s)}\right]e^{r(T-s)},\end{cases}$$

where $\hat{q}^*(t),\hat{\eta}^*(t),\hat{\pi}^*(t)$ were given in Corollary 4.3.1 and $\breve{q}^*(t),\breve{\eta}^*(t),\breve{\pi}^*(t)$ were given in Corollary 4.3.2.

4.5 Numerical examples

In this section, we shall provide numerical examples for a sensitivity analysis of the robust equilibrium reinsurance and investment strategies derived in Section 4.3 and the utility losses presented in Section 4.4. The model parameters used as our benchmark are shown in Table 4.1 unless otherwise stated. In each of the following

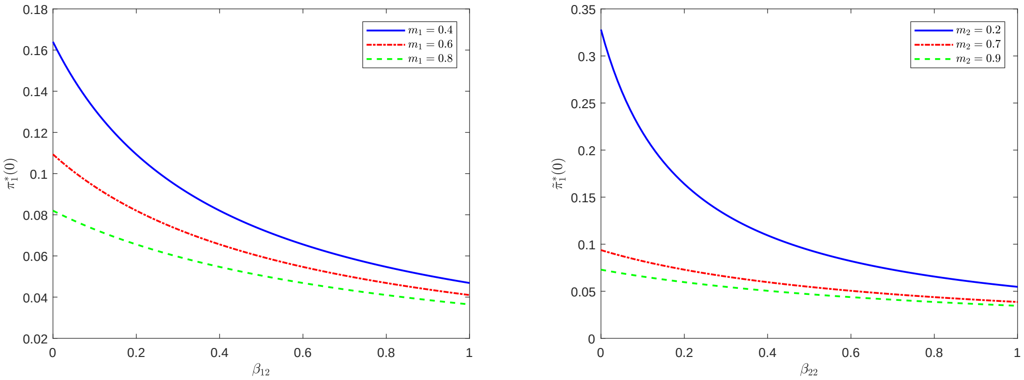


Figure 4.1: Effects of the ambiguity aversion parameters β_{k2}, for $k \in \{1, 2\}$, on the robust equilibrium investment strategies of the insurer and the reinsurer for the risky asset.

figures, we vary the value of one parameter and study the sensitivity of robust equilibrium reinsurance-investment strategies and utility losses with respect to the change in the value of that parameter. It has been guaranteed that the conditions in Assumption 4.3.1 are satisfied when the parameters vary in the sensitivity analysis here.

Table 4.1: Values of parameters in numerical experiments

t	T	r	$\tilde{\mu}$	$\tilde{\sigma}$	λ	μ	σ	θ
0	15	0.05	0.1	0.6	3	2	1	0.2
m_1	m_2	β_{11}	β_{12}	β_{21}	β_{22}	x	y	
0.5	0.6	0.1	0.8	0.9	0.2	10	20	

Figure 4.1 depicts the effects of the ambiguity aversion coefficient β_{k2}, for $k \in \{1, 2\}$, and the risk aversion parameter m_k on the robust equilibrium investment strategies of the insurer and the reinsurer in the risky asset. From Figure 4.1, we note that if an AAI (or an AAR) has a higher level of ambiguity aversion, they would reduce the amount invested in the risky asset. Intuitively this conclusion appears to be reasonable because the decision-makers would invest less wealth in an asset that they have less information about the underlying mechanism that generates the price movements to mitigate financial risks. This conclusion also indicates that an AAI (or an AAR) would be more conservative to financial risks than an ANI (or an ANR), which is reflected in the decrement in the investment

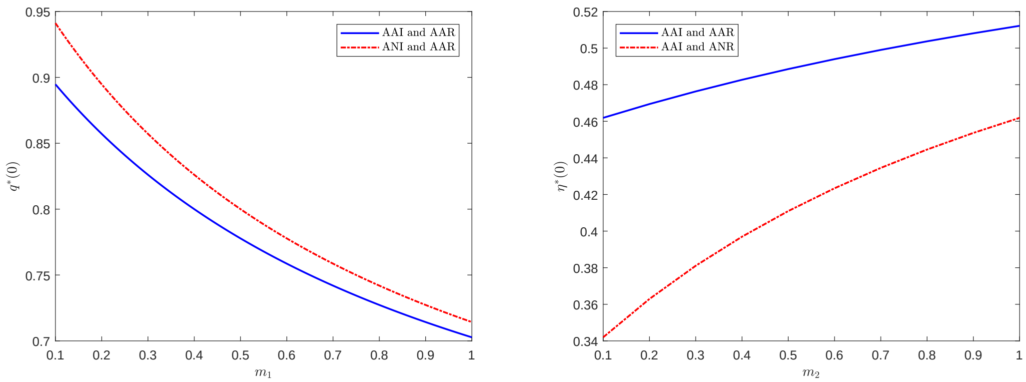


Figure 4.2: Effects of the risk aversion parameters m_k, for $k \in \{1, 2\}$, on the robust equilibrium reinsurance strategies of the insurer and the reinsurer.

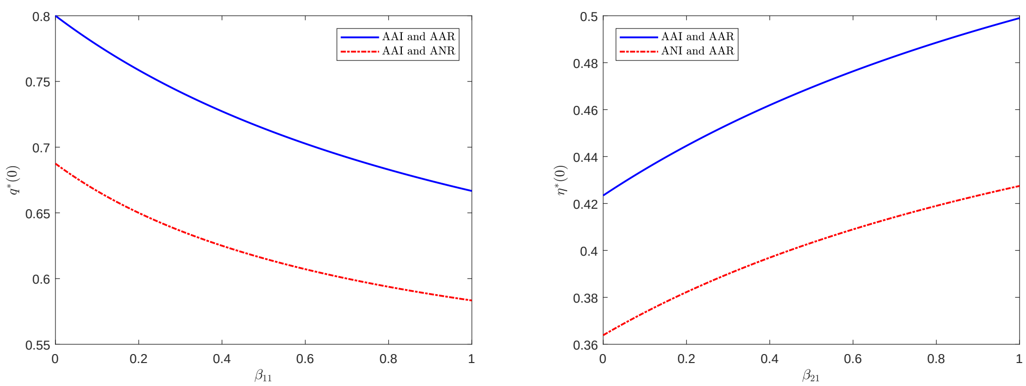


Figure 4.3: Effects of the ambiguity aversion parameters β_{k1}, for $k \in \{1, 2\}$, on the robust reinsurance contracts.

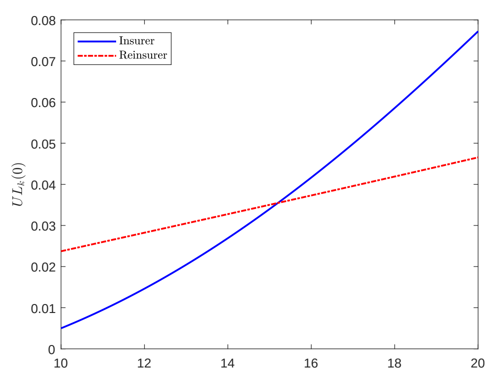


Figure 4.4: Effects of the time horizon T on the utility losses of the insurer and the reinsurer.

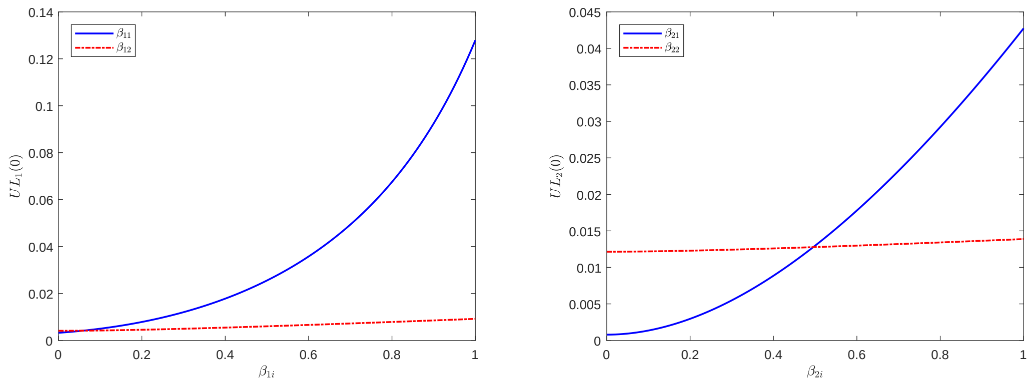


Figure 4.5: Effects of the ambiguity aversion parameters β_{ki}, for $k, i \in \{1, 2\}$, on the utility losses of the insurer and the reinsurer.

demand for the risky asset. Additionally, for a fixed ambiguity aversion parameter, the robust equilibrium investment strategies in the stock decrease as the parameter m_k increases. In other words, the more risk-averse the insurer (or the reinsurer) is, the less the wealth the insurer (or the reinsurer) would like to invest in the risky asset.

In Figure 4.2, we show the effects of the risk aversion coefficients m_k, for $k \in \{1, 2\}$, on the robust equilibrium reinsurance strategies of the insurer and the reinsurer under different scenarios. We find that the insurer's equilibrium retention level $q^*(0)$ decreases as m_1 increases. This can be explained by that a more risk-averse insurer is less willing to undertake the insurance risks and so the insurer tends to cede more insurance risks to the reinsurer. For the same level of risk aversion, an AAI retains less insurance risk than an ANI, which indicates that ambiguity aversion attitudes render the insurer more conservative to the insurance risks. In regard to the reinsurer, we observe that the equilibrium reinsurance premium $\eta^*(0)$ is an increasing function of the reinsurer's risk aversion parameter m_2. This may be attributed to that the more risk-averse the reinsurer is, the less the insurance risks the reinsurer would like to bear. Consequently, the reinsurer tends to increase the reinsurance premium with a view to compensating the additional insurance risks to be undertaken. Finally, for a fixed risk aversion parameter, an AAR charges a higher reinsurance premium than an ANR, which indicates that the consideration of model uncertainty induces the reinsurer to select more conservative and cautious strategies. The results appear to indicate that the impact of ambiguity aversion on financial risks and that on insurance risks are consistent with each other.

Figure 4.3 illustrates the effects of the ambiguity aversion parameter β_{k1}, for $k \in \{1, 2\}$, on the robust equilibrium reinsurance strategies of the insurer and the reinsurer which were derived in Theorem 4.3.3, Corollary 4.3.1 and Corollary 4.3.2, respectively. Firstly, we can observe that the insurer decreases his optimal retention level $q^*(0)$ as the ambiguity aversion parameter corresponding to diffusion risk of the claims becomes larger. Moreover, the reinsurer is prone to increasing the reinsurance premium when her level of ambiguity aversion increases in order to offset the adverse effects of model misspecification. These appear to be in line with intuition. The left panel of Figure 4.3 shows that for a fixed ambiguity aversion parameter of the insurer, the optimal retention level of the insurer in the optimal reinsurance contract between an AAI and an AAR is higher than that in an optimal reinsurance contract between an AAI and ANR. This is mainly because, as analyzed in Figure 4.2, an AAR charges a higher reinsurance premium than an ANR. As discussed in Remark 4.3.3, if both the insurer and the reinsurer are ambiguity-averse, the impacts of their attitudes towards model uncertainty would be strengthened. Consequently, the reinsurer tends to adopt more conservative strategies, i.e., the reinsurer increases the reinsurance premium. This may explain why the blue curve is above the red curve in the right panel of Figure 4.3.

Figures 4.4-4.5 depict the results of the sensitivity analysis for the utility losses of the insurer and the reinsurer. Figure 4.4 illustrates that the utility loss $UL_1(0)$ of the insurer exhibits an upward trend as the horizon time T extends. One potential explanation could be that intuitively the insurer may face a higher level of model uncertainty when the reinsurance and investment horizon T is longer. Therefore, it seems to be relevant for the insurer to consider ambiguity aversion if the insurer intends to develop a medium or long-term cooperative relationship with the reinsurer and participates in long-term investment activities. The utility loss $UL_2(0)$ of the reinsurer also increases as T increases, but it is in a less extent compared with that of the insurer. Additionally, we find that the utility losses of the insurer and the reinsurer are increasing functions of their respective ambiguity aversion parameters. Figure 4.5 captures the effects of the ambiguity aversion coefficients β_{ki}, for $k, i \in \{1, 2\}$, on the utility losses of the insurer and the reinsurer. These results indicate that if a decision-maker with less information about the reference measure $\mathbb{P}$ ignores ambiguity aversion, the decision-maker would suffer a greater utility loss. Also, we can see that the utility losses of the insurer and the reinsurer show relatively less sensitivity with respect to the ambiguity aversion parameter β_{k2} compared with β_{k1}. This reflects that the decision-makers' ambiguity aversion attitudes towards the claim process play more important roles in their utility losses than these towards the financial market.

4.6 Concluding remarks

In this chapter, we study a robust optimal reinsurance contract problem under a continuous-time principal-agent framework. More specifically, we assume that the insurer and the reinsurer are both ambiguity-averse and intend to develop robust proportional reinsurance contract and robust investment strategies by considering a family of alternative models. Both the insurer and the reinsurer have the access to investment opportunities of a stock and a risk-free asset. Under the time-consistent mean-variance criterion, we consider two systems of extended HJB equations and obtain the explicit expressions for the equilibrium reinsurance-investment strategies and the corresponding equilibrium value functions of the insurer and the reinsurer. We also present some particular cases of our model and discuss the utility losses of the insurer and the reinsurer if they ignore model uncertainty.

The main implications from the results are summarized as follows. (1) The insurer and the reinsurer are prone to selecting more conservative investment strategies if they are more ambiguity-averse, or more risk-averse. This is reflected in the reduced amount invested in the risky asset. (2) The insurer tends to undertake less insurance risks and purchase more reinsurance if he is more ambiguity-averse, or more risk-averse. Besides, the reinsurer with a larger ambiguity aversion parameter or a risk aversion parameter would charge a higher reinsurance premium. (3) The utility losses of the insurer and the reinsurer increase as their ambiguity aversion parameters and the horizon for reinsurance and investment increase, which are consistent with the conclusions obtained by Hu et al. (2018a,b) and Hu and Wang (2019). This conclusion also indicates that it is important to stress model uncertainty for long-term decision-makers.

Chapter 5

Concluding Remarks and Further Research

In the present study, several investment and reinsurance optimization problems involving strategic interaction and model uncertainty were investigated. In Chapter 2 and Chapter 3, a competition between two AAIs was considered with the aim of maximizing the expected utility of relative surplus at terminal time and the mean-variance performance functional, respectively. Both criteria include an optimization procedure under the worst-case scenario of plausible alternative models to incorporate the ambiguity aversion attitudes of the decision makers. In Chapter 4, the bargaining between an AAI and an AAR was presented assuming that both aim to establish a reinsurance policy under the mean-variance criterion. Moreover, the insurers are permitted to invest their surpluses into a defaultable corporate zero-coupon bond in Chapter 2. This strategy is adopted because institutional investors are increasingly seeking high-yield corporate bonds, and default risk management has attracted considerable attention from both investors and regulators since the financial crisis of 2008. Throughout this work, the standard dynamic programming principle was adopted to derive the (extended) HJB equation, whereby the expressions for value functions and corresponding optimal strategies were obtained by solving (extended) HJB equations.

The results indicate that relative performance concerns make the insurers more risk-seeking while ambiguity aversion attitudes lead to more conservative decision-makers. This implies that the decision-maker with greater competition sensitivity parameter (or smaller ambiguity aversion parameter) tends to purchase less reinsurance protection and increase the amounts invested in the risky asset. Our study reveals that it is important for ambiguity-averse decision-makers to select robust optimal reinsurance and investment strategies. Otherwise, they would suffer great expected utility losses. Such findings seem to be in line with the previous studies. The research conducted in this thesis brings together elements of stochastic control theory, actuarial science and contract theory in economic sciences. It is hoped that such knowledge combination may provide novel insights into reinsurance and investment decision makings.

Several possible extensions of the work reported in this thesis may deserve further investigation. First, other features of the investment activities undertaken by the insurer and the reinsurer may be incorporated. For instance, the price of the risky asset can be assumed to follow the CEV model or Heston's SV model, as the constant volatility assumption is unrealistic. For risky assets, the SV models are particularly useful in this context, as they can explain a number of empirical findings, such as the implied volatility smile and volatility clustering. Second, empirical evidence has documented that moral hazard exists in the reinsurance market. This incentive problem may arise from information asymmetry because some actions of insurance company cannot be observed or monitored. Thus, another interesting research avenue may involve applying the framework adopted in the present study to investigate strategic interaction between an AAI and an AAR in presence of moral hazard. It may be of practical significance if we could provide some methods of designing reinsurance contracts to reduce the excess moral hazard. Finally, in this work, penalty-based multiple-priors utility model developed by Anderson et al. (2003) was adopted to capture the robust decision-making processes undertaken by decision-makers facing model uncertainty. It would thus be worthwhile to investigate strategic interactions among ambiguity-averse players by applying the recursive multiple-priors utility model proposed by Chen and Epstein (2002) or the smooth ambiguity method devised by Kronborg and Steffensen (2015) to capture their actions in presence of ambiguity, which may enhance the tractability of the problem and provide novel implications.

Appendix A

Derivation of relative entropy in Chapter 2

Given the reference probability measure $\mathbb{P}$ and an alternative measure $\mathbb{Q}_i$, the relative entropy of $\mathbb{Q}_i$ with respect to $\mathbb{P}$ is defined as the expectation under the alternative probability measure of the log Radon-Nikodym derivative defined in (2.2.4). A lower relative entropy implies that it is harder for the insurer to distinguish $\mathbb{P}$ from $\mathbb{Q}_i$ in statistic sense. Using Itô's formula, we obtain

$$\begin{aligned}
\mathrm{d}\ln\Lambda^{\phi_i}(t) = & -\phi_{i,1}(t)\mathrm{d}B_1(t) - \frac{1}{2}\left(\phi_{i,1}(t)\right)^2\mathrm{d}t - \phi_{i,2}(t)\mathrm{d}B_{i,0}(t) - \frac{1}{2}\left(\phi_{i,2}(t)\right)^2\mathrm{d}t \\
& + h^P(1-\phi_{i,3}(t))(1-Z(t))\mathrm{d}t + \ln\phi_{i,3}(t)\mathrm{d}Z(t)
\end{aligned}$$

The relative entropy over the interval from t to $t+\varepsilon$ is given by

$$\begin{aligned}
E_{\mathbb{Q}_i}\left[\ln\frac{\Lambda^{\phi_i}(t+\varepsilon)}{\Lambda^{\phi_i}(t)}\right] = & E_{\mathbb{Q}_i}\Bigg[-\int_t^{t+\epsilon}\phi_{i,1}(s)\left(\mathrm{d}B_{i,1}^{\mathbb{Q}_i}(s) - \phi_{i,1}(s)\mathrm{d}s\right) - \frac{1}{2}\int_t^{t+\epsilon}(\phi_{i,1}(s))^2\mathrm{d}s \\
& -\int_t^{t+\epsilon}\phi_{i,2}(s)\left(\mathrm{d}B_{i,0}^{\mathbb{Q}_i}(t) - \phi_{i,2}(s)\mathrm{d}s\right) - \frac{1}{2}\int_t^{t+\epsilon}(\phi_{i,2}(s))^2\mathrm{d}s + \int_t^{t+\epsilon}\ln\phi_{i,3}(s)\mathrm{d}M^{Q_i}(s) \\
& + \int_t^{t+\epsilon}h^P(1-\phi_{i,3}(s))(1-Z(s))\mathrm{d}s + \int_t^{t+\epsilon}\ln\phi_{i,3}(s)(1-Z(s))\phi_{i,3}(s)h^P\mathrm{d}s\Bigg] \\
= & E_{\mathbb{Q}_i}\left[\int_t^{t+\epsilon}\left(h^P(1-Z(s))(\phi_{i,3}(s)\ln\phi_{i,3}(s) + 1 - \phi_{i,3}(s)) + \frac{1}{2}\left(\phi_{i,1}(s)\right)^2 + \frac{1}{2}(\phi_{i,2}(s))^2\right)\mathrm{d}s\right].
\end{aligned}$$

Let $\epsilon \to 0$ and we obtain the continuous-time limit of the relative entropy given by

$$\frac{1}{2}\left(\phi_{i,1}(t)\right)^2\mathrm{d}t + \frac{1}{2}(\phi_{i,2}(t))^2\mathrm{d}t + h^P(1-Z(t))\left(\phi_{i,3}(t)\ln\phi_{i,3}(t) - \phi_{i,3}(t) + 1\right)\mathrm{d}t.$$

Appendix B

Proof of Lemma 2.3.1

Proof. (i) It is obvious that the optimal strategy (π_i^*, ϕ_i^*) is deterministic and state-independent, thus the alternative probability measure $\mathbb{Q}_i^*$ is well-defined and the first condition in Definition 2.2.1 is satisfied. The second condition in Definition 2.2.1 can be proved by property (ii).

(ii) Substituting (2.3.26) into (2.2.8), we have

$$
\begin{aligned}
\widehat{X}_i^{\pi_i^*,\pi_j^*}(t) = \hat{x}_i e^{rt} &+ \int_0^t e^{-r(s-t)} A_i(s)\mathrm{d}s + \int_0^t e^{-r(s-t)} \sigma \pi_{i,1}^*(s)\mathrm{d}B_{i,1}^{\mathbb{Q}_i^*}(s) \\
&- \int_0^t e^{-r(s-t)} n_i \sigma \pi_{j,1}^*(s)\mathrm{d}B_{j,1}^{\mathbb{Q}_j^*}(s) + \int_0^t e^{-r(s-t)} b_i q_i^*(s)\mathrm{d}B_{i,0}^{\mathbb{Q}_i^*}(s) \\
&- \int_0^t e^{-r(s-t)} n_i b_j q_j^*(s)\mathrm{d}B_{j,0}^{\mathbb{Q}_j^*}(s) \\
&- \int_0^t e^{-r(s-t)} (1 - Z(s-))\zeta \left(\pi_{i,2}^*(s) - n_i \pi_{j,2}^*(s)\right) \mathrm{d}Z(s), \qquad \text{(B.1)}
\end{aligned}
$$

where

$$
\begin{aligned}
A_i(s) = (\mu - r)&\left(\pi_{i,1}^*(s) - n_i \pi_{j,1}^*(s)\right) + (1 - Z(s-))\delta\left(\pi_{i,2}^*(s) - n_i \pi_{j,2}^*(s)\right) \\
&+ \lambda_i a_i - n_i \lambda_j a_j + a_i \gamma_i q_i^*(s) - n_i a_j \gamma_j q_j^*(s) \\
&- \sigma\left(\phi_{i,1}^*(s)\pi_{i,1}^*(s) - n_i \phi_{j,1}^*(s)\pi_{j,1}^*(s)\right) - b_i \phi_{i,2}^*(s) q_i^*(s) + n_i b_j \phi_{j,2}^*(s) q_j^*(s).
\end{aligned}
$$

For $i \in \{1, 2\}$, $A_i(s)$ is bounded since $(\pi_i^*(s), \phi_i^*(s))$ is deterministic. Inserting (B.1) into the candidate value function (2.3.25), we can obtain

$$
\begin{aligned}
&\left|\widetilde{W}_i\left(t, \widehat{X}_i^{\pi_i^*,\pi_j^*}(t), Z(t)\right)\right|^4 \\
=&\left|(1 - Z(t))W_i\left(t, \widehat{X}_i^{\pi_i^*,\pi_j^*}(t), 0\right) + Z(t)W_i\left(t, \widehat{X}_i^{\pi_i^*,\pi_j^*}(t), 1\right)\right|^4 \\
\leq& 4\left|W_i\left(t, \widehat{X}_i^{\pi_i^*,\pi_j^*}(t), 0\right)\right|^4 + 4\left|W_i\left(t, \widehat{X}_i^{\pi_i^*,\pi_j^*}(t), 1\right)\right|^4.
\end{aligned}
$$

Plugging (B.1) into (2.3.20), we obtain the following upper boundary with an appropriate constant $K > 0$,

$$
\begin{aligned}
&\left|W_i\left(t, \widehat{X}_i^{\pi_i^*,\pi_j^*}(t), 0\right)\right|^4 = \frac{\exp\{4g_{i,0}(t)\}}{m_i^4} \exp\left\{-4m_i e^{r(T-t)} \widehat{X}_i^{\pi_i^*,\pi_j^*}(t)\right\} \\
&= \frac{\exp\{4g_{i,0}(t)\}}{m_i^4} \exp\left\{ -4m_i \left(\hat{x}_i e^{rT} + \int_0^t e^{-r(s-T)} A_i(s)\mathrm{d}s\right)\right\} \\
&\quad \times \exp\left\{-4m_i \int_0^t e^{-r(s-T)} \sigma \pi_{i,1}^*(s) \mathrm{d}B_{i,1}^{\mathbb{Q}_i^*}(s)\right\} \\
&\quad \times \exp\left\{4m_i \int_0^t e^{-r(s-T)} n_i \sigma \pi_{j,1}^*(s) \mathrm{d}B_{j,1}^{\mathbb{Q}_j^*}(s)\right\} \\
&\quad \times \exp\left\{-4m_i \int_0^t e^{-r(s-T)} b_i q_i^*(s) \mathrm{d}B_{i,0}^{\mathbb{Q}_i^*}(s)\right\} \\
&\quad \times \exp\left\{4m_i \int_0^t e^{-r(s-T)} n_i b_j q_j^*(s) \mathrm{d}B_{j,0}^{\mathbb{Q}_j^*}(s)\right\} \\
&\quad \times \exp\left\{4m_i \int_0^t e^{-r(s-T)} (1 - Z(s-)) \zeta \left(\pi_{i,2}^*(s) - n_i \pi_{j,2}^*(s)\right) \mathrm{d}Z(s)\right\} \\
&\leq K \exp\left\{-4m_i \int_0^t e^{-r(s-T)} \sigma \pi_{i,1}^*(s) \mathrm{d}B_{i,1}^{\mathbb{Q}_i^*}(s)\right\} \\
&\quad \times \exp\left\{4m_i \int_0^t e^{-r(s-T)} n_i \sigma \pi_{j,1}^*(s) \mathrm{d}B_{j,1}^{\mathbb{Q}_j^*}(s)\right\} \\
&\quad \times \exp\left\{-4m_i \int_0^t e^{-r(s-T)} b_i q_i^*(s) \mathrm{d}B_{i,0}^{\mathbb{Q}_i^*}(s)\right\} \\
&\quad \times \exp\left\{4m_i \int_0^t e^{-r(s-T)} n_i b_j q_j^*(s) \mathrm{d}B_{j,0}^{\mathbb{Q}_j^*}(s)\right\} \\
&:= K E_1(t) E_2(t) E_3(t) E_4(t).
\end{aligned}
$$

The inequality follows from the fact that $g_{i,0}(t)$, $\int_0^t e^{-r(s-T)} A_i(s)\mathrm{d}s$ and $\int_0^t \zeta e^{-r(s-T)} (1-Z(s-))(\pi_{i,2}^*(s) - n_i \pi_{j,2}^*(s))\mathrm{d}Z(s)$ are deterministic and bounded functions. By Lemma 4.3 in Zeng and Taksar (2013), we have

$$
\begin{aligned}
E_1(t) = &\underbrace{\exp\left\{8m_i^2 \int_0^t e^{-2r(s-T)} \sigma^2 \left(\pi_{i,1}^*(s)\right)^2 \mathrm{d}s\right\}}_{constant} \\
&\underbrace{\times \exp\left\{-8m_i^2 \int_0^t e^{-2r(s-T)} \sigma^2 \left(\pi_{i,1}^*(s)\right)^2 \mathrm{d}s - 4m_i \int_0^t e^{-r(s-T)} \sigma \pi_{i,1}^*(s) \mathrm{d}B_{i,1}^{\mathbb{Q}_i^*}(s)\right\}}_{martingale}.
\end{aligned}
$$

Therefore, $\mathbb{E}_{\mathbb{Q}_i^*}[E_1(t)] < \infty$. Similarly, we can prove that $\mathbb{E}_{\mathbb{Q}_i^*}[E_l(t)] < \infty$, for $l = 2, 3, 4$. Consequently, $\mathbb{E}_{\mathbb{Q}_i^*}\left[\sup_{t\in[0,T]} \left|W_i\left(t, \widehat{X}_i^{\pi_i^*,\pi_j^*}(t), 0\right)\right|^4\right] < \infty$. From similar procedures, we can prove $\mathbb{E}_{\mathbb{Q}_i^*}\left[\sup_{t\in[0,T]}\left|W_i\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t),1\right)\right|^4\right]<\infty$, which verifies property (ii).

(iii) Let $\Gamma_i(t)=\frac{(\phi_{i,1}^*(t))^2m_i}{2\beta_{i,1}}+\frac{(\phi_{i,2}^*(t))^2m_i}{2\beta_{i,2}}+\frac{(\phi_{i,3}^*\ln\phi_{i,3}^*-\phi_{i,3}^*+1)h^P(1-z)m_i}{\beta_{i,3}}$, which is obviously bounded. We first define

$$\begin{cases} B_1(t):=\frac{\left(\phi_{i,1}^*(t)\right)^2}{2\psi_{i,1}\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t)\right)},\\ B_2(t):=\frac{\left(\phi_{i,2}^*(t)\right)^2}{2\psi_{i,2}\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t)\right)},\\ B_3(t):=\frac{\left(\phi_{i,3}^*\ln\phi_{i,3}^*-\phi_{i,3}^*+1\right)h^P(1-z)}{\psi_{i,3}\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t)\right)}.\end{cases}$$

Then substituting (2.2.12) into the expression in property (iii), we have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}_i^*}\left[\sup_{t\in[0,T]}\left|B_1(t)+B_2(t)+B_3(t)\right|^2\right]\\&=\mathbb{E}_{\mathbb{Q}_i^*}\left[\sup_{t\in[0,T]}|\Gamma_i(t)|^2\left|\widetilde{W}_i\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t),Z(t)\right)\right|^2\right]\\&\le\left\{\mathbb{E}_{\mathbb{Q}_i^*}\left[\sup_{t\in[0,T]}|\Gamma_i(t)|^4\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}_{\mathbb{Q}_i^*}\left[\sup_{t\in[0,T]}\left|\widetilde{W}_i\left(t,\widehat{X}_i^{\pi_i^*,\pi_j^*}(t),Z(t)\right)\right|^4\right]\right\}^{\frac{1}{2}}\le\infty.\end{aligned}$$

The first inequality follows from Cauchy-Schwarz inequality, and the latter inequality follows from the conclusion in property (ii).

□

Appendix C

Proof of Corollary 2.4.1

Proof. When insurer i is AAI and insurer j is ANI, for $i, j \in \{1, 2\}, i \neq j$, the value function of insurer i is given by (2.3.25), and the optimal reinsurance and investment strategies of insurer i are given by (2.3.26). While the value function and optimal reinsurance and investment strategies of insurer j are given in Proposition 2.4.1. As the previous analysis in Theorem 2.3.1, we can obtain the equilibrium strategy of investing in the stock by solving the following system of equations:

$$\begin{cases} \pi_{i,1}^* = \dfrac{\mu - r + m_i n_i \sigma^2 e^{r(T-t)} \pi_{j,1}^*}{(m_i + \beta_{i,1})\sigma^2 e^{r(T-t)}}, & t \in [0, T], \\ \pi_{j,1}^* = \dfrac{\mu - r}{m_j \sigma^2 e^{r(T-t)}} + n_j \pi_{i,1}^*, & t \in [0, T]. \end{cases}$$

Similarly, if we substitute $\phi_{i,2}^*$ in (2.3.27) into q_i^* in (2.3.26), we can get the expression of q_i^* as follows:

$$q_i^* = \frac{a_i \gamma_i + \rho b_i b_j m_i n_i e^{r(T-t)} q_j^*}{(m_i + \beta_{i,2}) b_i^2 e^{r(T-t)}}, \quad t \in [0, T].$$

Then the equilibrium reinsurance strategy is the solution of the following system of equations:

$$\begin{cases} q_i^* = \dfrac{a_i \gamma_i + \rho b_i b_j m_i n_i e^{r(T-t)} q_j^*}{(m_i + \beta_{i,2}) b_i^2 e^{r(T-t)}}, & t \in [0, T], \\ q_j^* = \dfrac{a_j \gamma_j + \rho b_i b_j m_j n_j e^{r(T-t)} q_i^*}{b_j^2 m_j e^{r(T-t)}}, & t \in [0, T]. \end{cases}$$

Finally, if we put the expressions of $\pi_{j,1}^*$ and q_j^* back into (2.3.27), we obtain (2.4.34). The Nash equilibrium investment strategy of the defaultable bond for the pre-default case can be derived by solving the system of equations as follows:

$$\begin{cases} \pi^*_{i,2} = \dfrac{C_i(t)}{m_i \zeta e^{r(T-t)}} + n_i \pi^*_{j,2}, & t \in [0, \tau \wedge T], \\ \pi^*_{j,2} = \dfrac{\widehat{C}_j(t)}{m_j \zeta e^{r(T-t)}} + n_j \pi^*_{i,2}, & t \in [0, \tau \wedge T]. \end{cases}$$

Since $n_i n_j \in [0, 1]$, and so $m_i \geq m_i n_i n_j$. Recalling that $\mu > r$, we have $\pi^*_{i,1}(t)$ and $\pi^*_{j,1}(t)$ in (2.4.32) are positive. Similarly, $\rho^2 \in [0, 1]$ implies that $m_i \geq \rho^2 m_i n_i n_j$, which leads to $q^*_i(t)$ and $q^*_j(t)$ in (2.4.33) are larger than 0. Then, we complete the proof of Corollary 2.4.1. □

Appendix D

Derivation of relative entropy in Chapter 3

The derivation is similar to that in Chapter 2, we need to determine the expectation under the alternative probability measure of the log Radon-Nikodym derivative defined in (3.2.4). Applying Itô's formula, we obtain

$$
\begin{aligned}
\mathrm{d}\ln\Lambda^{\phi_k}(t) = {} & \phi_{k1}(t)\mathrm{d}B(t) - \frac{1}{2}\phi_{k1}^2(t)\mathrm{d}t + \lambda_k(1-\phi_{k2}(t))\mathrm{d}t + \lambda(1-\phi_{k3}(t))\mathrm{d}t \\
& + \int_0^\infty \ln\phi_{k2}(t) N_k(\mathrm{d}t, \mathrm{d}z_k) + \int_0^\infty \ln\phi_{k3}(t) N(\mathrm{d}t, \mathrm{d}z_k).
\end{aligned}
$$

Then the relative entropy over the interval from t to $t+\varepsilon$ is given by

$$
\begin{aligned}
\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[\ln\frac{\Lambda^{\phi_k}(t+\varepsilon)}{\Lambda^{\phi_k}(t)}\right] = {} & \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\Bigg[\int_t^{t+\epsilon}\phi_{k1}(s)\left(\mathrm{d}B^{\mathbb{Q}_k}(s)+\phi_{k1}(s)\mathrm{d}s\right) \\
& + \int_t^{t+\epsilon}\left(\lambda_k\left(1-\phi_{k2}(s)\right)+\lambda\left(1-\phi_{k3}(s)\right)-\frac{1}{2}\phi_{k1}^2(s)\right)\mathrm{d}s \\
& + \int_t^{t+\epsilon}\int_0^\infty \ln\phi_{k2}(s)\widehat{N}_k(\mathrm{d}s,\mathrm{d}z_k) + \int_t^{t+\epsilon}\lambda_k\phi_{k2}(s)\ln\phi_{k2}(s)\mathrm{d}s \\
& + \int_t^{t+\epsilon}\int_0^\infty \ln\phi_{k3}(s)\widehat{N}(\mathrm{d}s,\mathrm{d}z_k) + \int_t^{t+\epsilon}\lambda\phi_{k3}(s)\ln\phi_{k3}(s)\mathrm{d}s\Bigg] \\
= \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\Bigg[\int_t^{t+\epsilon}\bigg(\frac{1}{2}\phi_{k1}^2(s) & +\lambda_k\left(\phi_{k2}(s)\ln\phi_{k2}(s)+1-\phi_{k2}(s)\right)+\lambda\left(\phi_{k3}(s)\ln\phi_{k3}(s)+1-\phi_{k3}(s)\right)\bigg)\mathrm{d}s\Bigg].
\end{aligned}
$$

Letting $\epsilon \to 0$, we can obtain the continuous-time limit of the relative entropy given by

$$
\left[\frac{1}{2}\phi_{k1}^2(t)+\lambda_k\left(\phi_{k2}(t)\ln\phi_{k2}(t)+1-\phi_{k2}(t)\right)+\lambda\left(\phi_{k3}(s)\ln\phi_{k3}(s)+1-\phi_{k3}(s)\right)\right]\mathrm{d}t.
$$

Appendix E

Proof of Theorem 3.3.1

Proof. The proof of this theorem consists of two steps. Firstly, we show that the solution of the extended HJB equation (3.3.13) is the value function corresponding to (u_k^*, ϕ_k^*), and that $g_k(t, \hat{x}_k)$ allows for the interpretation $g_k(t, \hat{x}_k) = \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right]$, where $\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}$ is the conditional expectation under the probability measure $\mathbb{Q}_k^*$ given that the current state of the controlled process is $\hat{x}_k$. In the second step, we need to prove that u_k^* is the equilibrium reinsurance-investment strategy of insurer k.

According to Dynkin's formula, see, for example, Oksendal (2003) and (3.3.16), we have

$$\begin{aligned}\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[g_k\left(T, \widehat{X}_k^{u_k^*,u_l^*}(T)\right)\right] &= g_k(t, \hat{x}_k) + \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\int_t^T \mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*} g_k\left(s, \widehat{X}_k^{u_k^*,u_l^*}(s)\right) \mathrm{d}s\right] \\ &= g_k(t, \hat{x}_k),\end{aligned}$$

where $\mathbb{Q}_k^*$ is the alternative probability measure describing the worst-case scenario and determined by the density generator ϕ_k^*. Taking into account (3.3.15), we obtain

$$g_k(t, \hat{x}_k) = \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[g_k\left(T, \widehat{X}_k^{u_k^*,u_l^*}(T)\right)\right] = \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right]. \tag{E.1}$$

Noting that (u_k^*, ϕ_k^*) solves the optimization problem in the left-hand side of (3.3.13) and $g_k(t, \hat{x}_k)$ satisfies (3.3.16), we can obtain

$$\mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*} W_k(t, \hat{x}_k) - \mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*} \frac{m_k}{2} g_k^2(t, \hat{x}_k) + \widehat{P}_k(\phi_k^*) = 0, \tag{E.2}$$

where

$$\widehat{P}_k(\phi_k^*) = \frac{(\phi_{k1}^*)^2}{2\beta_{k1}} + \frac{\lambda_k(\phi_{k2}^* \ln \phi_{k2}^* + 1 - \phi_{k2}^*)}{\beta_{k2}} + \frac{\lambda(\phi_{k3}^* \ln \phi_{k3}^* + 1 - \phi_{k3}^*)}{\beta_{k3}}.$$

We now use Dynkin's formula again, and combining with (3.3.14) we can obtain

$$\begin{aligned}\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[W_k\left(T,\widehat{X}_k^{u_k^*,u_l^*}(T)\right)\right] &= W_k(t,\hat{x}_k)+\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\int_t^T \mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*}W_k\left(s,\widehat{X}_k^{u_k^*,u_l^*}(s)\right)\mathrm{d}s\right]\\ &= \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right].\end{aligned} \tag{E.3}$$

Substituting (E.2) into (E.3), we have

$$\begin{aligned}W_k(t,\hat{x}_k) = {}& \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right] - \frac{m_k}{2}\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\int_t^T \mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*}g_k^2\left(s,\widehat{X}_k^{u_k^*,u_l^*}(s)\right)\mathrm{d}s\right]\\ &+\int_t^T \widehat{P}_k(\phi_k^*(s))\mathrm{d}s.\end{aligned} \tag{E.4}$$

Moreover, applying (3.3.15), Dynkin's formula and (E.1) yields

$$\begin{aligned}&\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\left(\widehat{X}_k^{u_k^*,u_l^*}(T)\right)^2\right]\\ &= \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[g_k^2\left(T,\widehat{X}_k^{u_k^*,u_l^*}(T)\right)\right]\\ &= g_k^2(t,\hat{x}_k)+\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\int_t^T \mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*}g_k^2\left(s,\widehat{X}_k^{u_k^*,u_l^*}(s)\right)\mathrm{d}s\right]\\ &= \left(\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right]\right)^2+\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\int_t^T \mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*}g_k^2\left(s,\widehat{X}_k^{u_k^*,u_l^*}(s)\right)\mathrm{d}s\right].\end{aligned} \tag{E.5}$$

Note that (E.5) is equivalent to

$$\mathrm{Var}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right] = \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\int_t^T \mathcal{L}^{u_k^*,u_l^*,\phi_k^*,\phi_l^*}g_k^2\left(s,\widehat{X}_k^{u_k^*,u_l^*}(s)\right)\mathrm{d}s\right]. \tag{E.6}$$

Finally, substituting (E.6) into (E.4), we have

$$\begin{aligned}W_k(t,\hat{x}_k) &= \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right] - \frac{m_k}{2}\mathrm{Var}_{t,\hat{x}_k}^{\mathbb{Q}_k^*}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right] + \int_t^T \widehat{P}_k(\phi_k^*(s))\mathrm{d}s\\ &= J_k^{u_k^*,u_l^*}(t,\hat{x}_k) = V_k(t,\hat{x}_k),\end{aligned} \tag{E.7}$$

where the last two equations hold due to (3.2.11).

Next, we are going to show that u_k^* is an equilibrium strategy. We consider the deterministic reinsurance-investment strategy

$$u_k^\epsilon(s) = \begin{cases}\tilde{u}_k, & t \le s < t+\varepsilon,\\ u_k^*(s), & t+\varepsilon \le s \le T,\end{cases}$$

where $\tilde{u}_k = (\tilde{q}_k, \tilde{\pi}_k) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\varepsilon > 0$. Similar to the Lemma A.2 in Li et al. (2016), it can be shown that the density generator function corresponding to $u_k^\epsilon(s)$ is given by

$$\phi_k^{u_k^\epsilon}(s) = \begin{cases} \phi_k^{\tilde{u}_k}, & t \le s < t+\varepsilon, \\ \phi_k^{u_k^*}(s), & t+\varepsilon \le s \le T, \end{cases} \tag{E.8}$$

and for any measurable function $f_k : \mathbb{R} \to \mathbb{R}$, we have

$$\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^\varepsilon}\left[f_k\left(\widehat{X}_k^{u_k^\varepsilon,u_l^*}(T)\right)\right] = \mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\mathbb{E}^{\mathbb{Q}_k^*}_{t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)}\left[f_k\left(\widehat{X}_k^{u_k^*,u_l^*}(T)\right)\right]\right]. \tag{E.9}$$

Here and thereafter, we use $\mathbb{Q}_k^\varepsilon$ and $\widetilde{\mathbb{Q}}_k$ to denote the alternative measures determined by $\phi_k^{u_k^\epsilon}$ and $\phi_k^{\tilde{u}_k}$, respectively. Here, we omit the proofs of these two conclusions. The interested readers can refer to Appendix A in Li et al. (2016).

According to Definition 3.2.2, it may be said that u_k^* is an equilibrium strategy if we can prove that

$$J_k^{u_k^\varepsilon,u_l^*}(t,\hat{x}_k) - J_k^{u_k^*,u_l^*}(t,\hat{x}_k) \le o(\varepsilon).$$

To this end, we first derive the expression of $J_k^{u_k^\varepsilon,u_l^*}(t,\hat{x}_k) - J_k^{u_k^*,u_l^*}(t,\hat{x}_k)$.

By (3.2.8), (3.2.9) and (E.9), we can derive

$$\begin{aligned}
& J_k^{u_k^\varepsilon,u_l^*}(t,\hat{x}_k) \\
={}& \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^\varepsilon}\left[\widehat{X}_k^{u_k^\varepsilon,u_l^*}(T) - \frac{m_k}{2}\left(\widehat{X}_k^{u_k^\varepsilon,u_l^*}(T)\right)^2\right] + \frac{m_k}{2}\left(\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k^\varepsilon}\left[\widehat{X}_k^{u_k^\varepsilon,u_l^*}(T)\right]\right)^2 \\
&+ \int_t^T \widehat{P}_k\left(\phi_k^{u_k^\varepsilon}(s)\right)\mathrm{d}s \\
={}& \mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\mathbb{E}^{\mathbb{Q}_k^*}_{t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)}\left[\widehat{X}_k^{u_k^*,u_l^*}(T) - \frac{m_k}{2}\left(\widehat{X}_k^{u_k^*,u_l^*}(T)\right)^2\right]\right] \\
&+ \frac{m_k}{2}\left(\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\mathbb{E}^{\mathbb{Q}_k^*}_{t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right]\right]\right)^2 + \int_t^{t+\varepsilon}\widehat{P}_k\left(\phi_k^{\tilde{u}_k}(s)\right)\mathrm{d}s \\
&+ \int_{t+\varepsilon}^T \widehat{P}_k\left(\phi_k^{u_k^*}(s)\right)\mathrm{d}s \\
={}& \mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[J_k^{u_k^*,u_l^*}\left(t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right] - \frac{m_k}{2}\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\left(\mathbb{E}^{\mathbb{Q}_k^*}_{t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right]\right)^2\right] \\
&+ \frac{m_k}{2}\left(\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\mathbb{E}^{\mathbb{Q}_k^*}_{t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)}\left[\widehat{X}_k^{u_k^*,u_l^*}(T)\right]\right]\right)^2 + \int_t^{t+\varepsilon}\widehat{P}_k\left(\phi_k^{\tilde{u}_k}(s)\right)\mathrm{d}s \\
={}& \mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[J_k^{u_k^*,u_l^*}\left(t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right] - \frac{m_k}{2}\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[g_k^2\left(t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right] \\
&+ \frac{m_k}{2}\left(\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[g_k\left(t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right]\right)^2 + \int_t^{t+\varepsilon}\widehat{P}_k\left(\phi_k^{\tilde{u}_k}(s)\right)\mathrm{d}s,
\end{aligned}$$

where the last identity is obtained by using (E.1). Then we have

$$J_k^{u_k^\varepsilon,u_l^*}(t,\hat{x}_k) - J_k^{u_k^*,u_l^*}(t,\hat{x}_k) := N_\varepsilon,$$

where N_ε is given by

$$\begin{aligned} N_\varepsilon = &\, \mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[J_k^{u_k^*,u_l^*}\left(t+\varepsilon, \widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right] - J_k^{u_k^*,u_l^*}(t,\hat{x}_k) \\ &- \frac{m_k}{2}\left\{\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[g_k^2\left(t+\varepsilon, \widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right] - g_k^2(t,\hat{x}_k)\right\} \\ &+ \frac{m_k}{2}\left\{\left(\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[g_k\left(t+\varepsilon, \widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right]\right)^2 - g_k^2(t,\hat{x}_k)\right\} \\ &+ \int_t^{t+\varepsilon} \widehat{P}_k\left(\phi_k^{\tilde{u}_k}(s)\right) \mathrm{d}s. \end{aligned} \tag{E.10}$$

In other words, we need to show that $N_\varepsilon \leq o(\varepsilon)$.

For notational simplicity, $\forall\, W_k(t,\hat{x}_k) \in C^{1,2}([0,T]\times\mathbb{R})$, we define an operator

$$\mathcal{L}_\varepsilon^{u_k,u_l,\phi_k,\phi_l} W_k(t,\hat{x}_k) := \mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[W_k\left(t+\varepsilon, \widehat{X}_k^{u_k,u_l}(t+\varepsilon)\right)\right] - W_k\left(t,\hat{x}_k\right),$$

where $u_k \in \mathcal{U}_k$, $\mathbb{Q}_k \in \mathcal{Q}$, and $\varepsilon > 0$. By the results in (E.8), we know

$$\mathcal{L}_\varepsilon^{\tilde{u}_k,u_l,\phi_k^{u_\varepsilon},\phi_l} W_k(t,\hat{x}_k) = \mathcal{L}_\varepsilon^{\tilde{u}_k,u_l,\phi_k^{\tilde{u}_k},\phi_l} W_k(t,\hat{x}_k). \tag{E.11}$$

Recalling that the definition of infinitesimal generator in (3.3.12) can be interpreted by

$$\mathcal{L}^{u_k,u_l,\phi_k,\phi_l} W_k(t,\hat{x}_k) = \lim_{\varepsilon\to 0} \frac{\mathbb{E}_{t,\hat{x}_k}^{\mathbb{Q}_k}\left[W_k\left(t+\varepsilon, \widehat{X}_k^{u_k,u_l}(t+\varepsilon)\right)\right] - W_k\left(t,\hat{x}_k\right)}{\varepsilon}, \tag{E.12}$$

we can further obtain

$$\lim_{\varepsilon\to 0} \frac{\mathcal{L}_\varepsilon^{\tilde{u}_k,u_l,\phi_k^{u_\varepsilon},\phi_l} W_k(t,\hat{x}_k)}{\varepsilon} = \mathcal{L}^{\tilde{u}_k,u_l,\phi_k^{\tilde{u}_k},\phi_l} W_k(t,\hat{x}_k).$$

Substituting (E.11) into (E.10), N_ε becomes

$$\begin{aligned} N_\varepsilon = &\, \mathcal{L}_\varepsilon^{\tilde{u}_k,u_l,\phi_k^{\tilde{u}_k},\phi_l} J_k^{u_k^*,u_l^*}(t,\hat{x}_k) - \frac{m_k}{2}\mathcal{L}_\varepsilon^{\tilde{u}_k,u_l,\phi_k^{\tilde{u}_k},\phi_l} g_k^2(t,\hat{x}_k) \\ &+ \frac{m_k}{2}\left\{\left(\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[g_k\left(t+\varepsilon, \widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right]\right)^2 - g_k^2(t,\hat{x}_k)\right\} + \int_t^{t+\varepsilon} \widehat{P}_k\left(\phi_k^{\tilde{u}_k}(s)\right) \mathrm{d}s. \end{aligned} \tag{E.13}$$

On the other hand, using Dynkin's formula yields

$$\begin{aligned} &\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[g_k\left(t+\varepsilon, \widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right] \\ &= g_k(t,\hat{x}_k) + \mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\int_t^{t+\varepsilon} \mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*} g_k\left(s, \widehat{X}_k^{\tilde{u}_k,u_l^*}(s)\right) \mathrm{d}s\right], \end{aligned}$$

which indicates that

$$\begin{aligned}&\left(\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[g_k\left(t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right]\right)^2-g_k^2(t,\hat{x}_k)\\&=2g_k(t,\hat{x}_k)\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\int_t^{t+\varepsilon}\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}g_k\left(s,\widehat{X}_k^{\tilde{u}_k,u_l^*}(s)\right)\mathrm{d}s\right]+o(\varepsilon).\end{aligned}\tag{E.14}$$

Plugging (E.14) into (E.13), we obtain

$$\begin{aligned}N_\varepsilon=&\,\mathcal{L}_\varepsilon^{\tilde{u}_k,u_l,\phi_k^{\tilde{u}_k},\phi_l}J_k^{u_k^*,u_l^*}(t,\hat{x}_k)-\frac{m_k}{2}\mathcal{L}_\varepsilon^{\tilde{u}_k,u_l,\phi_k^{\tilde{u}_k},\phi_l}g_k^2(t,\hat{x}_k)\\&+m_kg_k(t,\hat{x}_k)\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\int_t^{t+\varepsilon}\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}g_k\left(s,\widehat{X}_k^{\tilde{u}_k,u_l^*}(s)\right)\mathrm{d}s\right]\\&+\int_t^{t+\varepsilon}\widehat{P}_k\left(\phi_k^{\tilde{u}_k}(s)\right)\mathrm{d}s+o(\varepsilon).\end{aligned}\tag{E.15}$$

Considering the extended HJB equation in (3.3.13), we know

$$\begin{aligned}&\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}W_k(t,\hat{x}_k)-\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}\frac{m_k}{2}g_k^2(t,\hat{x}_k)\\&+m_kg_k(t,\hat{x}_k)\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}g_k(t,\hat{x}_k)+\widehat{P}_k(\phi_k^{\tilde{u}_k})\le 0.\end{aligned}$$

Additionally, from (E.7) we know that

$$\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}W_k(t,\hat{x}_k)=\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}J_k^{u_k^*,u_l^*}(t,\hat{x}_k).$$

As a consequence, we have

$$\begin{aligned}&\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}J_k^{u_k^*,u_l^*}(t,\hat{x}_k)-\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}\frac{m_k}{2}g_k^2(t,\hat{x}_k)\\&+m_kg_k(t,\hat{x}_k)\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}g_k(t,\hat{x}_k)+\widehat{P}_k(\phi_k^{\tilde{u}_k})\le 0.\end{aligned}\tag{E.16}$$

Applying (E.12) and Dynkin's formula, we can obtain

$$\begin{aligned}\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}g_k(t,\hat{x}_k)&=\lim_{\varepsilon\to0}\frac{\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[g_k\left(t+\varepsilon,\widehat{X}_k^{\tilde{u}_k,u_l^*}(t+\varepsilon)\right)\right]-g_k\left(t,\hat{x}_k\right)}{\varepsilon}\\&=\lim_{\varepsilon\to0}\frac{\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\int_t^{t+\varepsilon}\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}g_k\left(s,\widehat{X}_k^{\tilde{u}_k,u_l^*}(s)\right)\mathrm{d}s\right]}{\varepsilon}.\end{aligned}\tag{E.17}$$

Inserting (E.11) and (E.17) into (E.16), we have

$$\begin{aligned}&\mathcal{L}_\varepsilon^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}J_k^{u_k^*,u_l^*}(t,\hat{x}_k)-\mathcal{L}_\varepsilon^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}\frac{m_k}{2}g_k^2(t,\hat{x}_k)\\&+m_kg_k(t,\hat{x}_k)\mathbb{E}_{t,\hat{x}_k}^{\widetilde{\mathbb{Q}}_k}\left[\int_t^{t+\varepsilon}\mathcal{L}^{\tilde{u}_k,u_l^*,\phi_k^{\tilde{u}_k},\phi_l^*}g_k\left(s,\widehat{X}_k^{\tilde{u}_k,u_l^*}(s)\right)\mathrm{d}s\right]+\widehat{P}_k(\phi_k^{\tilde{u}_k})\le o(\varepsilon).\end{aligned}\tag{E.18}$$

Finally, if we put (E.18) back into (E.15), we will obtain

$$J_k^{u_k^\varepsilon,u_l^*}(t,\hat{x}_k)-J_k^{u_k^*,u_l^*}(t,\hat{x}_k)=N_\varepsilon\le o(\varepsilon),$$

which implies that u_k^* is an equilibrium strategy. This completes the proof. $\square$

Appendix F

Proof of Theorem 3.3.2

Proof. Simplifying Equation (3.3.13), we obtain

$$\sup_{u_k\in\mathcal{U}_k}\inf_{\phi_k\in\Sigma_k}\Bigg\{\frac{\partial W_k(t,\hat{x}_k)}{\partial t}+\big[r\hat{x}_k+(\mu-r)(\pi_k-n_k\pi_l^*)+\mu_k(\lambda_k+\lambda)(\gamma_k+(1+\eta_k)q_k)$$
$$-n_k\mu_l(\lambda_l+\lambda)(\gamma_l+(1+\eta_l)q_l^*)+\sigma\pi_k\phi_{k1}-\sigma n_k\pi_l^*\phi_{l1}^*\big]\frac{\partial W_k(t,\hat{x}_k)}{\partial\hat{x}_k}$$
$$+\frac{1}{2}\sigma^2(\pi_k^2+n_k^2(\pi_l^*)^2-2n_k\pi_k\pi_l^*)\left(\frac{\partial^2W_k(t,\hat{x}_k)}{\partial\hat{x}_k^2}-m_k\left(\frac{\partial g_k(t,\hat{x}_k)}{\partial\hat{x}_k}\right)^2\right)$$
$$+(\lambda_k\phi_{k2}+\lambda\phi_{k3})\mathbb{E}^{\mathbb{Q}_k}\Big[W_k(t,\hat{x}_k-q_kz_k)-\frac{m_k}{2}g_k^2(t,\hat{x}_k-q_kz_k)+m_kg_k(t,\hat{x}_k)g_k(t,\hat{x}_k-q_kz_k)\Big]$$
$$+(\lambda_l\phi_{l2}^*+\lambda\phi_{l3}^*)\mathbb{E}^{\mathbb{Q}_l}\Big[W_k(t,\hat{x}_k+n_kq_l^*z_l)-\frac{m_k}{2}g_k^2(t,\hat{x}_k+n_kq_l^*z_l)+m_kg_k(t,\hat{x}_k)$$
$$\times g_k(t,\hat{x}_k+n_kq_l^*z_l)\Big]-(\lambda_k\phi_{k2}+\lambda_l\phi_{l2}^*+\lambda\phi_{k3}+\lambda\phi_{l3}^*)\Big(W_k(t,\hat{x}_k)+\frac{m_k}{2}g_k^2(t,\hat{x}_k)\Big)$$
$$+\frac{\phi_{k1}^2}{2\beta_{k1}}+\frac{\lambda_k(\phi_{k2}\ln\phi_{k2}+1-\phi_{k2})}{\beta_{k2}}+\frac{\lambda(\phi_{k3}\ln\phi_{k3}+1-\phi_{k3})}{\beta_{k3}}\Bigg\}=0. \tag{F.1}$$

In order to solve (3.3.14) and (F.1), we conjecture that the solutions have the following forms:

$$W_k(t,\hat{x}_k)=A_k(t)\hat{x}_k+B_k(t),\ A_k(T)=1,\ B_k(T)=0,$$

$$g_k(t,\hat{x}_k)=\widetilde{A}_k(t)\hat{x}_k+\widetilde{B}_k(t),\ \widetilde{A}_k(T)=1,\ \widetilde{B}_k(T)=0.$$

By some simple calculation, we obtain the associated partial derivatives as follows:

$$\frac{\partial W_k(t,\hat{x}_k)}{\partial t}=A_k'(t)\hat{x}_k+B_k'(t),\quad\frac{\partial W_k(t,\hat{x}_k)}{\partial\hat{x}_k}=A_k(t),\quad\frac{\partial^2W_k(t,\hat{x}_k)}{\partial\hat{x}_k^2}=0, \tag{F.2}$$

$$\frac{\partial g_k(t,\hat{x}_k)}{\partial t}=\widetilde{A}_k'(t)\hat{x}_k+\widetilde{B}_k'(t),\quad\frac{\partial g_k(t,\hat{x}_k)}{\partial\hat{x}_k}=\widetilde{A}_k(t),\quad\frac{\partial^2g_k(t,\hat{x}_k)}{\partial\hat{x}_k^2}=0. \tag{F.3}$$

Substituting (F.2) and (F.3) into (F.1) yields

$$\sup_{u_k\in\mathcal{U}_k}\inf_{\phi_k\in\Sigma_k}\Bigg\{A_k'\hat{x}_k+B_k'+\big[r\hat{x}_k+(\mu-r)(\pi_k-n_k\pi_l^*)+\mu_k(\lambda_k+\lambda)(\gamma_k+(1+\eta_k)q_k)$$
$$-n_k\mu_l(\lambda_l+\lambda)(\gamma_l+(1+\eta_l)q_l^*)+\sigma\pi_k\phi_{k1}-\sigma n_k\pi_l^*\phi_{l1}^*-(\lambda_k\phi_{k2}+\lambda\phi_{k3})q_k\mu_k$$
$$+(\lambda_l\phi_{l2}^*+\lambda\phi_{l3}^*)n_kq_l^*\mu_l\big]A_k-\frac{m_k\widetilde{A}_k^2}{2}\big[\sigma^2(\pi_k^2+n_k^2(\pi_l^*)^2-2n_k\pi_k\pi_l^*)$$
$$+(\lambda_k\phi_{k2}+\lambda\phi_{k3})q_k^2\sigma_k^2+(\lambda_l\phi_{l2}^*+\lambda\phi_{l3}^*)n_k^2(q_l^*)^2\sigma_l^2\big]+\frac{\phi_{k1}^2}{2\beta_{k1}}$$
$$+\frac{\lambda_k(\phi_{k2}\ln\phi_{k2}+1-\phi_{k2})}{\beta_{k2}}+\frac{\lambda(\phi_{k3}\ln\phi_{k3}+1-\phi_{k3})}{\beta_{k3}}\Bigg\}=0. \tag{F.4}$$

Fixing u_k and letting the first-order derivative of the left-hand side of (F.4) with respect to ϕ_k equal zero, we can obtain the infimum point $\phi_k^*:=(\phi_{k1}^*,\phi_{k2}^*,\phi_{k3}^*)$ given by

$$\begin{cases}\phi_{k1}^*=-\beta_{k1}\sigma\pi_kA_k,\\ \phi_{k2}^*=e^{\beta_{k2}X_k},\\ \phi_{k3}^*=e^{\beta_{k3}X_k},\end{cases} \tag{F.5}$$

where

$$X_k=\mu_kq_kA_k+\frac{m_k\sigma_k^2q_k^2\widetilde{A}_k^2}{2}. \tag{F.6}$$

Putting (F.5) back into (F.4), we then obtain

$$\sup_{u_k\in\mathcal{U}_k}\Bigg\{A_k'\hat{x}_k+B_k'+A_k\big[r\hat{x}_k+(\mu-r)(\pi_k-n_k\pi_l^*)+\mu_k(\lambda_k+\lambda)(\gamma_k+(1+\eta_k)q_k)$$
$$-n_k\mu_l(\lambda_l+\lambda)(\gamma_l+(1+\eta_l)q_l^*)+n_k\sigma^2\beta_{l1}(\pi_l^*)^2A_l+n_k\mu_lq_l^*\left(\lambda_le^{\beta_{l2}X_l^*}+\lambda e^{\beta_{l3}X_l^*}\right)\big]$$
$$-\frac{m_k\widetilde{A}_k^2}{2}\left[\sigma^2(\pi_k^2+n_k^2(\pi_l^*)^2-2n_k\pi_k\pi_l^*)+n_k^2\sigma_l^2(q_l^*)^2\left(\lambda_le^{\beta_{l2}X_l^*}+\lambda e^{\beta_{l3}X_l^*}\right)\right]$$
$$-\frac{\beta_{k1}\sigma^2\pi_k^2A_k^2}{2}+\frac{\lambda_k}{\beta_{k2}}\left(1-e^{\beta_{k2}X_k}\right)+\frac{\lambda}{\beta_{k3}}\left(1-e^{\beta_{k3}X_k}\right)\Bigg\}=0, \tag{F.7}$$

where

$$X_l^*=\mu_lq_l^*A_l+\frac{m_l\sigma_l^2(q_l^*)^2\widetilde{A}_l^2}{2}.$$

Maximizing over u_k, we can obtain that the equilibrium reinsurance-investment strategy of insurer k should satisfy

$$\begin{cases}\pi_k^*=\dfrac{1}{\beta_{k1}A_k^2+m_k\widetilde{A}_k^2}\left[\dfrac{(\mu-r)A_k}{\sigma^2}+m_kn_k\pi_l^*\widetilde{A}_k^2\right],\\ \mu_k(\lambda_k+\lambda)(1+\eta_k)A_k=\left(\lambda_ke^{\beta_{k2}X_k^*}+\lambda e^{\beta_{k3}X_k^*}\right)(\mu_kA_k+m_k\sigma_k^2\widetilde{A}_k^2q_k^*),\end{cases} \tag{F.8}$$

where X_k^* is obtained by substituting q_k^* for q_k in (F.6).

We first verify that π_k^* in (F.8) derived by the first-order condition is definitely the optimal investment strategy of insurer k. Gather the terms of π_k in the left-hand side of (F.7) and let

$$h_1(\pi_k) = (\mu - r)\pi_k A_k - \frac{m_k \sigma^2 \widetilde{A}_k^2}{2}(\pi_k^2 - 2n_k \pi_l^* \pi_k) - \frac{\beta_{k1}\sigma^2 \pi_k^2 A_k^2}{2},$$

accordingly we have

$$h_1'(\pi_k) = (\mu - r)A_k - m_k \sigma^2 \widetilde{A}_k^2 (\pi_k - n_k \pi_l^*) - \beta_{k1}\sigma^2 A_k^2 \pi_k,$$

and

$$h_1''(\pi_k) = -m_k \sigma^2 \widetilde{A}_k^2 - \beta_{k1}\sigma^2 A_k^2.$$

It is obvious that we have $h_1''(\pi_k) < 0$ for any admissible investment strategy π_k, and hence the first-order optimality condition leads to the optimal investment strategy. Similarly, we let

$$h_2(q_k) = \mu_k(\lambda_k + \lambda)(\gamma_k + (1+\eta_k)q_k)A_k + \frac{\lambda_k}{\beta_{k2}}\left(1 - e^{\beta_{k2}X_k}\right) + \frac{\lambda}{\beta_{k3}}\left(1 - e^{\beta_{k3}X_k}\right),$$

which gathers the terms of q_k in the left-hand side of (F.7), and so we have

$$h_2'(q_k) = \mu_k(\lambda_k + \lambda)(1+\eta_k)A_k - (\mu_k A_k + m_k \sigma_k^2 \widetilde{A}_k^2 q_k)\left(\lambda_k e^{\beta_{k2}X_k} + \lambda e^{\beta_{k3}X_k}\right),$$

and

$$\begin{aligned} h_2''(q_k) = &- m_k \sigma_k^2 \widetilde{A}_k^2 \left(\lambda_k e^{\beta_{k2}X_k} + \lambda e^{\beta_{k3}X_k}\right) \\ &- (\mu_k A_k + m_k \sigma_k^2 \widetilde{A}_k^2 q_k)^2 \left(\lambda_k \beta_{k2} e^{\beta_{k2}X_k} + \lambda \beta_{k3} e^{\beta_{k3}X_k}\right) < 0, \end{aligned}$$

which indicates that the first-order optimality condition implies the optimal reinsurance strategy q_k^* given in (F.8).

Inserting π_k^* and q_k^* into (3.3.14) and (F.7), we obtain

$$\begin{aligned} &\hat{x}_k\left(\widetilde{A}_k' + r\widetilde{A}_k\right) + \widetilde{B}_k' + \big[(\mu - r)(\pi_k^* - n_k \pi_l^*) + \mu_k(\lambda_k + \lambda)(\gamma_k + (1+\eta_k)q_k^*) \\ &\quad - n_k \mu_l(\lambda_l + \lambda)(\gamma_l + (1+\eta_l)q_l^*) - \sigma^2 \beta_{k1}(\pi_k^*)^2 A_k + n_k \sigma^2 \beta_{l1}(\pi_l^*)^2 A_l\big]\widetilde{A}_k \\ &\quad - \mu_k \widetilde{A}_k q_k^* \left(\lambda_k e^{\beta_{k2}X_k^*} + \lambda e^{\beta_{k3}X_k^*}\right) + n_k \mu_l \widetilde{A}_k q_l^* \left(\lambda_l e^{\beta_{l2}X_l^*} + \lambda e^{\beta_{l3}X_l^*}\right) = 0, \end{aligned}$$

and

$$
\begin{aligned}
&\hat{x}_k \left(A_k' + rA_k\right) + B_k' + A_k\big[(\mu - r)(\pi_k^* - n_k\pi_l^*) + \mu_k(\lambda_k + \lambda)(\gamma_k + (1+\eta_k)q_k^*) \\
&- n_k\mu_l(\lambda_l + \lambda)(\gamma_l + (1+\eta_l)q_l^*) + n_k\sigma^2\beta_{l1}(\pi_l^*)^2 A_l + n_k\mu_l q_l^* \left(\lambda_l e^{\beta_{l2}X_l^*} + \lambda e^{\beta_{l3}X_l^*}\right)\big] \\
&- \frac{m_k\widetilde{A}_k^2}{2}\left[\sigma^2((\pi_k^*)^2 + n_k^2(\pi_l^*)^2 - 2n_k\pi_k^*\pi_l^*) + n_k^2\sigma_l^2(q_l^*)^2\left(\lambda_l e^{\beta_{l2}X_l^*} + \lambda e^{\beta_{l3}X_l^*}\right)\right] \\
&- \frac{\beta_{k1}\sigma^2(\pi_k^*)^2A_k^2}{2} + \frac{\lambda_k}{\beta_{k2}}\left(1 - e^{\beta_{k2}X_k^*}\right) + \frac{\lambda}{\beta_{k3}}\left(1 - e^{\beta_{k3}X_k^*}\right) = 0.
\end{aligned}
$$

By separating the variables with and without $\hat{x}_k$, we can obtain the following system of equations:

$$
\begin{cases}
\widetilde{A}_k' + r\widetilde{A}_k = 0, \qquad A_k' + rA_k = 0, \\
\widetilde{B}_k' + \big[(\mu - r)(\pi_k^* - n_k\pi_l^*) + \mu_k(\lambda_k + \lambda)(\gamma_k + (1+\eta_k)q_k^*) \\
\qquad - n_k\mu_l(\lambda_l + \lambda)(\gamma_l + (1+\eta_l)q_l^*) - \sigma^2\beta_{k1}(\pi_k^*)^2 A_k + n_k\sigma^2\beta_{l1}(\pi_l^*)^2A_l\big]\widetilde{A}_k \\
\qquad - \mu_k\widetilde{A}_k q_k^*\left(\lambda_k e^{\beta_{k2}X_k^*} + \lambda e^{\beta_{k3}X_k^*}\right) + n_k\mu_l\widetilde{A}_k q_l^*\left(\lambda_l e^{\beta_{l2}X_l^*} + \lambda e^{\beta_{l3}X_l^*}\right) = 0, \\
B_k' + A_k\big[(\mu - r)(\pi_k^* - n_k\pi_l^*) + \mu_k(\lambda_k + \lambda)(\gamma_k + (1+\eta_k)q_k^*) \\
\qquad - n_k\mu_l(\lambda_l + \lambda)(\gamma_l + (1+\eta_l)q_l^*) + n_k\sigma^2\beta_{l1}(\pi_l^*)^2A_l + n_k\mu_l q_l^*\left(\lambda_l e^{\beta_{l2}X_l^*} + \lambda e^{\beta_{l3}X_l^*}\right)\big] \\
\qquad - \frac{m_k\widetilde{A}_k^2}{2}\left[\sigma^2((\pi_k^*)^2 + n_k^2(\pi_l^*)^2 - 2n_k\pi_k^*\pi_l^*) + n_k^2\sigma_l^2(q_l^*)^2\left(\lambda_l e^{\beta_{l2}X_l^*} + \lambda e^{\beta_{l3}X_l^*}\right)\right] \\
\qquad - \frac{\beta_{k1}\sigma^2(\pi_k^*)^2A_k^2}{2} + \frac{\lambda_k}{\beta_{k2}}\left(1 - e^{\beta_{k2}X_k^*}\right) + \frac{\lambda}{\beta_{k3}}\left(1 - e^{\beta_{k3}X_k^*}\right) = 0.
\end{cases}
$$

Combining with the boundary conditions, we can obtain

$$
\widetilde{A}_k(t) = e^{r(T-t)}, \qquad A_k(t) = e^{r(T-t)},
$$

$$
\widetilde{B}_k(t) = \frac{n_k\mu_l\gamma_l(\lambda_l + \lambda) - \mu_k\gamma_k(\lambda_k + \lambda)}{r}\left(1 - e^{r(T-t)}\right) + \int_t^T \tilde{b}_{k1}(s)\mathrm{d}s + \int_t^T \tilde{b}_{k2}(s)\mathrm{d}s,
$$

$$
B_k(t) = \frac{n_k\mu_l\gamma_l(\lambda_l + \lambda) - \mu_k\gamma_k(\lambda_k + \lambda)}{r}\left(1 - e^{r(T-t)}\right) + \int_t^T b_{k1}(s)\mathrm{d}s + \int_t^T b_{k2}(s)\mathrm{d}s,
$$

where

$$
\begin{cases}
\tilde{b}_{k1}(s) = \left[\mu_k(\lambda_k + \lambda)(1+\eta_k)q_k^* - n_k\mu_l(\lambda_l + \lambda)(1+\eta_l)q_l^*\right]e^{r(T-s)} \\
\qquad - \left[\mu_k q_k^*\left(\lambda_k e^{\beta_{k2}X_k^*} + \lambda e^{\beta_{k3}X_k^*}\right) - n_k\mu_l q_l^*\left(\lambda_l e^{\beta_{l2}X_l^*} + \lambda e^{\beta_{l3}X_l^*}\right)\right]e^{r(T-s)}, \\
\tilde{b}_{k2}(s) = (\mu - r)\left(\pi_k^* - n_k\pi_l^*\right)e^{r(T-s)} - \sigma^2\left(\beta_{k1}(\pi_k^*)^2 - n_k\beta_{l1}(\pi_l^*)^2\right)e^{2r(T-s)}, \\
b_{k1}(s) = \big[\mu_k(\lambda_k + \lambda)(1+\eta_k)q_k^* - n_k\mu_l(\lambda_l + \lambda)(1+\eta_l)q_l^* \\
\qquad + n_k\mu_l q_l^*\left(\lambda_l e^{\beta_{l2}X_l^*} + \lambda e^{\beta_{l3}X_l^*}\right)\big]e^{r(T-s)} - \frac{m_k e^{2r(T-s)}}{2}n_k^2\sigma_l^2(q_l^*)^2\big(\lambda_l e^{\beta_{l2}X_l^*} \\
\qquad + \lambda e^{\beta_{l3}X_l^*}\big) + \frac{\lambda_k}{\beta_{k2}}\left(1 - e^{\beta_{k2}X_k^*}\right) + \frac{\lambda}{\beta_{k3}}\left(1 - e^{\beta_{k3}X_k^*}\right), \\
b_{k2}(s) = \left[(\mu - r)(\pi_k^* - n_k\pi_l^*) + n_k\sigma^2\beta_{l1}(\pi_l^*)^2 e^{r(T-s)}\right]e^{r(T-s)} \\
\qquad - \frac{m_k e^{2r(T-s)}}{2}\sigma^2\left((\pi_k^*)^2 + n_k^2(\pi_l^*)^2 - 2n_k\pi_k^*\pi_l^*\right) - \frac{\beta_{k1}\sigma^2(\pi_k^*)^2}{2}e^{2r(T-s)}.
\end{cases}
$$

Consequently, the robust Nash equilibrium reinsurance-investment strategy of insurer k should satisfy

$$\begin{cases} q_k^* = \dfrac{1}{m_k \sigma_k^2 e^{r(T-t)}} \left[\dfrac{\mu_k(\lambda_k + \lambda)(1+\eta_k)}{\lambda_k e^{\beta_{k2} X_k^*} + \lambda e^{\beta_{k3} X_k^*}} - \mu_k \right], \\ \pi_k^* = \dfrac{1}{\sigma^2(\beta_{k1} + m_k) e^{r(T-t)}} \left[\mu - r + m_k n_k \sigma^2 \pi_l^* e^{r(T-t)} \right]. \end{cases}$$

Explicit expressions for the robust equilibrium investment strategy of each insurer, which are presented in (3.3.18), can be obtained readily by solving the following system of equations:

$$\begin{cases} \pi_1^* = \dfrac{1}{\sigma^2(\beta_{11} + m_1) e^{r(T-t)}} \left[\mu - r + m_1 n_1 \sigma^2 \pi_2^* e^{r(T-t)} \right], \\ \pi_2^* = \dfrac{1}{\sigma^2(\beta_{21} + m_2) e^{r(T-t)}} \left[\mu - r + m_2 n_2 \sigma^2 \pi_1^* e^{r(T-t)} \right]. \end{cases}$$

This completes the proof. $\square$

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2. **Wang, N.** and Siu, T. K. (2020). Robust Reinsurance Contracts with Risk Constraint. *Scandinavian Actuarial Journal* **5**, 419-453.

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7. **Wang, N.**, Jin, Z., Siu, T. K. and Qiu, M. Household Consumption-investment-insurance Decisions with Uncertain Income and Market Ambiguity. Under revision.

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9. Thirurajah, S., Shen, Y., Sherris, M., **Wang, N.** and Ziveyi, J. A Combo

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